# A DISCRETE TIME APPROACH FOR EUROPEAN AND AMERICAN BARRIER OPTIONS

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Abstract. The extension of the Black–Scholes option pricing theory to the valuation of barrier options is reconsidered. Working in the binomial framework of CRR we show how various types of barrier options can be priced either by backward induction or by closed binomial formulas. We also consider analytically and numerically the convergence of the prices in discrete time to their continuous–time limits. The arising numerical problems are solved by quadratic interpolation. Furthermore, the case of American barrier options is analyzed in detail. For American barrier call options, binomial formulae and their limit results are given. Finally, the binomial approach is applied to contracts with local and partial barrier checks.

## 1. Introduction

Barrier options are very similar to standard call and put options. However, a final payoff can only occur if during a monitoring period the price of the underlying asset has – depending on the specific contract under consideration – either attained or failed to attain a prespecified upper or lower level, called the "barrier". Such contracts have indeed become the most popular types of exotic options.

Merton [1973] and in particular Conze, Viswanathan [1991] have extended the Black–Scholes model to obtain closed formulas for the valuation of several types of barrier options in continuous time. In general, approximate prices for options can be obtained with binomial models even in cases where it is not possible to derive closed formulas. Here we show that prices for the whole class of barrier options can be obtained within the binomial model, if the backward induction algorithm is suitably adjusted. Fortunately, in many cases the application of the reflection principle allows us to obtain binomial formulas and hence to avoid backward induction .

Similar to the limit result by Cox, Ross, and Rubinstein [1979] (CRR hereafter) for standard options we recover the well–known continuous time formulas for the price of some barrier options as limits of binomial formulas. The results can be seen as a justification for using a binomial model as a discrete approximation of the continuous–time setting. However, unfortunately simulations reveal that with an increasing refinement of the binomial lattice option prices converge in a very irregular manner. We explain and solve this problem using quadratic interpolation.

The pricing of American options continues to be of great interest to researchers. In the case of barrier options early exercise can be optimal even for call options because losses from the underlying hitting a knock–out barrier can thus be avoided. Consequently, the early–exercise–feature of such contracts is examined in detail. Exploiting special properties of the discrete–time set–up, we succeed in constructing binomial formulas for American barrier calls. In particular, a constant early exercise level can be derived in the discrete set–up. In the limit, we recover the formulas for European barrier call options with rebate at the barrier.

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Finally, we briefly extend the analysis to further contract variations. Special attention should be paid to options where the barrier is not continuously but only temporarily or locally checked, since such features, which occur frequently in practice, can result in considerable price differences.

The first binomial option pricing model was developed simultaneously by Cox, Ross and Rubinstein [1979] and Rendleman and Bartter [1979 ]. CRR presented the fundamental economic principles of option pricing by arbitrage considerations in the simplest manner. In addition, they showed that their binomial option pricing formula for a European call yields the Black–Scholes formula as a continuous–time limit.

The pricing of "down–and–out" options dates back to Merton [1973 ][1]. Cox and Rubinstein [1985] explain how the pathdependence of a down–and–out call can be resolved in the binomial model. However, they do not examine the more difficult case of in–options and American options. In a different but simular context, Sondermann [1988] imposes subjective price boundaries on the price path of the underlying in a discrete–time set–up. Using the reflection principle he obtains a binomial formula for which a limit result is derived. Conze, Viswanathan [1991] define several barrier options and derive exact replication and valuation formulas using the reflection principle in continuous time. In addition they derive some results for the corresponding American type options. Rubinstein, Reiner [1991] list continuous–time formulas for all the eight different barrier options. Recently, Boyle and Sok Hoon Lau [1994] have pointed out the irregularities in the convergence of prices of barrier options in binomial lattices which we mentioned above. They solved this difficulty by extracting a subset of refinements of the binomial lattice such that convergence is smooth. These findings where independently put forth in Reimer, Sandmann [1993] . However, we additionally propose a different method, because the method for computing fitting tree refinements may fail. A quadratic interpolation method exhibits stable pricing results for arbitrary barrier conditions and arbitrary, especially constant, tree refinements.

## 2. The discrete time model

Let $\underline{\underline{T}} = \{0 = t_0 < t_1 < ... < t_N = T\}$ be the equidistant discretization of the time axis. Suppose that $S(t_0)$ is the initial asset value at time $t_0$. The stochastic behaviour of the asset is then modeled by

$$S(t_n, i) = S(t_0)u^i d^{n-i} \quad \forall i = 1, ..., n; \quad \forall t_n \in \underline{\underline{T}} \tag{1}$$

where $S(t_n, i)$ denotes the asset price at time $t_n$ after $i$ up–movements and $u > d > 0$, with $u \cdot d = 1$, are the time and state independent proportional asset movements per period. Furthermore assume that the interest rate is constant during the time interval $[0, T]$ and let $r$ be the interest rate per period. The binomial model is arbitrage free iff there exists a probability measure $P$ such that the discounted asset price process is a martingale under $P$. This socalled equivalent martingale measure exists and is unique iff $u > 1 + r > d$ where the transition probability is given by

$$p := P[S(t_{n+1}, .) = S \cdot u | S(t_n, i) = S] = \frac{1 + r - d}{u - d} \tag{2}$$

Since the market structure is complete, the price of an Arrow–Debreu–security $\pi(n, i)$ at $t_0$ ,which pays one unit at time $t_n$ if the asset price is equal to $S(t_0)u^i d^{n-i}$ and otherwise nothing is equal to

$$\pi(n, i) := \left(\frac{1}{1 + r}\right)^n \binom{n}{i} p^i (1 - p)^{n-i} \tag{3}$$

The arbitrage price of any state dependent contingent claim $G$ whose payments are only conditioned on the asset price at time $t_N$ is therefore equal to

$$\pi(G) = \left(\frac{1}{1 + r}\right)^N E_P[G(S_T)] = \left(\frac{1}{1 + r}\right)^N \sum_{i=0}^{N} \binom{N}{i} p^i (1 - p)^{N-i} G\left(S(t_0)u^i d^{N-i}\right) \tag{4}$$

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[1] cp. also Ingersoll in "The New Palgrave"

where $\pi(.)$ is the unique arbitrage free price system. With barrier options, this general pricing principle cannot be applied in a straightforward manner. Due to the barrier condition the payoff depends on the whole price path and not only on the final asset price at time $t_N$. To overcome this problem, one method to calculate the arbitrage free price of European type barrier options is given by a backward induction argument[2]. Consider the case of a down–and–out put or call option with barrier H. Then the following recursive algorithm yields the arbitrage price of these barrier options. Denote by $G_T(t_n, i)$ the value of a down-and-out option issued at time $t_n \in \underline{\underline{T}}$ and state $i = 0, ..., n$ with fixed maturity $t_N = T$. Due to the contract specification at time $t_N = T$, the value of $G_T(t_N, i)$ must be equal to the immediate payoff for all states $i = 0, ..., N$ :

$$(5) \qquad G_T(t_N, i) := \begin{cases} [S(t_0)u^i d^{N-i} - K]^+ \text{ resp. } [K - S(t_0)u^i d^{N-i}]^+ & \text{if} \quad S(t_0)u^i d^{N-i} > H \\ 0 & \text{if} \quad S(t_0)u^i d^{N-i} \leq H \end{cases}$$

and $\forall t_n \in \underline{\underline{T}} \setminus \{t_N\}$ and $i = 0, ..., n$

$$(6) \qquad G_T(t_n, i) := \begin{cases} \frac{1}{1+r}[pG_T(t_{n+1}, i+1) + (1-p)G_T(t_{n+1}, i)] & \text{if} \quad S(t_0)u^i d^{n-i} > H \\ 0 & \text{if} \quad S(t_0)u^i d^{n-i} \leq H \end{cases}$$

The backward induction is based on the martingale property of the discounted price process $G_T$. A similar recursive algorithm can be applied to up–and–out put or call options. Due to the close relationship between the different European barrier options and standard options the backward induction method can be applied straightforward to compute the arbitrage price of all these options. Furthermore this algorithm can be modified easily for American type down–and–out resp. up–and–out put or call options. For example, consider the adjustment to (6) for an American down–and–out call:

$$(6a) \quad G_T(t_n, i) := \begin{cases} \max\{S(t_0)u^i d^{n-i} - K; \frac{1}{1+r}[pG_T(t_{n+1}, i+1) + (1-p)G_T(t_{n+1}, i)]\} & \\ & \text{if } S(t_0)u^i d^{n-i} > H \\ 0 & \text{if } S(t_0)u^i d^{n-i} \leq H \end{cases}$$

Again, a similar algorithm can be applied to American up–and–out put or call options. Unfortunately we cannot deduce the price of American type "in"– options from those of the American type "out"– options. To obtain a backward induction algorithm for American type "in"– options it is worthwile to consider the European case more closely.

For example, consider the down–and–in put option in more detail. Let $H$ be the lower barrier and assume that $H$ is a possible terminal realization of the asset at time $t_N = T$. Let $J_H \in \mathbb{N}$ such that $H = S(t_0)u^{J_H}d^{N-J_H}$. Furthermore, since $u \cdot d = 1$ the symmetry of the binomal lattice implies that $N - 2J_H$ is the minimum number of immediate down–movements such that the asset reaches the barrier for the first time. Since $H$ is a lower barrier (i.e. $H < S$) we have $N - 2J_H > 0$. Note that whenever $S(t_N) = H$ a down–and–in option issued at time $t_n$ with fixed maturity $t_N = T$ is equal to a standard European option issued at time $t_n$. With the same notation as before the following algorithm yields the arbitrage price of European down–and–in put options: $\quad \forall\ i = 0, ..., N$

$$(7) \qquad G_T(t_N, i) := \begin{cases} 0 & \text{if } S(t_0)u^i d^{N-i} > H \\ [K - S(t_0)u^i d^{N-i}]^+ & \text{if } S(t_0)u^i d^{N-i} \leq H \end{cases}$$

and $\forall\ t_n \in \underline{\underline{T}} \setminus \{t_N\};\ i = 0, ..., n$

$$(8)\ G_T(t_n, i) \ := \begin{cases} \frac{1}{1+r}[pG_T(t_{n+1}, i+1) + (1-p)G_T(t_{n+1}, i)] & \text{if } S(t_0)u^i d^{n-i} \neq H \\ \left(\frac{1}{1+r}\right)^{N-n} \sum\limits_{j=0}^{N-n} \binom{N-n}{j} p^j (1-p)^{N-n-j} \left[K - S(t_0)u^{i+j}d^{N-(i+j)}\right]^+ & \text{if } S(t_0)u^i d^{n-i} = H \end{cases}$$

[2]For the case of a down–and–out–call this was already demonstrated by Cox, Rubinstein [1985].

For the American case, the algorithm must be changed slightly. The early exercise of an in–option is only admissible if the price path has already satisfied the "in"–condition. The initial condition (7) is the same as before, whereas (8) is now changed to[3]

$$A_T(t_N, i) := [K - S(t_0)u^i d^{N-i}]^+ \quad \forall i = 0, \cdots, N$$

and $\forall n = N-1, \cdots, 0$

$$A_T(t_n, i) := \max\left\{K - S(t_0)u^i d^{n-i}, \frac{1}{1+r}\left[pA_T(t_{n+1}, i+1) + (1-p)A_T(t_{n+1}, i)\right]\right\}$$

$$G_T(t_n, i) := \begin{cases} \frac{1}{1+r}\left[p \quad G_T(t_{n+1}, i+1) + (1-p) \quad G_T(t_{n+1}, i)\right] & \text{if } S(t_0)u^i d^{n-i} > H \\ A_T(t_n, i) & \text{if } S(t_0)u^i d^{n-i} = H \\ \max\left\{K - S(t_0)u^i d^{n-i} \; ; \; \frac{1}{1+r}\left[pG_T(t_{n+1}, i+1) + (1-p)G_T(t_{n+1}, i)\right]\right\} & \\ & \text{if } S(t_0)u^i d^{n-i} < H \end{cases} \tag{8a}$$

## 3. Closed–form binomial formulae for European barrier options

As a general pricing principle, the backward induction method can be used to price European and American type barrier options in a somehow straightforward manner. To study the convergence behaviour of this method a closed–form binomial formula for barrier options can be constructed. Therefore we redefine the notion of Arrow–Debreu–securities, such that the barrier is reflected.

**Definition 1.**

i) A down–and–in–Arrow–Debreu–security for state $S(t_N) = x$ is defined by the payoff at time $t_N$

$$g_d(x, H) := \begin{cases} 1 & \text{iff } S(t_N) = x \text{ and } \exists\; t_n \in \underline{\underline{T}} \text{ such that } S(t_n) \leq H \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

ii) An up–and–in–Arrow–Debreu–security for state $S(t_N) = x$ is defined by the payoff at time $t_N$

$$g_u(x, H) := \begin{cases} 1 & \text{iff } S(t_N) = x \text{ and } \exists\; t_n \in \underline{\underline{T}} \text{ such that } S(t_n) \geq H \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Given the arbitrage prices of such conditioned Arrow–Debreu–securities at time $t_0$ we can immediately apply the argument which supports the pricing rule (4).

**Proposition 1.** Let $H$ be a possible terminal realization of the asset price at time $t_N$ and $J_H \in H$ such that $H = S(t_0)u^{J_H} d^{N-J_H}$

i) The arbitrage price $\pi_d(N, i, H)$ at $t_0$ of a down–and–in–Arrow–Debreu–security for state $S(t_N) = S(t_0)u^i d^{N-i}$ with barrier $H < S(t_0)$ is equal to

$$\pi_d(N, i, J_H) = \begin{cases} \left(\frac{1}{1+r}\right)^N \binom{N}{i} p^i (1-p)^{N-i} & \text{if } i \leq J_H \\ \left(\frac{1}{1+r}\right)^N \binom{N}{2J_H - i} p^i (1-p)^{N-i} & \text{if } J_H \leq i \leq 2J_H \\ 0 & \text{if } 2J_H < i \end{cases} \tag{11}$$

[3] With $A_T(t_n, i)$ we recursively calculate the arbitrage price of the standard American put option. For the down–and–in call and the up–and–in call or put, similar algorithms can be applied. If the underlying asset is dividend protected, then the early exercise for the American down–and–in call resp. up–and–in call option is not optimal (see section 5.1)

ii) The arbitrage price $\pi_u(N,i,H)$ at $t_0$ of an up–and–in–Arrow–Debreu–security for state $S(t_N) = S(t_0)u^i d^{N-i}$ with barrier $H > S(t_0)$ is equal to

$$\pi_u(N,i,J_H) := \begin{cases} 0 & \text{if } i < \frac{J_H}{2} \\ \left(\frac{1}{1+r}\right)^N \binom{N}{2J_H - i} p^i (1-p)^{N-i} & \text{if } \frac{J_H}{2} \leq i \leq J_H \\ \left(\frac{1}{1+r}\right)^N \binom{N}{i} p^i (1-p)^{N-i} & \text{if } i \geq J_H \end{cases} \tag{12}$$

**Remark.**

1. By arbitrage we have $\pi_u(N,i,H) + \pi_d(N,i,H) \geq \left(\frac{1}{1+r}\right)^N \binom{N}{i} p^i (1-p)^{N-i}$ since the payoff of the left hand side portfolio weakly dominates the unconditional payoff for $\frac{J_H}{2} \leq i \leq 2J_H$ and coincides otherwise.
2. For $J_H \geq \frac{N}{2}$ , i.e. $H \geq S(t_0)$ the down–and–in–Arrow–Debreu–security coincides with the unconditional Arrow–Debreu–security. If $J_H \leq \frac{N}{2}$, i.e. $H \leq S(t_0)$ the up–and–in–Arrow–Debreu–security is equal to an unconditional Arrow–Debreu–security.

**Proof.** For $H < S(t_0)$, i.e. $J_H < \frac{N}{2}$, the reflection principle (Feller [1968]) yields the number $Z_d(N,i,J_H)$ of price paths with terminal value $S(t_0)u^i d^{N-i}$ which touch or cross the barrier $H = S(t_0)u^{J_H} d^{N-J_H}$

$$Z_d(N,i,J_H) := \begin{cases} \binom{N}{i} & \text{if } i \leq J_H \\ \binom{N}{2J_H - i} & \text{if } J_H \leq i \leq 2J_H \\ 0 & \text{if } i > 2J_H \end{cases} \qquad \forall i = 0,..,N$$

Since the transition probability $p$ defines the unique equivalent martingal measure P, the arbitrage price of the down–and–in–Arrow–Debreu–security is given by

$$\pi_d(N,i,H) = \left(\frac{1}{1+r}\right)^N E_P[g_d(N,i,H)]$$

which yields (11). With simular arguments we can derive formula (12).

□

Consequently, these conditioned Arrow–Debreu–securities can be used to compute the binomial formulae of all European type barrier options. The following theorem summarizes this for a European down–and–out call. The remaining formulae are given in the appendix (Proposition 2).

**Theorem 1.** Suppose the barrier $H$ is a terminal knot of the binomial asset price process at time $t_N$, i.e. $\exists J_H \in \mathbb{N}_0$ such that $H = S(t_0)u^{J_H} d^{N-J_H}$, then the arbitrage price of an European down–and–out call[4] with $H < S(t_0)$ is equal to

$$\begin{aligned} C_{d_0}[S,K,T,H] \;=\; & S(t_0) \sum_{i=a\vee J_H}^{N} \binom{N}{i} \bar{p}^i (1-\bar{p})^{N-i} - \left(\frac{1}{1+r}\right)^N K \sum_{i=a\vee J_H}^{N} \binom{N}{i} p^i (1-p)^{N-i} \\ & - S(t_0) \sum_{i=a\vee J_H}^{2J_H} \binom{N}{2_{J_H} - i} \bar{p}^i (1-\bar{p})^{N-i} + \left(\frac{1}{1+r}\right)^N K \sum_{i=a\vee J_H}^{2J_H} \binom{N}{2J_H - i} (1-p)^{N-i} \end{aligned} \tag{13}$$

[4] A reasonable down barrier $H < S(t_0)$ should not be too low with respect to the strike level $K$. If $H$ is too small, no asset price path that touches or crosses the barrier can reach a terminal knot that yields a positive option payoff. Formally, $2J_H \geq a$, otherwise the down-and-out-call is equal to a standard call, i.e. the last two sums of equation (13) are by definition equal to zero.

where

$$\begin{aligned} a &= \inf\{i \in \mathbb{N} | S(t_0)u^i d^{N-i} \geq K\}, \\ a \vee J_H &:= \max\{a, J_H\} \\ \bar{p} &= \frac{pu}{1+r} \quad , \quad p = \frac{1+r-d}{u-d} \end{aligned}$$

**Proof.** By definition of the Arrow–Debreu-securities and the down-and-in Arrow–Debreu securities, we have

$$C_{do}[S,K,T,H] = \sum_{i=0}^{N} \pi(N,i) \left[S(t_0)u^i d^{N-i} - K\right]^+ - \sum_{i=0}^{N} \pi_d(N,i,J_H) \left[S(t_0)u^i d^{N-i} - K\right]^+$$

Since $H := S(t_0)u^{J_H} d^{N-J_H} < S(t_0)$ we have $J_H < \frac{N}{2}$ and by assumption $J_H \geq 0$.

□

Under the usual assumptions, these binomial formulas converge in distribution to the well known formulae for European type barrier options[5] in continuous time. As an example consider the European up–and–out put and down–and–out call[6].

**Theorem 2.** Let $\Delta t = \frac{T-t_0}{N}$ be the grid size of the binomial lattice. For $u = \exp\{\sigma\sqrt{\Delta t}\}$, $d = u^{-1}$ and $\tilde{r} = \frac{1}{\Delta t} ln(1+r)$ (the continuously compounded interest rate) the convergence in the distribution of the binomial formulae is given by

$$\begin{aligned} \lim_{\Delta t \to 0} C_{do}[S(t),K,T,H] &= S(t)N(x(K \vee H)) - Ke^{-\tilde{r}s}N(x(K \vee H) - \sigma\sqrt{s}) \\ &-S(t)\left(\frac{S(t)}{H}\right)^{-1-\alpha} N(y(K \vee H)) + Ke^{-\tilde{r}s} \cdot \left(\frac{S(t)}{H}\right)^{1-\alpha} N(y(K \vee H) - \sigma\sqrt{s}) \end{aligned} \tag{14}$$

$$\begin{aligned} \lim_{\Delta t \to 0} P_{uo}[S(t),K,T,H] &= Ke^{-\tilde{r}s}N\left(-x(K \wedge H) + \sigma\sqrt{s}\right) - S(t)N(-x(K \wedge H)) \\ &-Ke^{-\tilde{r}s}\left(\frac{S(t)}{H}\right)^{1-\alpha} \cdot N\left(-y(K \wedge H) + \sigma\sqrt{s}\right) + S(t)\left(\frac{S(t)}{H}\right)^{-1-\alpha} N\left(-y(K \wedge H)\right) \end{aligned} \tag{15}$$

where

$$\begin{aligned} s &:= T - t \quad \text{the time to maturity; } K \vee H = \max\{K,H\}; K \wedge H = \min\{K,H\} \\ \alpha &:= \frac{2\tilde{r}}{\sigma^2} \qquad \text{and} \\ x(z) &:= \left(\ln\left(\frac{S}{ze^{-\tilde{r}s}}\right) + \frac{1}{2}\sigma^2 s\right) \cdot \frac{1}{\sigma\sqrt{s}}; \quad y(z) := \left(\ln\left(\frac{H^2}{S \cdot ze^{-\tilde{r}s}}\right) + \frac{1}{2}\sigma^2 s\right) \cdot \frac{1}{\sigma\sqrt{s}} \end{aligned}$$

**Proof: see appendix.**

**Remark.**

1. The first two terms of (14) are just equal to the arbitrage price of a standard European call option. The remaining part of (14) corrects the price with respect to the barrier condition. This correction term gives the arbitrage price of a down-and-in call option in the case of $K > H$.
2. For $K < H$ the first two terms of (15) are equal to the arbitrage price of a standard European put option. In this situation the correction terms corresponds to the arbitrage price of a European up–and–in call option.

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[5] see Cox, Rubinstein [1985] and Rubinstein, Reiner [1991]

[6] For completeness, the remaining limit formulae are given in the appendix (Proposition 4)

## 4. Binomial approximation

The binomial formulae for barrier options cover only cases where the barrier $H$ is exactly an endpoint of the binomial tree. But application of the reflection principle requires nothing more than that the barrier $H$ is located within the tree lattice. For barrier levels at tree knots in between terminal knots, the binomial formula remains valid if we have $H = S(t_0)u^{J_H}d^{N-1-J_H}$ and the binomial coefficients in (13) are computed with $2 \cdot J_H + 1$ instead of $2 \cdot J_H$.
The arbitrage price computed by the binomial formulae with a fixed grid size remains constant for all barriers $H$ between two knot–levels of the binomial tree. Consequently, for a given parameter constellation only with a very small number of specific tree refinements the valuation algorithm behaves properly. With deviating refinements we cannot expect a monotonic convergence behaviour to the limit especially when there are small grid sizes. Consider a European down–and–out option. The endpoint condition on $H$ requires that there exists a $J_H \in \mathbb{N}$ such that $S(t_0)u^{J_H}d^{N-J_H} = H$. Define the number $k$ as the minimum number of immediate down movements such that $S(t_0)d^k = H$. Obviously we have $S(t_0)u^i d^{i+k} = H$ for all $i$. Now we can interprete the time grid or the tree refinement as a function of the number $k$ , i.e.

$$\Delta t = \left( \frac{ln \frac{S}{H}}{k \cdot \sigma} \right)^2 \quad \Leftrightarrow \quad N(k) = N = \frac{(T - t_0)k^2\sigma^2}{(\ln \frac{S}{H})^2}$$

The optimal refinement number for a down–and–out call with first touch after $k$ down movements is then defined as[7]

$$N^*(k) = \max \left\{ i \in \mathbb{N} | \quad i \leq N(k) = \frac{(T - t_0)k^2\sigma^2}{(\ln \frac{S}{H})^2}, \; i - k \text{ is an even number} \right\}$$

The following figure underlines the important role of these optimal refinement numbers.

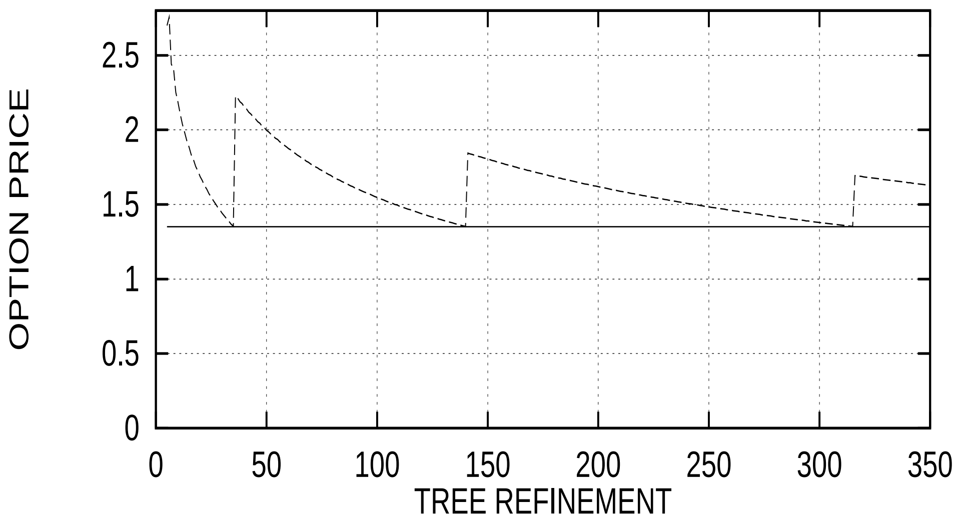



*Figure 1:* Binomial formula for a down–and–out call with $S(t_0) = 40, K = 40, r = 5\%, \sigma = 15\%, T = 365$ days, $H = 39$ and optimal refinement $N^*(k) = 35, 140, 315$ for $k = 1, 2, 3$.

The appropriate grid size in a binomial model depends in a crucial manner on the barrier $H$. This is obviously an unfortunate feature. If the discrete time framework is used to approximate the continuous time model, in some sense "better" or "quicker" approximations are desirable. Although closed–form solutions for European barrier options are known, a "better" numerical approximation technique is useful as a test for situations where closed–form solutions are not available or unknown.

[7] This has been observed by Boyle and Sok Hoon Lau [1994] in an independent study. They consider the recursive algorithms and define the optimal refinement number in a similar way.

In the case of a European down–and–out call the following technique appears to be very sucessful. For a fixed number of periods $N$ resp. grid size $\Delta t$ and a fixed barrier $H$ which is not a barrier level of the binomial tree we can select three barriers $H_1, H_2, H_3$ of the binomial tree lattice such that

$$\begin{aligned} H_1 &:= S(t_0)u^{J_H^*}d^{N-J_H^*} < H_2 = S(t_0)u^{J_H^*}d^{N-J_H^*-1} < H_3 = S(t_0)u^{J_H^*+1}d^{N-J_H^*-1} \\ \text{for } J_H^* &= \max\{i \in \mathbb{N} \mid S(t_0)u^i d^{N-i} \leq H\} \Rightarrow H_1 \leq H < H_3 \end{aligned}$$

Using the binomial formula we can compute the arbitrage prices of the down–and–out call options with these barriers. The price of a down–and–out call option with barrier $H \in [H_1, H_3]$ is now simply approximated by the Lagrange interpolation polynomium of degree 2, i.e.

$$\begin{aligned} C_{d_0}[S,K,T,H] \approx f(H) &= \sum_{i=1}^{3} L_i(H) \cdot C_{d_0}[S,K,T,H_i] \\ L_i(H) &= \prod_{j \neq i}^{3}(H - H_j) \;/\; \prod_{j \neq i}^{3}(H_i - H_j) \end{aligned} \tag{16}$$

Figure 2 gives a typical example of the success of this approximation for a fixed grid size and barriers $H$ between 35 and the initial asset price $S$. There is basically no difference between the continuous time solution and the approximation. Actually, you cannot recognize the result, because of the precision of the approximation.

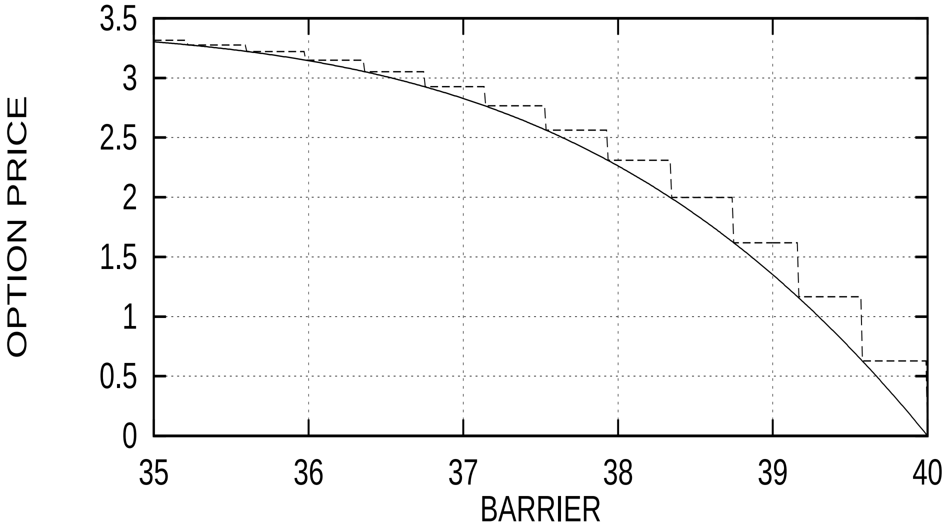



*Figure 2:* Approximation of the continuous time solution for a down-and-out call with a binomial model of $N = 200$ periods with and without Lagrange interpolation. $S(t_0) = 40, K = 40, r = 5\%, \sigma = 15\%$ and $T = 365$ days.

## 5. American barrier options on dividend protected securities

From Merton (1973) we know that a standard American type call option on a dividend protected asset is always more worth alive than dead, i.e. early exercise does not occur. In the case of an out barrier option, this is not always true, since when the underlying asset reaches the barrier, the contract becomes worthless. Thus in general, there is an incentive for early exercise just before reaching the barrier. The following proposition extends Merton's result to the case of barrier call options:

**Proposition 5.** Let the underlying security be a dividend protected security, then

a) an American down-and-in and an American up-and-in call option will never be exercised before maturity.

b) for an American up-and-out call option with barrier $H > S(t_0)$ early exercise can become optimal if and only if $H > K$ .

c) for an American down-and-out call option with barrier $H < S(t_0)$ and continuous price paths of the underlying security early exercise can become optimal if and only if $H > K$.

**Proof.**

ad a) By definition, the option can only be exercised if it is already "in". In this situation, the barrier option is equivalent to a standard call option for which Merton's result applies.

ad b) If $H < K$ a European up-and-out call is worthless, and furthermore whenever the inner value $\text{Max}\{S_t - K, 0\}$ at time $t < T$ is greater than zero, the barrier condition implies that the contract is already out.

Suppose $H > K$ and that the option is still alive at time $t$ before maturity. The inner value at time $t$ is then by definition equal to

$$g(S_t) = \begin{cases} S_t - K & \forall\ K < S_t < H \\ 0 & \forall\ S_t \geq H, S_t < K \end{cases}$$

Due to $H > K$ the early exercise payoff is discontinuous at $S_t = H$ and bounded by $H - K$ from above. A sufficient condition for early exercise at time $t$ is therefore given by

$$\begin{aligned} (S_t - K)e^{\tilde{r}(T-t)} &\geq H - K > g(S_T) \quad \forall \quad S_T \\ \Leftrightarrow S_t &\geq He^{-\tilde{r}(T-t)} - K\left(1 - e^{-\tilde{r}(T-t)}\right) \end{aligned}$$

ad c) i) [8] First consider the situation $H \leq K$. Let $t \in [0, T[$ and assume that the option is still alive, i.e. $S_t* > H\ \forall\ t^* \in [0, t]$. In the case $H < S_t \leq K$, there is no early exercise, since the inner value $[S_t - K]^+$ is equal to zero. For $H \leq K < S_t$ consider the following portfolio: buy the down-and-out call with the barrier $H$ and time to maturity $T - t$, sell the underlying asset and place the exercise price into the money account. At $t$, the portfolio is worth

$$C_{do}[S_t, K, T - t, H] - S_t + K$$

Now, in case the barrier is not reached until time $T$, the down-and-out call yields the same payoff as the standard call, and therefore the final payoff at time $T$ is given by

$$[S_T - K]^+ - S_T + Ke^{\tilde{r}(T-t)} = \begin{cases} K(e^{r(T-t)} - 1) & \geq & 0 & \text{if } S_T \geq K \\ Ke^{r(T-t)} - S_T & > & 0 & \text{if } S_T < K \end{cases}$$

Now assume the barrier is reached at time $t^* \in ]t, T[$ for the first time. Since by assumption $S_t* = H$, the value of the portfolio at time $t^*$ is equal to $-H + Ke^{\tilde{r}(t^*-t)} \geq 0$, which can be placed into the money account until time $T$. Thus, the final payoff of this portfolio strategy yields a non-negative payoff (even positive if $r > 0$) and by means of no arbitrage, this implies a non-negative initial value of the portfolio: $C_{do} \geq S_t - K$

ii) Second, consider the situation $H > K$. Suppose that the down-and-out call is still alive at time $t < T$, i.e. $S_t* > H\ \forall\ t^* \in [0, t]$. With the same portfolio argument as in case i), where instead of $K$ the discounted exercise price $Ke^{-\tilde{r}(T-t)}$ is placed into the money account, we can conclude that for the European down-and-out call the following boundary conditions must be satisfied:

$$C_{do}[S_t, K, T - t, H] \leq S_t - Ke^{-\tilde{r}(T-t)} \quad \text{if} \quad S_t > H > K$$

[8] The portfolio argument and the proof of Proposition 5 part c was first given by Daniel Sommer.

and furthermore that

$$C_{do}[S_t, K, T-t, H] \geq S_t - H \quad \text{if} \quad S_t > H > K$$

where both bounds are tight. Since $S_t - K > S_t - H$ for $H > K$, there are situations possible, when early exercise is optimal for the option holder.

□

We can now apply these distribution free results to the special structure of the binomial model.

**Theorem 3.**

a) The arbitrage price of an American up-and-out call option with barrier $Su^{J_H}d^{N-J_H} = H > K$ and a grid size $\Delta t = \frac{T-t_0}{N}$ such that $dH > K$ is equal to

$$C_{uo}^{am}[S,K,T,H] = C_{uo}^{eur}[S,K,T,dH] + \frac{S}{Hd}\sum_{i=1}^{N(h)}\left[\binom{h-2+2i}{i} - \binom{h-2+2i}{i-1}\right]\bar{p}^{h-1+i}(1-\bar{p})^i[dH-K] \tag{17}$$

where $h = 2J_H - N$ for $H > S, \bar{p} = \frac{pu}{1+r}$, and $N(h) = \sup\{i \in \mathbb{N} | i \leq \frac{N+2-h}{2}\}$

b) The arbitrage price of an American down-and-out call option with barrier $Su^{J_H}d^{N-J_H} = H > K$ and grid size $\Delta t = \frac{T-t_0}{N}$ such that $uH < K$ is equal to

$$C_{do}^{am}[S,K,T,H] = C_{do}^{eur}[S,K,T,uH] + \frac{S}{Hu}\sum_{i=1}^{N(h)}\left[\binom{h-2+2i}{i} - \binom{h-2+2i}{i-1}\right]\bar{p}^{i}(1-\bar{p})^{h-1+i}[uH-K] \tag{18}$$

where $h = N - 2J_H$ for $H < S$.

**Proof:** see appendix

**Remark.**

1) The reason for these binomial closed–form solution is the existence of a constant early exercise boundary. Thus American type barrier options are in some cases equivalent to European barrier options with a constant rebate.
2) Applying the continuous time closed–form solutions for European barrier options with a constant rebate (Rubinstein,Reiner (1991)) we have the following limit results:

$$\lim_{\substack{\Delta t\to 0\\ H>K}} C_{uo}^{am}[S,K,T,H] = \lim_{\substack{\Delta t\to 0\\ H>K}} C_{uo}^{eur}[S,K,T,H] + [H-K]\cdot\left[\left(\frac{S}{H}\right)^{-\alpha} N\left(-y_1(H)\right) + \left(\frac{S}{H}\right) N\left(-y_2(H)+\alpha\sigma\sqrt{s}\right)\right] \tag{19}$$

$$\lim_{\substack{\Delta t\to 0\\ H>K}} C_{do}^{am}[S,K,T,H] = \lim_{\substack{\Delta t\to 0\\ H>K}} C_{do}^{eur}[S,K,T,H] + [H-K]\left[\left(\frac{S}{H}\right)^{-\alpha} N\left(y_1(H)\right) + \left(\frac{S}{H}\right) N\left(y_2(H)-\alpha\sigma\sqrt{s}\right)\right] \tag{20}$$

with $\alpha = \frac{2\tilde{r}}{\sigma^2}$, $s = T - t_0$ and $y_{1,2}(z) = \left(\frac{\ln\left(\frac{H^2}{S\cdot z e^{-\tilde{r}s}}\right)\pm\sigma^2 s}{\sigma\sqrt{s}}\right)$.

3) The argument for American put options is similar but we can't expect to find closed form solutions for all cases. The basic difficulty is that it can be optimal to exercise a standard put option when the value of the underlying is small. Thus for the case of the up-and-out and up-and-in put option, it is not possible to find a closed–form binomial expression. In the case of a down-and-out put or down-and-in put, it is possible to find closed–form solutions for some barriers $H$. If the barrier $H < K$ is greater than the critical value $S^*(t)$ of the underlying, which indicates the early exercise for standard put option at time $t$, then the American down-and-out put will be exercised just before the barrier. This can be expressed by a binomial formula, which includes again a rebate of $K - H$. For the American down-and-in put, a binomial formula can be constructed in the case, where $H < S^*(t) < K$ where $S^*(t)$ is again the critical value for early exercise at time $t$ in the standard case. In both cases, the limit result is given by the corresponding European type down barrier puts plus a rebate of $[K - H]$. In all the other cases, we have to apply a recursive algorithm.

## 6. European options with local or partial barrier condition

We consider now situations, where the barrier condition has to be satisfied only on a subset of spots, but not on the whole time interval. We restrict the analysis to the following three basic cases, which we define in the discrete framework.

**Definition 2.** Let $\underline{\underline{T}} = \{0 = t_0 < t_1 < ... < t_{N_1} < t_{N_1+1} < ... < t_N = T\} \subset [0, T]$ be an equidistant discretization of the time axis.

a) A barrier option with maturity $T$, underlying security $S$, and barrier $H$ is called
   i) a *front partial barrier option* with barrier period $\underline{\underline{T}}(t_0, t_{N_1}) = \{t_0 < ... < t_{N_1}\} \subset \underline{\underline{T}}$, if the path dependency of the payoff is restricted to the period $\underline{\underline{T}}(t_0, t_{N_1})$ and independent of the security realizations at times $t \in \{t_{N_1+1} < ... < t_{N-1}\}$.
   ii) a *back partial barrier option* with barrier period $\underline{\underline{T}}(t_{N_1}, t_N) = \{t_{N_1} < ... < t_N\}$, if the path dependency of the payoff is restricted to the period $\underline{\underline{T}}(t_{N_1}, t_N)$ and independent of the security realizations at times $t_i \in \{t_0 < ... < t_{N_1-1}\}$.

b) A barrier option with maturity $T$ is called a *local barrier option* with barrier times $\underline{\underline{T}}^H = \{t_0' < t_1' < ... < t_n'\} \subset \underline{\underline{T}}$ if the path dependency of the payoff is restricted to the set $\underline{\underline{T}}^H$ and independent of the security realizations at times $t \in \underline{\underline{T}} \setminus \underline{\underline{T}}^H$.

The payoff of a front partial down–and–out call option with maturity $T > t_{N_1}$, barrier $H$, and barrier period is defined by $\underline{\underline{T}}(t_0, t_{N_1})$ is given by

$$\begin{cases} [S_T - K]^+ & \text{if} \quad S_{t_i} > H \ \forall\, t_i \in \underline{\underline{T}}(t_0, t_{N_1}) \\ 0 & \text{if} \quad \exists\, t^* \in \underline{\underline{T}}(t_0, t_{N_1}) \quad \text{with} \quad S_{t_*} \le H \end{cases}$$

With reference to the previous discussion we can compute a binomial formula for a partial barrier option if we can compute the corresponding prices of partial down-and-in, resp. partial up–and–in, Arrow–Debreu–securities. Given the binomial model for the underlying security, these prices can be computed by applying the reflection principle (see proposition 5 in the appendix). With these prices, we can compute the arbitrage prices of all partial down barrier options. The following theorem demonstrates this for the partial down–and–out call option.

**Theorem 4.**

i) Let the barrier $H$ be a knot of the binomial security process at time $t_{N_1}$, i.e. $\exists J_H \in N_0$ such that $H = S(t_0) u^{J_H} d^{N_1 - J_H}$. The arbitrage price of a European front partial down-and-out call with barrier period $\underline{\underline{T}}(t_0, t_{N_1}) = \{t_0 < ... < t_{N_1}\}$ is equal to

$$C_{do}^{fp}[S,K,T,H,\underline{\underline{T}}(t_0,t_N)]$$

$$= \left(\frac{1}{1+r}\right)^N \sum_{i=J_H+1}^{N} \binom{N}{i} p^i(1-p)^{N-i}\left[Su^id^{N-i}-K\right]^+ \tag{21}$$

$$- \left(\frac{1}{1+r}\right)^N \sum_{i=J_H+1}^{J_H+N-N_1} \sum_{k=0\vee(i+N_1-N)}^{J_H} \binom{N_1}{k}\binom{N-N_1}{i-k} \cdot p^i(1-p)^{N-i}[Su^id^{N-i}-K]^+$$

$$- \left(\frac{1}{1+r}\right)^N \sum_{i=J_H+1}^{2J_H+N-N_1} \sum_{k=(J_H+1)\vee(i+N_1-N)}^{2J_H\wedge i} \binom{N_1}{2J_H-k}\binom{N-N_1}{i-k} \cdot p^i(1-p)^{N-i}[Su^id^{N-i}-K]^+$$

ii) Let the barrier $H$ be a knot of the binomial security price process at time $t_N$, i.e. $\exists J_H \in N_0$ such that $H = S(t_0)u^{J_H}d^{N-J_H}$. The arbitrage price of a European back partial down–and–out call with barrier $H$, maturity $t_N = T$, exercise price $K$ and barrier period $\underline{\underline{T}}(t_{N_1},t_N) = \{t_{N_1} < \dots < t_N\}$ is equal to

$$C_{do}^{bp}[S,K,T,H,\underline{\underline{T}}(t_{N_1},t_N)]$$

$$= \left(\frac{1}{1+r}\right)^N \sum_{i=J_H}^{N} \binom{N}{i} p^i(1-p)^{N-i}[Su^id^{N-i}-K]^+ \tag{22}$$

$$- \left(\frac{1}{1+r}\right)^N \sum_{i=J_H}^{2J_H\wedge(J_H+\frac{N-N_1}{2})} \sum_{k=0\vee(i+N_1-N)}^{N_1\wedge(2J_H-i)} \binom{N_1}{k}\binom{N-N_1}{2J_H-i-k} \cdot p^i(1-p)^{N-i}[Su^id^{N-i}-K]^+$$

As the last extension of the binomial approach, we consider the situation of local barrier options. Let $\underline{\underline{T}}^H = \{t'_1 < \dots < t'_n\} \subseteq \underline{\underline{T}}$ be a given subset of $\underline{\underline{T}}$. For each local barrier option, there exists a recursive algorithm to compute the arbitrage price. Consider for example a local down-and-out call with barrier $H$. As in section 2 denote by $G_T(t_n,i)$ the value of such a local down-and-out call with fixed maturity $t_N = T$ issued at time $t_n \in \underline{\underline{T}}$ and state $i$, i.e. $S(t_n,i) = S(t_0)u^id^{n-i}$. The initial condition of the algorithm is therefore

$$G_T(T,i) = \begin{cases} [S(t_0)u^id^{N-i}-K]^+ & \text{if } \; T = t_N \notin \underline{\underline{T}}^H \\ [S(t_0)u^id^{N-i}-K]^+ & \text{if } \; t_N \in \underline{\underline{T}}^H \text{ and } S(t_0)u^id^{N-i} > H \\ 0 & \text{if } \; t_N \in \underline{\underline{T}}^H \text{ and } S(t_0)u^id^{N-i} \le H \end{cases} \tag{23}$$

By backward induction we have $C_{do}^{local}[S,K,T,H,\underline{\underline{T}}^H] = G_T(t_0,0)$ with $\forall\ k = 0,\dots,N-1$ and $i = 0,\dots,k$

$$G_T(t_k,i) = \begin{cases} \frac{1}{1+r}[pG_T(t_{k+1},i+1)+(1-p)G_T(t_{k+1},i)] & \text{if } \; t_k \notin \underline{\underline{T}}^H \\ \frac{1}{1+r}[pG_T(t_{k+1},i+1)+(1-p)G_T(t_{k+1},i)] & \text{if } \; t_k \in \underline{\underline{T}}^H \\ & \quad \text{and } S(t_k,i) > H \\ 0 & \text{if } \; t_k \in \underline{\underline{T}}^H \\ & \quad \text{and } S(t_k,i) \le H \end{cases} \tag{24}$$

Furthermore if we assume that $H$ is a knot of the binomial security price process at any time $t'_k \in \underline{\underline{T}}^H$ it is possible to construct a binomial formula. For simplicity let us assume that both sets $\underline{\underline{T}}$ and $\underline{\underline{T}}^H$ are equidistant sets, i.e. there exists a number $N_H \in \mathbb{N}$ such that $t'_j \in \underline{\underline{T}}^H$ is equal to $t_{jN_H} \in \underline{\underline{T}}$. Thus $t_{j+1} - t_j = \Delta t \cdot N_H \;\; \forall\ t_j \in \underline{\underline{T}}^H$, $t'_n = t_{n\cdot N_H} = t_N \in \underline{\underline{T}}$ and $\Delta t$ is the grid size of the set $\underline{\underline{T}}$. Furthermore assume that $N_H$ is an even number. Let $H$ be a terminal knot, i.e. $\exists\ J_H \in \mathbb{N}$ such that

$H = S(t_0)u^{J_H}d^{N-J_H}$ and since $N_H$ is even, $H$ is also a knot at time $t'_j \in \underline{\underline{T}}^H$. With these simplifications the arbitrage price of a local down–and–out call is equal to

$$C_{do}^{local}[S, K, T, H, \underline{\underline{T}}^H]$$
$$= \left(\frac{1}{1+r}\right)^N \sum_{i_1=j_H(1)+1}^{N_H} \sum_{i_2=0\vee 1+j_H(2)-i_1}^{N_H} \cdots \sum_{i_n=0\vee 1+j_H(n)-i_{n-1}}^{N_H} \binom{N_H}{i_1}\binom{N_H}{i_2}\cdots\binom{N_H}{i_n}$$

$$\cdot p^{\sum_{k=1}^n i_k}(1-p)^{N-\sum_{k=1}^n i_k}\left[S(t_0)u^{\sum_{k=1}^n i_k}d^{N-\sum_{k=1}^n i_k} - K\right]^+ \tag{25}$$

where $j_H(n) = J_H$ and $j_H(k) = j_H(k+1) - \frac{N_H}{2} = j_H - (n-k)\frac{N_H}{2}$ is the number of up movements needed at time $t_k$ such that $S(t_k, j_H(k) = H$. Obviously this formula is only useful in situations where the number of local checks of the barrier is small.

## 7. Summary

Cox, Ross, Rubinstein [1979] and Rendleman, Bartter [1979] have developed a binomial model for the pricing of European and American type standard options. For European type options they derived closed–form binomial formulae which converge to the Black–Scholes formulae under the usual assumptions. Within the binomial framework we have derived recursive algorithms which can be used for both European and American barrier options. Furthermore the general argument supporting these algorithms can be used in the case of modifications of the contract definition or/and to dividend paying securities. In analogy to Cox, Ross, Rubinstein and Rendleman, Bartter we give binomial formulae for European barrier options and prove the convergence towards the continuous time solutions. In addition the convergence behaviour is analyzed and a robust approximation with Lagrange interpolation is proposed. This interpolation method reduces the complexity of the lattice and is therefore of practical use for the implementation of numerical procedures. Furthermore we solve the case of American barrier options explicitly and derive closed–form solutions within the binomial and continuous time framework. The Merton (1973) result for American type call options is extended to American barrier call options. As a consequence the binomial approach choosen can be generalized immediately to European type barrier options with rebate. Finally barrier options with local or partial barrier condition are discussed within the binomial framework.

## 8. Appendix

**Proposition 2.** Suppose the barrier $H$ is a terminal knot of the binomial asset price process at time $t_N$; i.e. $\exists J_H \in \mathbb{N}$ such that $H = S(t_0)u^{J_H}d^{N-J_H}$. Define for $a, b \in \mathbb{N}$ the following binomial sums:

$$B(p,a,b) := \begin{cases} 0 & \text{for } a > b \\ \sum_{i=a}^{b} \binom{N}{i} p^i (1-p)^{N-i} & \text{for } a \le b \le N \end{cases}$$

$$\tilde{B}(p,a,b) := \begin{cases} 0 & \text{for } a > b \\ \sum_{i=a}^{b} \binom{N}{2J_H - i} p^i (1-p)^{N-i} & \text{for } a \le b \le 2J_H \end{cases}$$

Under these assumptions the arbitrage price of the following barrier options (where we assume $H < S(t_0)$ in the down case and $H > S(t_0)$ in the up case) is equal to

$$\begin{aligned}
C_{d_i}[S,K,T,H] &= S(t_0)\cdot B(\bar p, a, J_H-1) - \hat K \cdot B(p,a,J_H-1) \\
&\quad + S(t_0)\cdot \tilde B(\bar p, a\vee J_H, 2J_H) - \hat K\cdot \tilde B(p, a\vee J_H, 2J_H) \\
C_{uo}[S,K,T,H] &= S(t_0)\cdot B(\bar p, a, J_H) - \hat K \cdot B(\bar p, a, J_H) \\
&\quad - S(t_0)\cdot \tilde B(\bar p, a\vee \frac{[J_H]}{2}, J_H) + \hat K\cdot B(p, a\vee \frac{[J_H]}{2}, J_H) \\
C_{ui}[S,K,T,H] &= S(t_0)\cdot B(\bar p, a\vee (J_H+1), N) - \hat K\cdot B(\bar p, a\vee(J_H+1), N) \\
&\quad + S(t_0)\cdot \tilde B(\bar p, a\vee \frac{[J_H]}{2}, J_H) - \hat K\cdot \tilde B(p, a\vee \frac{[J_H]}{2}, J_H) \\
P_{do}[S,K,T,H] &= \hat K\cdot B(p, J_H, b) - S(t_0)\cdot B(\bar p, J_H, b) \\
&\quad - \hat K\cdot \tilde B(p, J_H, 2J_H\wedge b) + S(t_0)\cdot \tilde B(\bar p, 2J_H\wedge k) \\
P_{di}[S,K,T,H] &= \hat K\cdot B(p,0,(J_H-1)\wedge b) - S(t_0)\cdot B(\bar p, 0, (J_H-1)\wedge b) \\
&\quad + \hat K\cdot \tilde B(p, J_H, 2J_H\wedge b) - S(t_0)\tilde B(\bar p, J_H, 2J_H\wedge b) \\
P_{uo}[S,K,T,H] &= \hat K\cdot B(p,0,J_H\wedge b) - S(t_0)\cdot B(\bar p, 0, J_H\wedge b) \\
&\quad - \hat K\cdot \tilde B(p, \frac{[J_H]}{2}, b\wedge J_H) + S(t_0)\tilde B(\bar p, \frac{[J_H]}{2}, b\wedge J_H) \\
P_{ui}[S,K,T,H] &= \hat K\cdot B(p,(J_H+1),b) - S(t_0)\cdot B(\bar p, (J_H+1), b) \\
&\quad + \hat K\cdot \tilde B(p, \frac{[J_H]}{2}, b\wedge J_H) - S(t_0)\tilde B(\bar p, \frac{[J_H]}{2}, b\wedge J_H)
\end{aligned}$$

where

$$\begin{aligned}
a &:= \inf\{i\in \mathbb{N} | S(t_0)u^i d^{N-i} \geq K\} & b &:= \sup\{i\in\mathbb{N}|S(t_0)u^i d^{N-i}\leq K\} \\
\bar p &:= \frac{p\cdot u}{1+r} & p &:= \frac{1+r-d}{u-d} \\
a\vee J_H &:= \max\{a, J_H\} & a\wedge J_H &:= \min\{a, J_H\} \\
\hat K &:= \left(\frac{1}{1+r}\right)^N\cdot K.
\end{aligned}$$

**Proof of Theorem 2.**

1) Consider the binomial formula (13) for the European down–and–out–call. Since for $H\leq K \Leftrightarrow a\geq J_H$ the first two terms coincide with the usual binomial formula for European call options for which we already know that under the given assumptions the limit in distribution of

$$S(t_0)\sum_{i=a\vee J_H}^{N}\binom{N}{i}\bar p^i(1-p)^{N-i} - \left(\frac{1}{1+r}\right)^N\cdot K\sum_{i=a\vee J_H}^{N}\binom{N}{i}p^i(1-p)^{N-i} \quad \text{for } H\leq K$$

is given by[9]

$$S(t_0)N(x) - Ke^{-r(T-t_0)}N(x-\sigma\sqrt{T-t_0})$$

with

$$x(K) = \left(\ln\left(\frac{S(t)}{Ke^{-r\sqrt{T-t_0}}}\right)+\frac{1}{2}\sigma^2(T-t_0)\right)\frac{1}{\sigma\sqrt{T-t_0}}$$

For $H>K$ it is easy to see that we only have to consider $x(H)$ instead of $x(K)$ as the argument of the standard normal distribution. It remains to proof that under the assumption of theorem 2 the correction term (for $N$ sufficiently large)

$$(*)\qquad S(t_0)\sum_{i=a\vee J_H}^{2J_H}\binom{N}{2J_H-i}\bar p^i(1-\bar p)^{N-i} - \left(\frac{1}{1+r}\right)^N\cdot K\sum_{i=a\vee J_H}^{2J_H}\binom{N}{2J_H-i}p^i(1-p)^{N-i}$$

with $N\geq 2J_H$ converges in distribution to

$$S(t_0)\left(\frac{S(t)}{H}\right)^{-(\alpha+1)}N(y(K\vee H)) - Ke^{-r(T-t)}\left(\frac{S(t)}{H}\right)^{1-\alpha}N(y(K\vee H))-\sigma\sqrt{T-t_0})$$

---

[9] See for example Cox, Rubinstein [1985] , for simplicity let $r$ be the continuously compounded interest rate.

For simplicity let us assume $K \geq H$ and[10] therefore $a \geq J_H$. By index transformation $(*)$ can be rewritten as

$$
\begin{aligned}
& S(t_0)\left(\frac{1-\bar{p}}{\bar{p}}\right)^{N-2J_H} \sum_{i=0}^{2J_H-a} \binom{N}{i} \bar{p}^{(N-i)}(1-\bar{p})^i \\
& -\frac{K}{(1+r)^N}\left(\frac{1-p}{p}\right)^{N-2J_H} \sum_{i=0}^{2J_H-a} \binom{N}{i} p^{(N-i)}(1-p)^i
\end{aligned}
$$

For sufficiently small $\Delta t = \frac{T-t_0}{N}$ the martingale transition probability $p$ can be approximated by

$$p = \frac{e^{r\Delta t}-d}{u-d} \approx \frac{1}{2} + \frac{1}{2}\frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{\Delta t}$$

which yields

$$
\begin{aligned}
\lim_{\Delta t\to 0} \quad & E_p\left[\ln\frac{S(T)}{S(t_0)}\right] = (r-\frac{\sigma^2}{2})(T-t_0) \\
\lim_{\Delta t\to 0} \quad & V_p\left[\ln\frac{S(T)}{S(t_0)}\right] = \sigma^2(T-t_0)
\end{aligned}
$$

Furthermore we have the following approximation for the ratio $\dfrac{p}{1-p}$:

$$
\begin{aligned}
\frac{p}{1-p} &= \frac{1+\frac{r-\sigma^2/2}{\sigma}\sqrt{\Delta t}}{1-\frac{r-\sigma^2/2}{\sigma}\sqrt{\Delta t}} + o(\Delta t) \\
&= 1+2\left(\frac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}\cdot\frac{1}{1-\frac{r-\sigma^2/2}{\sigma}\sqrt{\Delta t}} + o(\Delta t) \\
&= 1+2\left(\frac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}\cdot\sum_{i=0}^{\infty}\left(\frac{n-\sigma^2/2}{\sigma}\sqrt{\Delta t}\right)^i + o(\Delta t) \\
&= 1+2\sum_{i=0}^{\infty}\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{\Delta t}\right)^{i+1} + o(\Delta t) \\
&= 1+2\left(\frac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}+2\left(\frac{r-\sigma^2/2}{\sigma}\right)^2\Delta t + o(\Delta t)
\end{aligned}
$$

Observing that for small $\Delta t$ the Taylor-expansion of the exponential function is given by

$$
\begin{aligned}
\exp\left\{2\left(\frac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}\right\} &= \sum_{i=0}^{\infty}\frac{1}{i!}\left(2\left(\frac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}\right)^i \\
&= 1+2\left(\frac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}+2\left(\frac{r-\sigma^2/2}{\sigma}\right)\Delta t + o(\Delta t)
\end{aligned}
$$

yields the approximation $\frac{p}{1-p} \approx \exp\left\{2\left(\frac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}\right\}$
Therefore we obtain the following results[11]:

$$
\begin{aligned}
\text{i)} \quad & S(t_0)u^{J_H}d^{N-J_H} = H \Rightarrow N-2J_H = \frac{\ln(\frac{H}{S(t_0)})}{\ln d} \\
& J_H = \frac{\ln(\frac{H}{S(t_0)}d^N)}{\ln(\frac{u}{d})} \\
\text{ii)} \quad & \lim_{\Delta t\to 0}\left(\tfrac{1-p}{p}\right)^{N-2J_H} = \lim_{\Delta t\to 0}\exp\left\{-2\left(\tfrac{r-\sigma^2/2}{\sigma}\right)\sqrt{\Delta t}\cdot\ln\tfrac{H}{S(t_0)}\cdot\tfrac{1}{-\sigma\sqrt{\Delta t}}\right\} \\
& = \left(\tfrac{S(t_0)}{H}\right)^{1-\frac{2r}{\sigma^2}} \\
\text{iii)} \quad & \lim_{\Delta t\to 0}\left(\tfrac{d}{u}\right)^{N-2J_H} = \lim_{\Delta t\to 0}\exp\left\{-2\sigma\sqrt{\Delta t}\cdot\ln\tfrac{H}{S(t_0)}\cdot\tfrac{1}{-\sigma\sqrt{\Delta t}}\right\} = \left(\tfrac{H}{S(t_0)}\right)^2 \\
\Rightarrow \quad & \lim_{\Delta t\to 0}\left(\tfrac{1-\bar{p}}{\bar{p}}\right)^{N-2J_H} = \lim_{\Delta t\to 0}\left(\tfrac{1-p}{p}\cdot\tfrac{d}{u}\right)^{N-2J_H} = \left(\tfrac{S(t_0)}{H}\right)^{-1-\frac{2r}{\sigma^2}}
\end{aligned}
$$

[10] The case $H < K$ is similar and can be done by a change of variables.

[11] Now we explicitly use the assumption that $H$ is an endpoint of the binomial tree.

Finally we have to consider the two sums. Let $J(N)$ be the sum of $N$ independent binomially distributed variables with up and down probabilities $1-p$ resp. $p$. Thus we have

$$E_p[J(N)] = N(1-p) \quad \text{and} \quad V_p[J(N)] = Np(1-p).$$

$$\Rightarrow \sum_{i=0}^{2J_H-a} \binom{N}{i} p^{N-i}(1-p)^i = \text{prob}[J(N) \le 2J_H - a]$$

Obviously, the Central Limit Theorem can be applied. By construction we have[12]

$$\begin{array}{llcl} \text{a)} & \frac{J(N)-E_p[J(N)]}{\sqrt{V_p[J(N)]}} & = & \frac{\ln\left(\frac{S_T}{S(t_0)}\right)-E_p[\ln\frac{S_T}{S(t_0)}]}{\sqrt{V_p[\ln\frac{S_T}{S(t_0)}]}} \\ \text{b)} & 2J_H - a & = & \frac{2\left(\ln\frac{H}{S(t_0)d^N}\right)-\ln\frac{K}{S(t_0)d^N}}{\ln\frac{u}{d}} - \varepsilon \\ & \Rightarrow \frac{2J_H-a-E_p[J(N)]}{\sqrt{V_p[J(N)]}} & = & \frac{2\ln H-\ln S(t_0)-\ln K-\varepsilon\ln\frac{u}{d}+E_p[\ln\frac{S_T}{S(t_0)}]}{\sqrt{V_p[\ln\frac{S_T}{S(t_0)}]}} \\ & & \overset{\Delta t\to 0}{\longrightarrow} & \frac{2\ln H-\ln S(t_0)-\ln K+(r-\sigma^2/2)(T-t_0)}{\sigma\sqrt{(T-t_0)}} =: y_2 \end{array}$$

By the Central Limit Theorem therefore we have

$$\lim_{\Delta t\to 0} \sum_{i=0}^{2J_H-a} \binom{N}{i} p^{N-i}(1-p)^i = N(y_2)$$

For the second sum we can use the same argument. The only change concerns the transition probability $\bar{p}$. The Taylor-expansion for $\bar{p}$ yields

$$\bar{p} \approx \frac{1}{2} + \frac{1}{2}\frac{r+\sigma^2/2}{\sigma}\sqrt{\Delta t}$$

$$\begin{array}{rcl} \Rightarrow \lim\limits_{\Delta t\to 0} E_{\bar{p}}[\ln\frac{S_T}{S(t_0)}] & = & \left(r+\frac{\sigma^2}{2}\right)(T-t_0) \\ \lim\limits_{\Delta t\to 0} V_{\bar{p}}[\ln\frac{S_T}{S(t_0)}] & = & \sigma^2(T-t_0) \end{array}$$

Again, by the Central Limit Theorem we obtain

$$\lim_{\Delta t\to 0} \sum_{i=0}^{2J_H-a} \binom{N}{i} \bar{p}^{N-i}(1-\bar{p})^{N-i} = N(y_1)$$

$$\text{with} \quad y_1 := \frac{2\ln H - \ln S(t_0) - \ln K + \left(r+\frac{\sigma^2}{2}\right)(T-t_0)}{\sigma\sqrt{(T-t_0)}}$$

For the case $H > K$ the same analysis can be done. The only change concerns the summation.

2) In the case of a European up–and–out put, again two cases have to be considered: $H \ge K$ and $H < K$. For simplicity let us assume $H \ge K$ and thus $J_H \ge b = \sup\{i \in \mathbb{N} | S(t_0)u^i d^{N-i} \le K\}$. Again we only consider the correction term in (14) since the first two terms will converge in distribution to the Black–Scholes formula for put options. The correction term can be rewritten as:

$$\begin{aligned} & \left(\frac{1}{1+r}\right)^N K \sum_{i=\lceil J_H/2\rceil}^{b} \binom{N}{2J_H-i} p^i(1-p)^{N-i} - S(t_0) \sum_{i=\lceil J_H/2\rceil}^{b} \binom{N}{2J_H-i} \bar{p}^i(1-\bar{p})^{N-i} \\ = & \left(\frac{1-p}{p}\right)^{N-2J_H} \left[\left(\frac{1}{1+r}\right)^N K \sum_{i=2J_H-b}^{2J_H-\lceil J_H/2\rceil} \binom{N}{i} p^{N-i}(1-p)^i \right. \\ & \left. -S(t_0) \sum_{i=2J_H-b}^{2J_H-\lceil J_H/2\rceil} \binom{N}{2J_H-i} \bar{p}^{N-i}(1-\bar{p})^i \right] \end{aligned}$$

Given the results in 1) we only have to consider these two sums. The first sum is equal to

$$\sum_{i=2J_H-b}^{N} \binom{N}{i} p^{N-i}(1-p)^i - \sum_{i=2J_H+1-\lceil J_H/2\rceil}^{N} \binom{N}{i} p^{N-i}(1-p)^i$$

$$\text{prob}[J(N) \ge 2J_H - b] - \text{prob}[J(N) > 2J_H - \lceil J_H/2\rceil]$$

---

[12] We use the fact that $a = \inf\{\in \mathbb{N} | S(t_0)u^i d^{N-i} \ge K\}$.

where $J(N)$ is the sum of $N$ independent binomially distributed variables with up and down probabilities $1-p$ resp. $p$. From the definition of $J_H$ and $b$ we have

$$\begin{aligned}
\frac{2J_H - b - E_p[J(N)]}{\sqrt{V_p[J(N)]}} &= \frac{2\ln H - \ln S(t_0) - \ln K - \varepsilon \ln u/d + E_p[\ln S_T/S(t_0)]}{\sqrt{V_p[\ln S_T/S(t_0)]}} \\
&\overset{\Delta t \to 0}{\longrightarrow} \frac{2\ln H - \ln S(t_0) - \ln K + (r - \sigma^2/2)(T-t_0)}{\sigma\sqrt{T-t_0}} =: y_2 \\
\frac{2J_H - \lceil J_H/2 \rceil - E_p[J(N)]}{\sqrt{V_p[J(N)]}} &= \frac{3/2\,(\ln H/S(t_0)) - 1/2 N \ln d - \varepsilon \ln u/d + E_p[\ln S_T/S(t_0)]}{\sqrt{V_p[\ln S_T/S(t_0)]}} \\
&\overset{\Delta t \to 0}{\longrightarrow} +\infty \qquad \text{since } -N\ln d = \frac{T-t_0}{\sqrt{\Delta t}}
\end{aligned}$$

Therefore the Central Limit Theorem yields

$$\text{prob}[J(N) \geq 2J_H - b] - \text{prob}\left[J(N) > 2J_H - \frac{\lceil J_H \rceil}{2}\right] \overset{\Delta t \to 0}{\longrightarrow} N(-y_2)$$

The same argument applies for the second sum where again the transition probability $p$ has to be replaced by $\bar{p} = \frac{p \cdot u}{1+r}$.

□

**Proposition 4.** Let $\Delta t = \frac{T-t_0}{N}$ the grid size of the binomial lattice. For $u = \exp\{\sigma\sqrt{\Delta t}\}$, $d = \exp\{-\sigma\sqrt{\Delta t}\}$ and $\tilde{r} = \frac{1}{\Delta t}\ln(1+r)$ the convergence in distribution of the binomial formulae in theorem 1 are given by[13]

a) for $K > H$, $S > H$

$$\lim_{\Delta t \to 0} C_{di}[S,K,T.H] = S \cdot \left(\frac{S}{H}\right)^{-1-\alpha} N(y_1(K)) - \hat{K}\left(\frac{S}{H}\right)^{1-\alpha} N(y_2(K))$$

for $K < H$, $S > H$

$$\begin{aligned}
\lim_{\Delta t \to 0} C_{di}[S,K,T,H] &= S[N(x_1(K)) - N(x_1(H))] - \hat{K}[N(x_2(K)) - N(x_2(H))] \\
&+ S \cdot \left(\frac{S}{H}\right)^{-1-\alpha} N(y_1(H)) - \hat{K}\left(\frac{S}{H}\right)^{1-\alpha} N(y_2(H))
\end{aligned}$$

b) for $K > H$, $S < H$

$$\lim_{\Delta t \to 0} C_{uo}[S,K,T,H] = 0$$

for $K < H$, $S < H$

$$\begin{aligned}
\lim_{\Delta t \to 0} C_{uo}[S,K,T,H] &= S[N(x_1(K)) - N(x_1(H))] - \hat{K}[N(x_2(K)) - N(x_2(H))] \\
&-S \cdot \left(\frac{S}{H}\right)^{-1-\alpha} [N(y_1(K)) - N(y_1(H))] \\
&+\hat{K} \cdot \left(\frac{S}{H}\right)^{1-\alpha} [N(y_2(K)) - N(y_2(H))]
\end{aligned}$$

c) for $K > H$, $S < H$

$$\lim_{\Delta t \to 0} C_{ui}[S,K,T,H] = SN(x_1(K)) - \hat{K}N(x_2(K))$$

for $K < H$, $S < H$

$$\begin{aligned}
\lim_{\Delta t \to 0} C_{ui}[S,K,T,H] &= SN(x_1(H)) - \hat{K}N(x_2(H)) \\
&+S \cdot \left(\tfrac{S}{H}\right)^{-1-\alpha} [N(y_1(K)) - N(y_1(H))] \\
&-\hat{K} \cdot \left(\tfrac{S}{H}\right)^{1-\alpha} [N(y_2(K)) - N(y_2(H))]
\end{aligned}$$

[13] We assume $H < S(t_0)$ in all down–cases and $H > S(t_0)$ for all up–cases since otherwise the value of an out–option is equal to zero and the value of an in–option coincides with that of a standard option.

d) for $K < H$, $S > H$

$$\lim_{\Delta t \to 0} P_{do}[S,K,T,H] = 0$$

for $K > H$, $S > H$

$$\begin{aligned}\lim_{\Delta t \to 0} P_{do}[S,K,T,H] &= \hat{K}[N(x_2(H)) - N(x_2(K))] - S[N(x_1(H)) - N(x_1(K))] \\ &\quad -\hat{K} \cdot \left(\frac{S}{H}\right)^{1-\alpha} [N(y_2(H)) - N(y_2(K))] \\ &\quad +S \cdot \left(\frac{S}{H}\right)^{-1-\alpha} [N(y_1(H)) - N(y_1(K))]\end{aligned}$$

e) for $K < H$, $S > H$

$$\lim_{\Delta t \to 0} P_{di}[S,K,T,H] = \hat{K}N(-x_2(K)) - SN(-x_1(K))$$

for $K > H$, $S > H$

$$\begin{aligned}\lim_{\Delta t \to 0} P_{di}[S,K,T,H] &= \hat{K}N(-x_2(H)) - SN(-x_1(H)) \\ &\quad +\hat{K} \cdot \left(\frac{S}{H}\right)^{1-\alpha} [N(y_2(H)) - N(y_2(K))] \\ &\quad -S \cdot \left(\frac{S}{H}\right)^{-1-\alpha} [N(y_1(H)) - N(y_1(K))]\end{aligned}$$

f) for $K < H$, $S < H$

$$\lim_{\Delta t \to 0} P_{ui}[S,K,T,H] = \hat{K} \cdot \left(\frac{S}{H}\right)^{1-\alpha} N(-y_2(K)) - S \cdot \left(\tfrac{S}{H}\right)^{-1-\alpha} N(-y_1(K))$$

for $K > H$, $S < H$

$$\begin{aligned}\lim_{\Delta t \to 0} P_{ui}[S,K,T,H] &= \hat{K}[N(x_2(H)) - N(x_2(K))] - S[N(x_1(H)) - N(x_1(K))] \\ &\quad +\hat{K} \cdot \left(\frac{S}{H}\right)^{1-\alpha} N(-y_2(H)) - S \cdot \left(\frac{S}{H}\right)^{-1-\alpha} N(-y_1(H))\end{aligned}$$

where $\hat{K} = Ke^{-rs}$, $\alpha = \frac{2\tilde{r}}{\sigma^2}$, $s = T - t_0$ and

$$x_{1,2}(z) = \frac{\ln\left(\frac{S}{ze^{-\tilde{r}s}}\right) \pm \sigma^2 s}{\sigma\sqrt{s}} \qquad y_{1,2}(z) = \frac{\ln\left(\frac{H^2}{S \cdot e^{-\tilde{r}s}}\right) \pm \sigma^2 s}{\sigma\sqrt{s}}.$$

**Proof.** The proof of the above formulae is an application of the Central Limit Theorem already demonstrated in Theorem 2.

**Proof of Theorem 4.**

a) Let $C_{uo}^{am}[S,K,H,T]$ be the arbitrage price of an American up-and-out call option which is still alive at time $t$. Let $H > K$ be the barrier. By assumption $H$ is an endpoint of the binomial tree. Thus at time $t_n \in \underline{\underline{T}} = \{t_0 < \ldots < t_N\}$ the option is still alive if $S(t_n) = S(t_0)u^j d^{n-j} < H$. There are two possible cases of interest. First $S(t_n) \le d^2 H$ and second $S(t_n) = dH$. Suppose $S(t_n) \le d^2 H$ which implies that at time $t_{n+1}$ the option is still alive. Consider now the difference between immediate exercise or exercise at time $t_{n+1}$. Since we know that $H > K$ we have

$$\begin{aligned}C_{uo}^{am}[S,K,T,H] &\ge C_{uo}^{eur}[S,K,H,T] && \forall\ S(t_n) \le d^2 H \\ &\ge \frac{1}{1+r} E_P[[S(t_{n+1}) - K]^+ \mid S(t_n) \le d^2 H] && \forall\ S(t_n) \le d^2 H \\ &= \operatorname{Max}\left\{0,\ S(t_n) - \frac{K}{1+r}\right\} > \operatorname{Max}\{0,\ S(t_n) - K\}\end{aligned}$$

and therefore it is not optimal to exercise the option at time $t_n$.
Suppose now $S(t_n) = dH$. Since $d = \exp\left\{-\sigma\sqrt{\Delta t}\right\}$ and $H > K$ implies that for $\Delta t < \left(\frac{1}{\sigma}\ln\frac{H}{K}\right)^2 \Leftrightarrow N >$

$(T-t_0)\frac{\sigma^2}{(ln(H/K))^2}$ the inner value $dH-K$ is positiv. Since $dH-K$ is also the maximum possible payoff of the contract at time $T=t_N$, early exercise at any time $t_n<T$ is optimal in the situation $S(t_n)=dH$. This implies that within the binomial setup the arbitrage price of an American up–and–out call option with barrier $H$ is equal to the European up–and –out call option with the barrier $dH$ plus a rebate of $dH-K$ when the barrier $dH$ is reached, assuming that the grid size is small enough such that $dH>K$. Define $h\in\mathbb{N}$ such that $Su^h=H$ with $0<h<N$ since $S<H$ and $H$ is an element of the binomial tree. Furthermore since $H=Su^{J_H}d^{N-J_H}$ we have $h=2J_H-N$. From the reflection principle we know that for $h\geq 2$

$$\binom{h-2+2i}{h-2+i}-\binom{h-2+2i}{(h-1)+i}\qquad\text{for } i=1,\ldots,\left\lfloor\frac{N+2-h}{2}\right\rfloor\ =:N(h)\ ,$$

is equal to the number of paths which at time $t_{h-2+2i}\in\underline{\underline{T}}$ end in the knot $Su^{h-2+i}d^i=Su^hd^2=Hd^2$ and have not crossed or touched the barrier $Su^{h-1}=Hd$. This is then equal to the number of paths which at time $t_{h-1+2i}$ reach for the first time the knot $Hd$. Summing up, the arbitrage price of the American up–and–out call with barrier $H$ is equal to

$$\begin{aligned}C^{am}_{uo}[S,K,T,H] &= C^{eur}_{uo}[S,K,T,dH]\\ &+ \sum_{i=1}^{N(h)}\binom{h-2+2i}{i}p^{h-1+i}(1-p)^i\left(\frac{1}{1+r}\right)^{h-1+2i}[dH-K]\\ &- \sum_{i=1}^{N(h)}\binom{h-2+2i}{i-1}p^{h-1+i}(1-p)^i\left(\frac{1}{1+r}\right)^{h-1+2i}[dH-K]\end{aligned}$$

where the grid size $\Delta t<\left(\frac{1}{\sigma}\ln\frac{H}{K}\right)^2$.

b) Let $C^{am}_{do}[S,K,T,H]$ with $S>H$ be the arbitrage price of an American down–and–out call which is still alive at time $t$. Suppose $H>K$, then we know that $\forall\ S(t_i)\geq Hu^2$ immediate exercise is not optimal. Therefore consider the situation $S(t_i)=uH$. Suppose that there are $N_1<N$ periods left and define $J_H$ such that $(uH)u^{J_H}d^{N_1-J_H}=Hu\Rightarrow 2J_H=N_1$. For the European down–and–out call we can now use the binomial formulae (13) with $\underline{\underline{T}}'=\{t_i<\ldots<t_N\}$

$$\begin{aligned}&C^{eur}_{do}[uH,K,T,H]\\ &= \left(\frac{1}{1+r}\right)^{N_1}\left[\sum_{i=J_H}^{N_1}\binom{N_1}{i}p^i(1-p)^{N_1-i}[(uH)u^id^{N_1-i}-K]^+\right.\\ &\quad\left.-\sum_{i=J_H}^{2J_H}\binom{N_1}{2J_H-i}p^i(1-p)^{N_1-i}[(uH)u^id^{N_1-i}-K]^+\right]\\ &= \left(\frac{1}{1+r}\right)^{N_1}\left[\sum_{i=J_H}^{N_1}\binom{N_1}{i}p^i(1-p)^{N_1-i}((uH)u^id^{N_1-i}-K)\right.\\ &\quad\left.-\sum_{i=J_H}^{N_1}\binom{N_1}{N_1-i}p^i(1-p)^{N_1-i}((uH)u^id^{N_1-i}-K)\right]\\ &= \left(\frac{1}{1+r}\right)^{N_1}\left[\sum_{i=0}^{N_1}\binom{N_1}{i}p^i(1-p)^{N_1-i}((uH)u^id^{N_1-i}-K)\right.\\ &\quad-\sum_{i=0}^{J_H-1}\binom{N_1}{i}p^i(1-p)^{N_1-i}(uHu^id^{N_1-i}-K)\\ &\quad\left.-\sum_{i=J_H}^{N_1}\binom{N_1}{i}p^{i-1}(1-p)^{N_1+1-i}(Hu^id^{N_1+1-i}-K)\right]\end{aligned}$$

Since $uH>H>K$ and for $\Delta t$ such that $dH>K$ we have

$$\frac{1-p}{p}((dH)u^id^{N_1-i}-K)>Ku^id^{N_1-i}-K\qquad\forall\ i\geq J_H$$

and the European arbitrage price in this situation is bounded from above by

$$\begin{aligned} C_{do}^{eur}[uH,K,H,T] &< \left(\frac{1}{1+r}\right)^{N_1}\left[\sum_{i=0}^{N_1}\binom{N_1}{i}p^i(1-p)^{N_1-i}(uH-K)u^id^{N_1-i}\right] \\ &= (uH-K) \end{aligned}$$

which implies that early exercise is optimal in the situation $S(t_i) = uH > dH > K$. With this we can now use the same counting algorithm as for the up–and–out option, where

$$\begin{pmatrix} h-2+2i \\ h-2+i \end{pmatrix} - \begin{pmatrix} h-2+2i \\ h-1+i \end{pmatrix} \qquad \text{for } i = 1,\dots, \left\lfloor \frac{N+2-h}{2} \right\rfloor =: N(h)$$

and $Sd^h = H$ is equal to the number of paths which at time $t_{h-2+2i}$ end in a knot $uH$ for the first time.

□

**Proposition 5.**

i) Let $H$ be a barrier such that there exists a $J_H \in \mathbb{N}$ with $S(t_0)u^{J_H}d^{N_1-J_H} = H$. The arbitrage price of a front partial down–and–in Arrow–Debreu–security $\Pi_d^{fp}(\underline{\underline{T}}(t_0,t_{N_1}), i, J_H)$ with payoff at time $t_N$

$$\begin{cases} 1 & \text{if} \quad S_{t_N} = S_{t_0}u^id^{N-i} \ \text{ and } \ S_{t_i} \le H \quad \forall\, t_i \in \underline{\underline{T}}(t_0,t_{N_1}) \\ 0 & \text{otherwise} \end{cases}$$

is given by:

$$\Pi_d^{fp}\left(\underline{\underline{T}}(t_0,t_{N_1}); i, J_H\right) =$$

$$\begin{cases} \left(\frac{1}{1+r}\right)^N \binom{N}{i}\, p^i(1-p)^{N-i} & \text{for } 0 \le i \le J_H \\ \left(\frac{1}{1+r}\right)^N \left[\sum\limits_{k=0\vee(i-(N-N_1))}^{J_H}\binom{N_1}{k}\binom{N-N_1}{i-k} + \sum\limits_{k=J_H+1}^{2J_H\wedge i}\binom{N_1}{2J_H-k}\binom{N-N_1}{i-k}\right]\cdot p^i(1-p)^{N-i} & \\ & \text{for } J_H < i \le J_H+N_2 \\ \left(\frac{1}{1+r}\right)^N \left[\sum\limits_{k=i-(N-N_1)}^{2J_H\wedge i}\binom{N_1}{2J_H-k}\binom{N-N_1}{i-k}\right] p^i(1-p)^{N-i} & \\ & \text{for } J_H+N_2 \le i \le 2J_H+N_2 \\ 0 & \text{for } i > 2J_H+N_2 \end{cases}$$

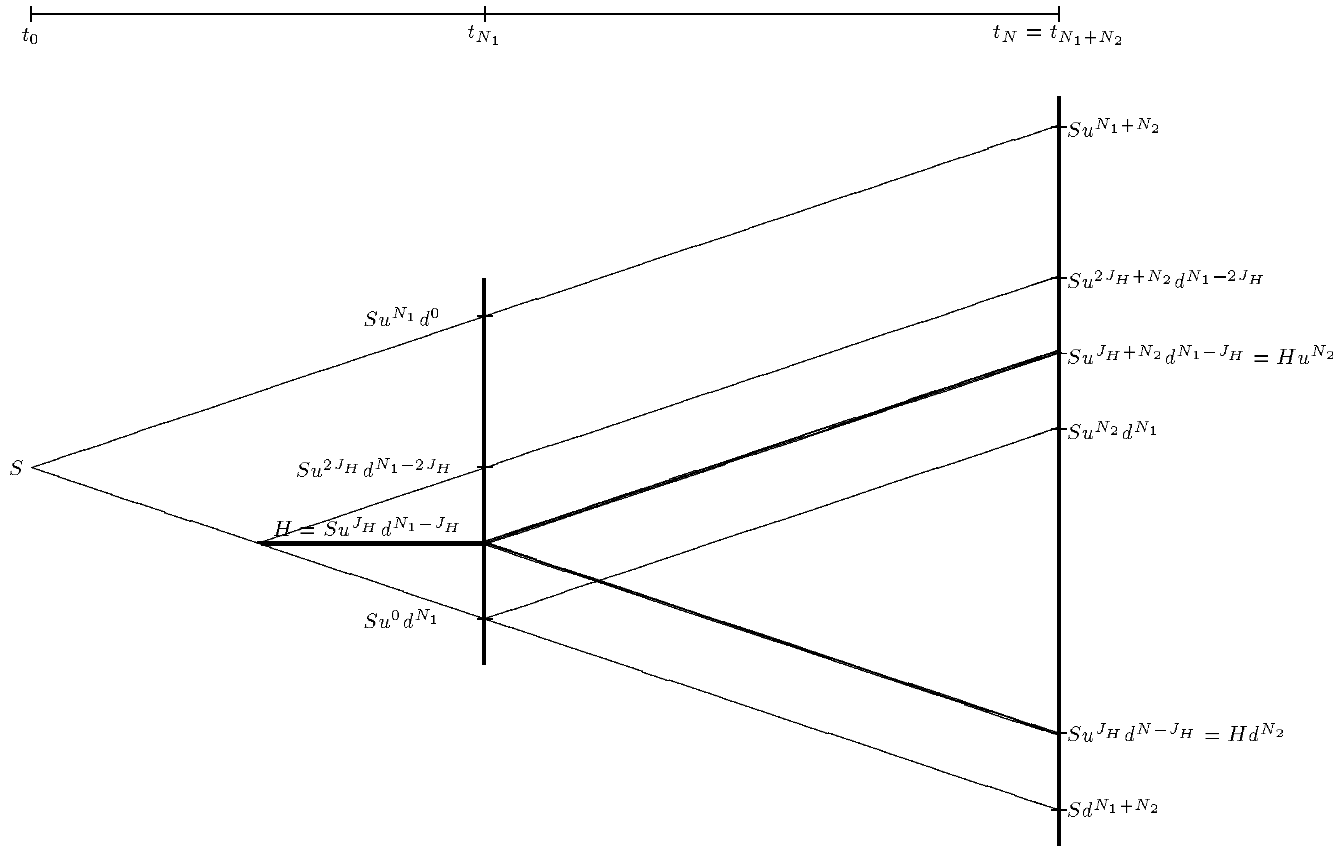
ii) Let $J_H \in N$ such that $S(t_0)u^{J_H}d^{N-J_H} = H$. The arbitrage price of a back–partial down–and–in Arrow–Debreu–security $\Pi_d^{bp}\left(\underline{\underline{T}}(t_{N_1},t_N), i, J_H\right)$ is given by

$$\Pi_d^{bp}\left(\underline{\underline{T}}(t_{N_1},t_{N_2}); i, J_H\right) =$$

$$\begin{cases} \left(\frac{1}{1+r}\right)^N \binom{N}{i}\, p^i(1-p)^{N-i} & \text{for } 0 \le i \le J_H \\ \left(\frac{1}{1+r}\right)^N \left[\sum\limits_{k=0\vee(i-(N-N_1))}^{N_1\wedge(2J_H-i)}\binom{N_1}{k}\binom{N-N_1}{2J_H-i-k}\right]\cdot p^i(1-p)^{N-i} & \\ & \text{for } J_H \le i \le \min\left\{2J_H, J_H = \frac{N-N_1}{2}\right\} \\ 0 & \text{for } i > \min\left\{2J_H, J_H + \frac{N-N_1}{2}\right\} \end{cases}$$

**Proof of Proposition 5.**

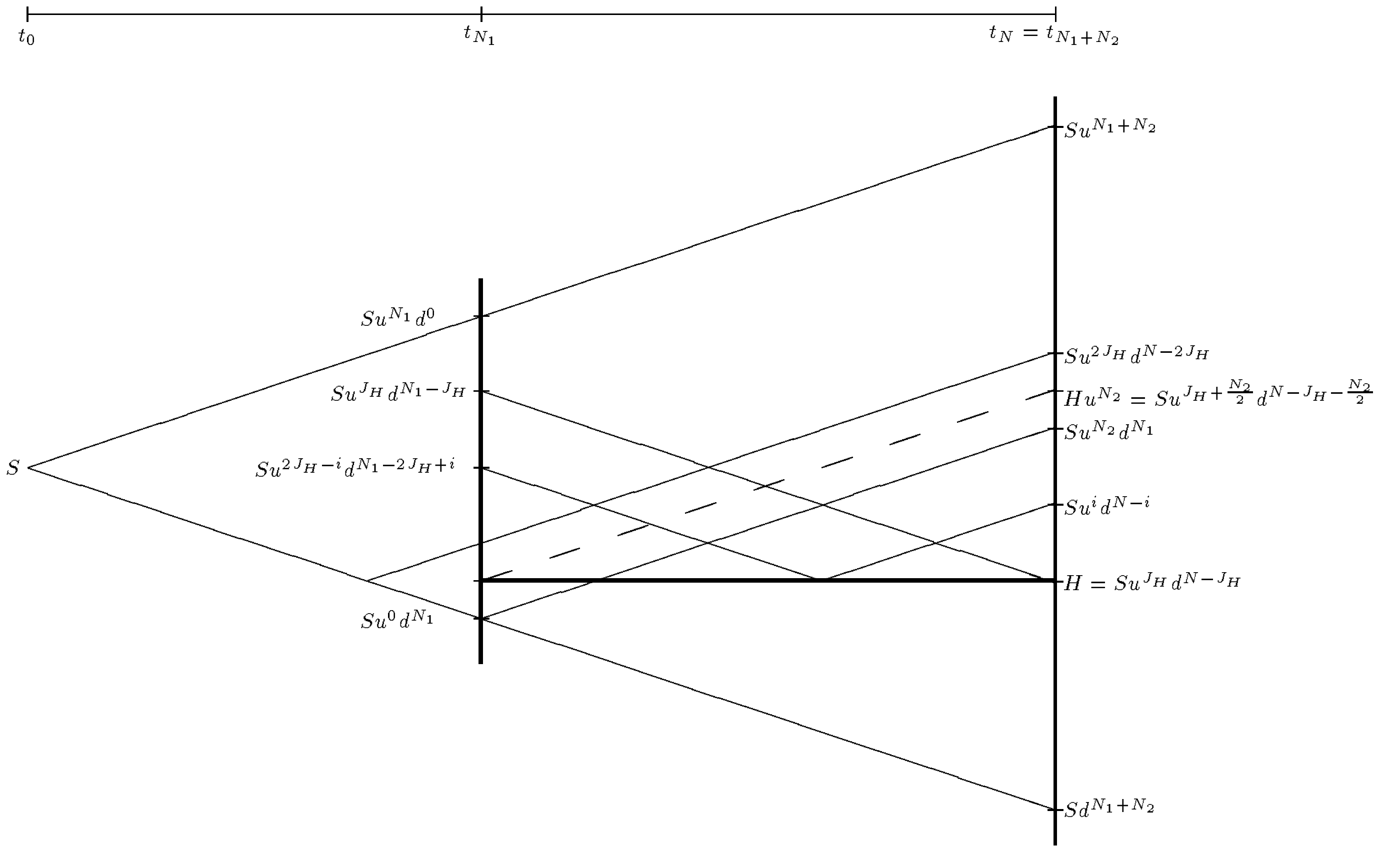
ad i) Define $K_d^{bp}(0, N_1, i, J_H)$ as the number of paths from the origin to the knot $S(t_0)u^i d^{N-i}$ which reach or cross the barrier $H = S(t_0)u^{J_H} d^{N_1 - J_H}$ at least at one time $t \in \{t_0 < \ldots < t_{N_1}\}$. Set $N_2 = N - N_1$, then the following picture summarizes the arguments:




$\Rightarrow K^{fp}(0, N_1, i, J_H) =$

$$
\begin{cases}
\binom{N}{i} & 0 \le i \le J_H \\
\displaystyle\sum_{k=0 \vee i-N_2}^{J_H} \binom{N_1}{k}\binom{N_2}{i-k} + \sum_{k=J_H+1}^{2J_H \wedge i} \binom{N_1}{2J_H - k}\binom{N_2}{i-k} & J_H < i \le J_H + N_2 \\
\displaystyle\sum_{k=i-N_2}^{2J_H \wedge i} \binom{N_1}{2J_H - k}\binom{N_2}{i-k} & J_H + N_2 < i \le 2J_H + N_2 \\
0 & i > 2J_H + N_2
\end{cases}
$$

ad ii) The argument is the same in both cases. Consider now the back partial case. The problem is to compute the number of paths $K_d^{bp}(N_1, N_2, i, J_H)$ from the origin to the knot $S(t_0)u^i d^{N-i}$ which reach or cross the barrier $H = S(t_0)u^{J_H} d^{N-J_H}$ at least at one time $t \in \{t_{N_1} < \ldots < t_N\}$. For simplicity let $N_2 := N - N_1$ be an even number.




$\Rightarrow K_d^{bp}(N_1, N_2, i, J_H) =$

$$\begin{cases} \binom{N}{i} & 0 \le i \le J_H \\ \displaystyle\sum_{k=0\vee(i-(N-N_1))}^{N_1\wedge(2J_H-i)} \binom{N_1}{k}\binom{N-N_1}{2J_H-i-k} & J_H \le i \le \min\left\{2J_H, J_H + \frac{N_2}{2}\right\} \\ 0 & i > \min\left\{2J_H, J_H + \frac{N_2}{2}\right\} \end{cases}$$

where for $i \le N - N_1 = N_2$

$$\sum_{k=0}^{N_1\wedge 2J_H-i} \binom{N_1}{k}\binom{N-N_1}{2J_H-i-k} = \binom{N}{2J_H-i}$$

□

## References


**Black, F.; J. Cox [1976]:** Valuing Corporate Securities: Some Effects of Bond Indenture Provisions, *Journal of Finance,* Vol. 31, No.2, May 1976, pp. 351–376.

**Black, F.; M. Scholes [1973]:** The Pricing of Options and Corporate Liabilities, *Journal of Political Economy,* Vol. 81, May–June 1973, pp. 637–659.

**Boyle, P.P.; Sok Hoon Lau [1994]:** Bumping Up Against the Barrier with the Binomial Method, *The Journal of Derivatives,* Vol. 1, No. 4, 1994, pp.6–14.

**Conze, A.; Viswanathan [1991]:** Path Dependent Options: The Case of Lookback Options, *Journal of Finance,* Vol. 16, No. 5, 1991, pp. 1893–1907.

**Cox, J.; St.A. Ross; M. Rubinstein [1979]:** Option Pricing: A Simplified Approach, *Journal of Financial Economics,* Vol. 7, September 1979, pp. 229–263.

**Cox, J.; M. Rubinstein [1985]:** Options Markets, 1985, *Prentice–Hall,* New Jersey.

**Feller, W. [1968]:** An Introduction to Probability Theory and its Applications, *Wiley Series in Probability and Mathematical Statistics,* Vol. I+II, 1968, Wiley & Son, Inc., New York, London, Sidney.

**Goldman, M.B.; H.B. Sosin; M. Gatto [1979]:** Path Dependent Options: Buy at the Low, Sell at the High, it Journal of Finance, Vol. 34, December 1979, pp. 1111-1127.

**Harrison, J.; D.M. Kreps [1979]:** Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory,[1979/cb* Vol. 20, 1979, pp. 381–408.

**Ingersoll, J.E.jr. [1987]:** Option Pricing Theory, in Eatwell, J; M. Milgate; P. Newman: *The New Palgrave,* , Vol 3, 1987, pp. 745–751.

**Merton, R.C. [1973]:** Theory of Rational Option Pricing, it Bell Journal of Economics and Management, 1973, pp. 141–183.

**Reimer, M.; Sandmann, K. [1993]:** Down–and–out–Call: Bewertungstheorie, numerische Verfahren und Simulationsstudie, *Discussion Paper B–239* , 1993, University of Bonn.

**Rendlemann, R.; B. Bartter [1979]:** Tow–State Option Pricing, *Journal of Finance,* Vol. 34, December 1979, pp. 1093–1110.

**Rubinstein, M.; E. Reiner [1991]:** Breaking Down the Barriers, *RISK,* September 1991.

**Sondermann, D. [1988]:** Option Pricing with Bounds on the Underlying Securies, in Rudolph, B.; J. Wilhelm (Eds.): *Bankpolitik, finanzielle Unternehmensführung und die Theorie der Finanzmärkte,* 1988, Berlin



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