A TRACTABLE TERM STRUCTURE MODEL WITH ENDOGENOUS INTERPOLATION AND POSITIVE INTEREST RATES

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ABSTRACT. This paper presents the one– and the multifactor versions of a term structure model in which the factor dynamics are given by Cox/Ingersoll/Ross (CIR) type "square root" diffusions with piecewise constant parameters. The model is fitted to initial term structures given by a finite number of data points, interpolating endogenously. Closed form and near–closed form solutions for a large class of fixed income contingent claims are derived in terms of a noncentral chi-square distribution whose noncentrality parameter is in turn noncentral chi-square distributed. Implementation details on this distribution are given in the appendix.

1. INTRODUCTION

Three often cited requirements for term structure models applied in practice are

(i) fit to the initial term structure observed in the market
(ii) analytical tractability for fast solutions for derivative pricing and hedging
(iii) non-negative interest rates.

Requirement (i) is often extended to fitting an initial term structure of volatility.

Our approach to constructing a model which fulfills these requirements is based on the Cox, Ingersoll jr. and Ross (1985) (CIR) model, also called a "square root" model because of the way the volatility of the diffusion processes depends on the realizations of the state variables, which in the single factor version of the model is the short rate. We are thus in the class of affine yield factor models[1], where for constant drift parameters closed form or, in the multifactor case, very nearly closed form solutions can be obtained[2]. Unfortunately, if one extends the original CIR model as in Hull and White (1990) and allows for a time dependent drift in order to calibrate to an observed initial term structure, the closed form solutions do not carry over from the constant parameter case, so we are faced with a tradeoff between requirements (i) and (ii). Alternatively one could construct an affine model with state independent volatility, but such a model would assign a positive probability to negative interest rates, which in some important cases yields unrealistic results[3].

Two recent papers also address the tradeoff between fitting initial term structure data and closed form solutions in a "square root" model.

In what he calls a "simple class of square root models", Jamshidian (1995) restricts the CIR model with time–varying coefficients to a class satisfying the condition that the ratio

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[1] see Duffie and Kan (1992, 1996)

[2] see also Chen and Scott (1995)

[3] see Rogers (1996)

of the mean reversion level $\theta(t)$ and the square of the volatility parameter $\sigma^2(t)$ is identical for all t. This permits the derivation of explicit formulae for a large number of assets, but implies that the initial yield curve determines the volatility of the short rate.

Scott (1995) shifts the short rate realizations of a constant parameter CIR model by a deterministic component in order to fit the initial term structure, thus avoiding the need to perform calculations with a time dependent drift parameter. The yield curve interpolation is exogenous to this model and the short rate volatility parameter is constant across time.

In the present paper we will take a different approach to fit a "square root" term structure model to an initial term structure while still obtaining fast solutions for a large class of assets. Looking at a market for fixed income instruments, we observe interest rates or bond prices for only a finite number of maturities. If one considers the money market and or swap market, this number is quite small. This presents a possibility to avoid the tradeoff between fitting initial term structure data and fast analytical solutions: The observed data points divide the time line into intervals. On these intervals we inductively construct short rate processes of the CIR type with constant parameters, chosen so as to give an exact fit of the observed term structure. The processes are pieced together to yield a continuous short rate process.

In taking this approach, we consider the CIR stochastic differential equation with nondeterministic initial conditions and show that the solution does not explode under the same restrictions on the parameters as in the deterministic case; thus interest rates are almost surely strictly positive.

The inductive construction of the short rate process allows us to fit the model to an initial term structure consisting of a finite number of data points. By taking the relevant expectations, the model yields a complete initial term structure for the continuum of maturities, endogenously interpolating between the observed data. Thus it is more parsimonious in its assumptions in the sense that there is no need for an exogenous interpolation rule and the interpolation is consistent with the assumed short rate dynamics.

In the next section we introduce the model and show how to fit it to initial zero coupon bond prices, as well as discussing how historical or implied volatility structures can be input into the model, this being an additional requirement often put forth by practitioners in addition to the three already mentioned. The formulae for contingent claims pricing and hedging are derived in section 3, interpolated zero coupon bond prices being given as a special case. Some examples using market data are presented in section 4, and section 5 shows how the results in the one-factor case can be extended to a multifactor model.

2. The Model

2.1. **The Short Rate Process.** We wish to specify the model so that the dynamics of the short rate process are given by a generalized CIR equation with piecewise constant coefficients. We call this the *segmented square root model.* To formalize, let points in time $0 = T_0 < T_1 < \ldots < T_N$ and constants $\theta_1, \ldots, \theta_N, a_1, \ldots, a_N, \sigma_1, \ldots, \sigma_N \in \mathbb{R}_{++}$ be given. We define a step function $\theta : \mathbb{R}_+ \to \mathbb{R}_{++}$ in the following manner:

$$\theta(t) := \theta_1\ \chi_{\{0\}}(t) + \sum_{i=1}^{N} \theta_i\ \chi_{]T_{i-1},T_i]}(t) + \theta_N \sqrt{x}\ \chi_{]T_N,+\infty[}(t)$$

Step functions $\sigma, a : \mathbb{R}_+ \to \mathbb{R}_{++}$ are defined analogously. Now let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis satisfying the usual hypotheses on which a standard one-dimensional Brownian motion W is defined. The short rate r is a continuous $\mathbb{F}$-adapted stochastic process defined on Ω. We specify the dynamics of r by demanding that r is a solution of the following stochastic differential equation (SDE):

$$dy_t = (\theta(t) - a(t)y_t)dt + \sigma(t)\sqrt{y_t}dW_t \tag{1}$$

We shorten notation by defining two functions $\alpha, \beta : \mathbb{R}_+ \times]0, +\infty[\to \mathbb{R}$ as follows:

$$\begin{aligned} \alpha(t,x) &:= \sigma_1\sqrt{x}\,\chi_{\{0\}}(t) + \sum_{i=1}^{N} \sigma_i\sqrt{x}\,\chi_{]T_{i-1},T_i]}(t) + \sigma_N\sqrt{x}\,\chi_{]T_N,+\infty[}(t) \\ \beta(t,x) &:= (\theta_1 - a_1x)\chi_{\{0\}}(t) + \sum_{i=1}^{N}(\theta_i - a_ix)\chi_{]T_{i-1},T_i]}(t) + (\theta_N - a_Nx)\chi_{]T_N,+\infty[}(t) \end{aligned}$$

The dynamic equation now takes the form:

$$dy_t = \beta(t, y_t)dt + \alpha(t, y_t)dW_t \tag{2}$$

We stress the fact that a solution of this equation by definition only assumes values in $]0, +\infty[$, so that the short rate is automatically strictly positive at all times. We will show that solving equation (2) is equivalent to iteratively solving classical Cox/Ingersoll/Ross equations. When referring to classical CIR equations we mean equations of the following type:

$$dy_t = (\theta - ay_t)dt + \sigma\sqrt{y_t}dW_t. \tag{3}$$

Here θ, a, σ are strictly positive constants. If these fulfill the inequality $2\theta \geq \sigma^2$, then using Feller's test for explosions (cf. Karatzas and Shreve (1988), Proposition 5.5.22 and Theorem 5.5.29) one can prove that solutions of (3) with nonrandom initial conditions cannot explode. Using the result shown in appendix A, it follows that solutions of (3) with random initial conditions do not explode either.

2.1.1. Theorem. *For each $i \in \{1, \ldots, N\}$ let the constants $\theta_i, \sigma_i \in]0, +\infty[$ fulfill the inequality $2\theta_i \geq \sigma_i^2$. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis fulfilling the usual hypotheses, carrying a standard, one–dimensional Brownian motion W. Let $f : \Omega \to]0, +\infty[$ be an $\mathcal{F}_0$–measurable random variable. Then there exists a continuous, $\mathbb{F}$–adapted process r with values in $]0, +\infty[$, so that for each $t \in \mathbb{R}_+$, we have*

$$r_t = f + \int_0^t \alpha(s, r_s)dW_s + \int_0^t \beta(s, r_s)ds \quad P\text{–a.s.}$$

PROOF: We proceed inductively. We define $r^{(0)} : \{0\} \times \Omega \to]0, +\infty[$ by setting $r^{(0)}(0, \omega) := f(\omega)$. Now let $i \in \{1, \ldots, n\}$ be given. Suppose that we have already constructed a continuous, $\mathbb{F}$–adapted process $r^{(i-1)} : [0, T_{i-1}] \times \Omega \to]0, +\infty[$, so that for every $t \in [0, T_{i-1}]$ the following holds:

$$r_t^{(i-1)} = f + \int_0^t \alpha\left(s, r_s^{(i-1)}\right) dW_s + \int_0^t \beta\left(s, r_s^{(i-1)}\right) ds \quad P\text{–a.s.}$$

In the case of $i = 1$ this equation is trivial. We now introduce a deterministic time change $\langle\tau\rangle = \{\tau_t\}_{t\in\mathbb{R}_+}$ by setting $\tau_t := t + T_{i-1}$ for every $t \in \mathbb{R}_+$. We define a new filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ by setting $\mathcal{G}_t = \mathcal{F}_{\tau_t} = \mathcal{F}_{t+T_{i-1}}$ and define a process $\tilde{W} = \left(\tilde{W}_t\right)_{t\in\mathbb{R}_+}$ via $\tilde{W}_t := W_{t+T_{i-1}} - W_{T_{i-1}}$. The stochastic basis $(\Omega, \mathcal{F}, \mathbb{G}, P)$ also fulfills the usual hypotheses and it is easy to see that $\tilde{W}$ is a standard $(P, \mathbb{G})$–Brownian motion. The random variable

$r^{(i-1)}_{T_{i-1}}$ is $\mathcal{G}_0$–measurable. From our above remarks it follows that there is a continuous, $\mathbb{G}$–adapted process y with values in $]0,+\infty[$ so that $y_0 = r^{(i-1)}_{T_{i-1}}$ and

$$dy_t = (\theta_i - a_i y_t)dt + \sigma_i\sqrt{y_t}d\tilde{W}_t.$$

We define the process $r^{(i)} : [0,T_i]\times\Omega \to]0,+\infty[$ as follows:

$$r^{(i)}_t(\omega) := \begin{cases} r^{(i-1)}_t(\omega) & \text{if } t\in[0,T_{i-1}], \\ y_{t-T_{i-1}}(\omega) & \text{if } t\in]T_{i-1},T_i]. \end{cases} \tag{4}$$

The process $r^{(i)}$ is $\mathbb{F}$–adapted and possesses continuous paths. Obviously for $t\in[0,T_{i-1}]$ we have

$$r^{(i)}_t = f + \int_0^t \alpha\left(s, r^{(i)}_s\right)dW_s + \int_0^t \beta\left(s, r^{(i)}_s\right)ds \quad P\text{–a.s.} \tag{5}$$

Now assume $t\in]T_{i-1},T_i]$. Then:

$$\begin{aligned} & f + \int_0^t \beta\left(s, r^{(i)}_s\right)ds + \int_0^t \alpha\left(s, r^{(i)}_s\right)dW_s \\ = \; & r^{(i)}_{T_{i-1}} + \int_{T_{i-1}}^t \beta\left(s, r^{(i)}_s\right)ds + \int_{T_{i-1}}^t \alpha\left(s, r^{(i)}_s\right)dW_s \quad P\text{–a.s.} \end{aligned}$$

We will now write the right hand side of this equation somewhat differently. First of all:

$$\begin{aligned} \int_{T_{i-1}}^t \beta\left(s, r^{(i)}_s\right)ds &= \int_{T_{i-1}}^t \left(\theta_i - a_i r^{(i)}_s\right)ds \\ &= \int_0^{t-T_{i-1}} \left(\theta_i - a_i r^{(i)}_{s+T_{i-1}}\right)ds \\ &= \int_0^{t-T_{i-1}} \left(\theta_i - a_i y_s\right)ds. \end{aligned}$$

Furthermore:

$$\int_{T_{i-1}}^t \alpha\left(s, r^{(i)}_s\right)dW_s = \int_{T_{i-1}}^t \sigma_i\sqrt{r^{(i)}_s}dW_s \quad P\text{–a.s.}$$

Denoting stochastic integration by a $\bullet$, the transformation property of the stochastic integral under time change implies the following identity up to indistinguishability

$$\left(\sigma_i\sqrt{r^{(i)}}\bullet W\right)_{\langle\tau\rangle} - \left(\sigma_i\sqrt{r^{(i)}}\bullet W\right)_{\tau_0} = \sigma_i\sqrt{r^{(i)}_{\langle\tau\rangle}}\bullet W_{\langle\tau\rangle}.$$

The stochastic integral on the right hand side of this equation is taken with respect to the stochastic basis $(\Omega,\mathcal{F},\mathbb{G},P)$. Explicitly, we have for $t\in]T_{i-1},T_i]$:

$$\begin{aligned} & \int_{T_{i-1}}^t \sigma_i\sqrt{r^{(i)}_s}dW_s \\ = \; & \int_0^{t-T_{i-1}} \sigma_i\sqrt{r^{(i)}_{s+T_{i-1}}}d\tilde{W}_s \\ = \; & \int_0^{t-T_{i-1}} \sigma_i\sqrt{y_s}d\tilde{W}_s \quad P\text{–a.s.} \end{aligned}$$

Therefore

$$
\begin{aligned}
& f+\int_0^t \beta\left(s, r_s^{(i)}\right) ds+\int_0^t \alpha\left(s, r_s^{(i)}\right) dW_s \\
= \; & r_{T_{i-1}}^{(i)}+\int_{T_{i-1}}^t \beta\left(s, r_s^{(i)}\right) ds+\int_{T_{i-1}}^t \alpha\left(s, r_s^{(i)}\right) dW_s \\
= \; & r_{T_{i-1}}^{(i)}+\int_0^{t-T_{i-1}}\left(\theta_i-a_i y_s\right) ds+\int_0^{t-T_{i-1}} \sigma_i \sqrt{y_s} d\tilde{W}_s \\
= \; & y_{t-T_{i-1}}=r_t^{(i)} \quad P\text{–a.s.}
\end{aligned}
$$

We have now shown that the process $r^{(i)}$ satisfies (5) for all $t \in [0, T_i]$. After N such induction steps we obtain a solution of the equation (2) on $[0, T_N]$. One more induction step (with a trivial change of notation) allows us to extend the solution to all of $\mathbb{R}_+$. □

2.1.2. Remark. *The coefficients of the classical CIR equation are locally Lipschitz, therefore pathwise uniqueness holds for the equation (3). By an iteration procedure analogous to the one used in the proof of the theorem above, it follows that pathwise uniqueness also holds for the equation (2). Therefore, in our model the short rate is uniquely determined up to indistinguishability by the coefficients in the dynamic equation and the initial interest rate.*

2.2. Fitting the Model to Zero Coupon Bond Prices.

2.2.1. Proposition. *In the segmented square root model with time segments $[T_{j-1}; T_j]$, $j \in \{1; \ldots; N\}$, the time T_{j-1} prices of zero coupon bonds with maturity T_k, $j \le k \le N$, are exponential affine functions of the short rate realization $r(T_{j-1})$:*

$$
B(r(T_{j-1}), T_{j-1}, T_k) = \mathcal{C}_{j-1,k} \exp\{-\mathcal{D}_{j-1,k}\, r(T_{j-1})\} \tag{6}
$$

with $\mathcal{C}_{j-1,k}$ and $\mathcal{D}_{j-1,k}$ recursively defined as

$$
\begin{aligned}
\mathcal{C}_{j-1,k} &:= \mathcal{C}_{j,k} \cdot \mathcal{A}_j(T_{j-1}, T_j) \cdot \left(\frac{b_j}{b_j + 2\mathcal{D}_{j,k}}\right)^{\frac{1}{2}\nu_j} \\
\mathcal{D}_{j-1,k} &:= \mathcal{B}_j(T_{j-1}, T_j) + \eta_j \frac{\mathcal{D}_{j,k}}{b_j + 2\mathcal{D}_{j,k}}, \qquad \textit{where } \mathcal{C}_{k,k} := 1 \textit{ and } \mathcal{D}_{k,k} := 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_j(T_{j-1}, T_j) &:= \left(2c_j w_j(T_{j-1}, T_j) \exp\left\{\frac{1}{2}(c_j + a_j)(T_j - T_{j-1})\right\}\right)^{\frac{1}{2}\nu_j} \\
\mathcal{B}_j(T_{j-1}, T_j) &:= 2w_j(T_{j-1}, T_j)\left(\exp\{c_j(T_j - T_{j-1})\} - 1\right) \\
w_j(T_{j-1}, T_j) &:= \left((c_j + a_j)\exp\{c_j(T_j - T_{j-1})\} + c_j - a_j\right)^{-1} \\
c_j &:= \sqrt{a_j^2 + 2\sigma_j^2} \\
\eta_j &:= \frac{16 w_j(T_{j-1}, T_j)^2 c_j^2 \exp\{c_j(T_j - T_{j-1})\}}{\sigma_j^2 \mathcal{B}_j(T_{j-1}, T_j)} \\
b_j &:= \frac{4}{\sigma_j^2}\mathcal{B}_j(T_{j-1}, T_j)^{-1} \\
\nu_j &:= \frac{4\theta_j}{\sigma_j^2}
\end{aligned}
$$

PROOF: We prove the proposition by induction.
Let $k=j$. Then (6) becomes

$$B(r(T_{j-1}),T_{j-1},T_j)=\mathcal{A}_j(T_{j-1},T_j)\exp\{-\mathcal{B}_j(T_{j-1},T_j)r(T_{j-1})\}$$

which is the original CIR formula for the price of a zero coupon bond, valid since we have a constant parameter CIR process on $[T_{j-1},T_j]$. Now consider some $k>j$. If (6) is valid over $(n-1)$ segments then (6) determines all zero coupon bond prices

$$B(r(T_j),T_j,T_k) \qquad \forall\, k-j<n.$$

If there are n segments there remains one bond price not determined by (6). Since under the time T_1 forward measure all assets valued with respect to the zero coupon bond maturing in T_1 are martingales, we can write this remaining bond price as

$$B(r(T_0),T_0,T_n) \;=\; B(r(T_0),T_0,T_1)E^{T_1}\left[\frac{B(r(T_1),T_1,T_n)}{B(r(T_1),T_1,T_1)}\right], \tag{7}$$

where the bond price in T_1 is given by (6) with a suitable shift of indices. Setting $\tilde{r}(T_1):=b_1r(T_1)$ we can then employ the result in Jamshidian (1987) that $\tilde{r}(T_1)$ conditioned on $r(T_0)$ is noncentral chi–square distributed under the T_1 forward measure, with ν_1 degrees of freedom and non–centrality parameter $\eta_1 r(T_0)$. Thus (7) becomes

$$\begin{aligned} B(r(T_0),T_0,T_n) &= B(r(T_0),T_0,T_1)\cdot \\ &\int_0^\infty \mathcal{C}_{1,n}\exp\{-\mathcal{D}_{1,n}\,r(T_1)\}q_{\chi^2}((b_1r(T_1),\nu_1,\eta_1r(T_0))d(b_1r(T_1)). \end{aligned} \tag{8}$$

Applying lemma B.1.1 in the appendix to (8) we get

$$\begin{aligned} B(r(T_0),T_0,T_n) &= B(r(T_0),T_0,T_1)\mathcal{C}_{1,n}\exp\left\{-\frac{\mathcal{D}_{1,n}}{b_1+2\mathcal{D}_{1,n}}\eta_1 r(T_0)\right\}\left(\frac{b_1}{b_1+2\mathcal{D}_{1,n}}\right)^{\frac{1}{2}\nu_1} \\ &\quad \cdot\underbrace{\int_0^\infty q_{\chi^2}\left((b_1+2\mathcal{D}_{1,n})r(T_1),\nu_1,\frac{b_1}{b_1+2\mathcal{D}_{1,n}}\eta_1 r(T_0)\right)d((b_1+2\mathcal{D}_{1,n})r(T_1))}_{=1} \\ &= \mathcal{A}_1(T_0,T_1)\mathcal{C}_{1,n}\left(\frac{b_1}{b_1+2\mathcal{D}_{1,n}}\right)^{\frac{1}{2}\nu_1}\exp\left\{-\left(\mathcal{B}(T_0,T_1)+\frac{\mathcal{D}_{1,n}}{b_1+2\mathcal{D}_{1,n}}\eta_1\right)r(T_0)\right\} \\ &= \mathcal{C}_{0,n}\exp\{-\mathcal{D}_{0,n}\,r(T_0)\}. \end{aligned}$$

□

Given proposition 2.2.1, we can fit the model to initial zero coupon bond prices $B(r(T_0), T_0, T_j)$ by choosing the drift parameters θ_j on the segments $[T_{j-1}, T_j]$ accordingly:

$$
\begin{aligned}
B(r(T_0), T_0, T_j) &= \mathcal{C}_{0,j} \exp\{-\mathcal{D}_{0,j}\, r(T_0)\} \\
&= \left(\prod_{i=0}^{j-1} \mathcal{A}_{j-i}(T_{j-i-1}, T_{j-i}) \left(\frac{b_{j-i}}{b_{j-i} + 2\mathcal{D}_{j-i,j}} \right)^{\frac{2\theta_{j-i}}{\sigma_{j-i}^2}} \right) \exp\{-\mathcal{D}_{0,j}\, r(T_0)\} \\
\Leftrightarrow \qquad \theta_j &= \frac{1}{2}\sigma_j^2 \left(\frac{1}{2}(c_j + a_j)(T_j - T_{j-1}) + \ln 2c_j w_j(T_{j-1}, T_j) \right)^{-1} \\
&\quad \cdot \left(\ln B(r(T_0), T_0, T_j) + \mathcal{D}_{0,j}\, r(T_0) - \right. \\
&\quad \left. \sum_{i=1}^{j-1} \frac{2\theta_{j-i}}{\sigma_{j-i}^2} \ln \left(\frac{b_{j-i}}{b_{j-i} + 2\mathcal{D}_{j-i,j}} \right) + \ln \mathcal{A}_{j-i}(T_{j-i-1}, T_{j-i}) \right).
\end{aligned}
$$

Starting with $j = 1$ we can thus successively calculate all θ_j for $j \in \{1; \dots ; N\}$. Note, however, that strongly downward sloping initial forward rate curves can lead to negative θ_j, and thus this model shares the disadvantage of the extended CIR model with a time dependent drift coefficient in that it cannot fit all possible initial term structures.

2.3. **Initial Volatility Term Structures.** The problem of calibrating a model to observed volatility structures has two dimensions, of which the term structure of volatilities for forward rates or zero coupon bonds of different maturities is most often cited. Besides this maturity dimension, however, there is the temporal dimension of how volatilities evolve. Historical estimates of deterministic volatility coefficients usually assume that these coefficients do not change over time. When calibrating a model to implied volatilities, the first dimension is given by for example prices of options on zero coupon bonds of different maturities, and the temporal dimension of the volatility structure is determined by prices of options on zero coupon bonds with the same time to maturity, but different option expiries.

In the "simple class of square root models" of Jamshidian (1995), the maturity dimension is covered by choosing the (time–dependent) speed of mean reversion to match input volatilities for forward rates. Instantaneous forward rates are given by

$$r_c(r(t), t, T) = -\frac{\partial}{\partial T} \ln B(r(t), t, T)$$

By Itô's Lemma, initial forward rate volatilities in our model are therefore $\sigma_1 \sqrt{r(T_0)} \frac{\partial}{\partial T_k} \mathcal{D}_{0,k}$ and we can state

2.3.1. Proposition. *Initial forward rate volatilities in the segmented square root model are* $\sigma_1 \sqrt{r(T_0)} \frac{\partial}{\partial T_k} \mathcal{D}_{0,k}$, *with*

$$\frac{\partial}{\partial T_k} \mathcal{D}_{0,k} = \left(\prod_{i=1}^{k-1} \eta_i b_i (b_i + 2\mathcal{D}_{i,k})^{-2} \right) \frac{\partial}{\partial T_k} \mathcal{B}_k(T_{k-1}, T_k) \tag{9}$$

and $\mathcal{D}$, η, b *and* $\mathcal{B}$ *defined as in proposition 2.2.1.*

Note that forward rate curves in our model are continuous (see proposition 3.2.1).

PROOF: By induction, we show the validity of the more general version of (9)

$$\frac{\partial}{\partial T_k}\mathcal{D}_{j,k} = \left(\prod_{i=j+1}^{k-1}\eta_i b_i(b_i+2\mathcal{D}_{i,k})^{-2}\right)\frac{\partial}{\partial T_k}\mathcal{B}_k(T_{k-1},T_k) \tag{10}$$

For $k-j=1$, (10) becomes

$$\frac{\partial}{\partial T_k}\mathcal{D}_{j,k} = \frac{\partial}{\partial T_k}\mathcal{B}_k(T_{k-1},T_k)$$

which is obviously true since $\mathcal{D}_{j,j+1} = \mathcal{B}_{j+1}(T_j,T_{j+1})$. Now let (10) be valid for some $k-j\geq 1$. Then for $k+1$ (or by a simple change of notation $j-1$) we have

$$\begin{aligned}
\frac{\partial}{\partial T_{k+1}}\mathcal{D}_{j,k+1} &= \frac{\partial}{\partial T_{k+1}}(\mathcal{B}_{j+1}(T_j,T_{j+1})+\eta_{j+1}(b_{j+1}\mathcal{D}^{-1}_{j+1,k+1}+2)^{-1}) \\
&= \eta_{j+1}(b_{j+1}\mathcal{D}^{-1}_{j+1,k+1}+2)^{-2}b_{j+1}\mathcal{D}^{-2}_{j+1,k+1}\frac{\partial}{\partial T_{k+1}}\mathcal{D}_{j+1,k+1} \\
&= \eta_{j+1}b_{j+1}(b_{j+1}+2\mathcal{D}_{j+1,k+1})^{-2}\left(\prod_{i=j+2}^{k}\eta_i b_i(b_i+2\mathcal{D}_{i,k+1})^{-2}\right)\frac{\partial}{\partial T_{k+1}}\mathcal{B}_{k+1}(T_k,T_{k+1}) \\
&= \left(\prod_{i=j+1}^{k}\eta_i b_i(b_i+2\mathcal{D}_{i,k+1})^{-2}\right)\frac{\partial}{\partial T_{k+1}}\mathcal{B}_{k+1}(T_k,T_{k+1})
\end{aligned}$$

□

For given σ_k, we can thus use (9) to inductively calculate the speed of mean reversion parameters a_k to match initial forward rate volatilities for the maturities T_1 to T_n. Alternatively, one could calculate the σ_k for given parameters a_k.

Calibrating the models to an initial volatility structure along the maturity dimension, we therefore still retain a degree of freedom that is already taken in the "simple class". Thus our model differs from the simple class in how the temporal dimension of the volatility structure is specified: In the simple class, the way initial forward rates are interpolated determines how volatilities evolve over time, while in the segmented model the factor volatility, be it the short rate or some yield of forward rate[4], determines the endogenous interpolation.

3. Pricing and Hedging

3.1. Contingent Claim Valuation. Having calibrated the model, we can now proceed to price other assets relative to the initial term structure, given the assumptions of the model. We will consider contingent claims whose payoffs can be expressed as linear combinations of European (exchange) options on securities whose terminal function is a simple exponential in r, thus allowing us to apply lemma B.1.1. Consider the value $V_j(t)$ of such a security at time t:

$$V_j(t,r(t)) \;=\; f_j(t)\exp\{-g_j(t)r(t)\}, \tag{11}$$

where f_j and g_j are deterministic functions of t. The payoff $C(t_m)$ at expiry t_m of a European exchange option on two such securities is defined as

$$C\left(t_m,r(t_m)\right) \;:=\; \left[V_1\left(t_m,r(t_m)\right)-V_2\left(t_m,r(t_m)\right)\right]^+ . \tag{12}$$

[4]Note that affine models can be reparameterized in any yield or forward rate instead of the short rate (see Duffie and Kan (1992, 1996)).

Note that for $f_2(t_m) = K$ and $g_2(t_m) = 0$ we have a European call option on V_1. Alternatively, setting $f_1(t_m) = K$ and $g_1(t_m) = 0$ yields a European put option on V_2. Let

$$k \;:=\; \max\{n \in \{0;\ldots;N\} \mid T_n < t_m\}\,. \tag{13}$$

On $[T_k; T_{k+1}]$ we have a constant parameter CIR process, and following Jamshidian (1987) we know that $\tilde{r}(t_m) := \tilde{b}\cdot r(t_m)$ conditioned on $r(T_k)$ is noncentral chi–square distributed under the t_m forward measure, with ν_{k+1} degrees of freedom and noncentrality parameter $\tilde{\eta}\cdot r(T_k)$, where

$$\begin{aligned}
\tilde{b} &:= \frac{4}{\sigma_{k+1}^2}\mathcal{B}_{k+1}(T_k,t_m)^{-1}\\
\tilde{\eta} &:= \frac{16 w_{k+1}(T_k,t_m)^2\, c_{k+1}^2 \exp\{c_{k+1}(t_m - T_k)\}}{\sigma_{k+1}^2\mathcal{B}_{k+1}(T_k,t_m)}
\end{aligned}$$

with $\mathcal{B}$, c and w defined as in proposition 2.2.1. Therefore

$$\begin{aligned}
C(T_k, r(T_k)) =& B(r(T_k),T_k,t_m)E^{t_m}\left[[V_1(t_m,r(t_m)) - V_2(t_m,r(t_m))]^+ \,|\mathcal{F}_{T_k}\right]\\
=& B(r(T_k),T_k,t_m)\left(\int_Z f_1(t_m)\exp\{-g_1(t_m)r(t_m)\}q_{\chi^2}\left(\tilde{b}r(t_m),\nu_{k+1},\tilde{\eta}r(T_k)\right)d(\tilde{b}r(t_m))\right.\\
&\left. - \int_Z f_2(t_m)\exp\{-g_2(t_m)r(t_m)\}q_{\chi^2}\left(\tilde{b}r(t_m),\nu_{k+1},\tilde{\eta}r(T_k)\right)d(\tilde{b}r(t_m))\right)
\end{aligned} \tag{14}$$

with

$$Z := \{r(t_m) > 0 \mid V_1(t_m,r(t_m)) > V_2(t_m,r(t_m))\}.$$

Given the functional form of V_1 and V_2, we have either $Z =]0;r^*[$ or $Z =]r^*;\infty[$ for some deterministic $r^* > 0$. We consider $Z =]0;r^*[$; the calculations for $Z =]r^*;\infty[$ are analogous. Applying lemma B.1.1, (14) becomes

$$\begin{aligned}
&C(T_k, r(T_k))\\
=& B(r(T_k),T_k,t_m)\left(f_1(t_m)\exp\left\{-\frac{g_1(t_m)}{\tilde{b}+2g_1(t_m)}\,\tilde{\eta}\,r(T_k)\right\}\left(\frac{\tilde{b}}{\tilde{b}+2g_1(t_m)}\right)^{\frac{1}{2}\nu_{k+1}}\right.\\
&\int_0^{(\tilde{b}+2g_1(t_m))r^*} q_{\chi^2}\left((\tilde{b}+2g_1(t_m))r(t_m),\ \nu_{k+1},\ \frac{\tilde{b}}{\tilde{b}+2g_1(t_m)}\,\tilde{\eta}\,r(T_k)\right)d\left((\tilde{b}+2g_1(t_m))r(t_m)\right)\\
&- f_2(t_m)\exp\left\{-\frac{g_2(t_m)}{\tilde{b}+2g_2(t_m)}\,\tilde{\eta}\,r(T_k)\right\}\left(\frac{\tilde{b}}{\tilde{b}+2g_2(t_m)}\right)^{\frac{1}{2}\nu_{k+1}}\\
&\left.\int_0^{(\tilde{b}+2g_2(t_m))r^*} q_{\chi^2}\left((\tilde{b}+2g_2(t_m))r(t_m),\ \nu_{k+1},\ \frac{\tilde{b}}{\tilde{b}+2g_2(t_m)}\,\tilde{\eta}\,r(T_k)\right)d\left((\tilde{b}+2g_2(t_m))r(t_m)\right)\right)
\end{aligned}$$

which we can write as

$$C(T_k, r(T_k)) = \hat{f}_1 \exp\{-\hat{g}_1 r(T_k)\} \int_0^{\hat{b}_1 r^*} q_{\chi^2}\left(\hat{b}_1 r(t_m), \nu_{k+1}, \hat{\eta}_1 r(T_k)\right) d(\hat{b}_1 r(t_m)) - \hat{f}_2 \exp\{-\hat{g}_2 r(T_k)\} \int_0^{\hat{b}_2 r^*} q_{\chi^2}\left(\hat{b}_2 r(t_m), \nu_{k+1}, \hat{\eta}_2 r(T_k)\right) d(\hat{b}_2 r(t_m)) \tag{15}$$

with

$$\hat{f}_1 := \mathcal{A}_{k+1}(T_k, t_m) f_1(t_m) \left(\frac{\tilde{b}}{\tilde{b} + 2g_1(t_m)}\right)^{\frac{1}{2}\nu_{k+1}}$$

$$\hat{f}_2 := \mathcal{A}_{k+1}(T_k, t_m) f_2(t_m) \left(\frac{\tilde{b}}{\tilde{b} + 2g_2(t_m)}\right)^{\frac{1}{2}\nu_{k+1}}$$

$$\hat{g}_1 := \mathcal{B}_{k+1}(T_k, t_m) + \frac{g_1(t_m)\tilde{\eta}}{\tilde{b} + 2g_1(t_m)}$$

$$\hat{g}_2 := \mathcal{B}_{k+1}(T_k, t_m) + \frac{g_2(t_m)\tilde{\eta}}{\tilde{b} + 2g_2(t_m)}$$

$$\hat{b}_1 := \tilde{b} + 2g_1(t_m)$$

$$\hat{b}_2 := \tilde{b} + 2g_2(t_m)$$

$$\hat{\eta}_1 := \frac{\tilde{b}}{\tilde{b} + 2g_1(t_m)}\tilde{\eta}$$

$$\hat{\eta}_2 := \frac{\tilde{b}}{\tilde{b} + 2g_2(t_m)}\tilde{\eta}$$

and

$$\int_0^{\hat{b}_j r^*} q_{\chi^2}\left(\hat{b}_j r(t_m), \nu_{k+1}, \hat{\eta}_j r(T_k)\right) d\left(\hat{b}_j r(t_m)\right) = \chi^2_{\nu_{k+1}, \hat{\eta}_j r(T_k)}(\hat{b}_j r^*)$$

the value of the noncentral chi–square distribution function. (15) is the formula for pricing the exchange option defined by (12) and (11) in a constant parameter CIR model. Given this price at T_k, the next lower segment boundary to option expiry t_m, in analogy to proposition 2.2.1 we now state the pricing formula for the earlier segment boundaries T_n, $n < k$:

3.1.1. Proposition. *In the segmented square root model with time segments* $[T_{n-1}; T_n]$, $n \in \{1; \ldots; N\}$, *consider an exchange option defined by (12) and (11), k defined by (13).*

For $n \leq k$, the time T_n price of the option is given by

$$
\begin{aligned}
(16) \qquad C(T_n, r(T_n)) &= \mathcal{C}^{(1)}_{n,k} \exp\left\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\right\} P\left(b^{(1)}_{n,k+1} r(t_m) \leq z^*_m\right) \\
&\quad -\mathcal{C}^{(2)}_{n,k} \exp\left\{-\mathcal{D}^{(2)}_{n,k} r(T_n)\right\} P\left(b^{(2)}_{n,k+1} r(t_m) \leq z^*_m\right) \\
\text{if} \quad V_1(t_m, r(t_m)) &> V_2(t_m, r(t_m)) \quad \text{for} \quad r(t_m) \in]0; r^*[\quad \text{and} \\
(17) \qquad C(T_n, r(T_n)) &= \mathcal{C}^{(1)}_{n,k} \exp\left\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\right\} \left(1 - P\left(b^{(1)}_{n,k+1} r(t_m) \leq z^*_m\right)\right) \\
&\quad -\mathcal{C}^{(2)}_{n,k} \exp\left\{-\mathcal{D}^{(2)}_{n,k} r(T_n)\right\} \left(1 - P\left(b^{(2)}_{n,k+1} r(t_m) \leq z^*_m\right)\right) \\
\text{if} \quad V_1(t_m, r(t_m)) &> V_2(t_m, r(t_m)) \quad \text{for} \quad r(t_m) \in]r^*; \infty[,
\end{aligned}
$$

with $\mathcal{C}^{(j)}_{n,k}$ and $\mathcal{D}^{(j)}_{n,k}$ recursively defined as in proposition 2.2.1, however with $\mathcal{C}^{(j)}_{k,k}$ and $\mathcal{D}^{(j)}_{k,k}$ given by (15):

$$\mathcal{C}^{(j)}_{k,k} := \hat{f}_j \qquad \mathcal{D}^{(j)}_{k,k} := \hat{g}_j.$$

$P\left(b^{(j)}_{n,k+1} r(t_m) \leq z^*_m\right) = \int_0^{z^*_m} p\left(b^{(j)}_{n,k+1} r(t_m) = z_m\right) dz_m$ *is the distribution function of a $(k-n+1)$–times multiply compound*[5] *noncentral χ^2 distributed random variable with degrees of freedom ν_{n+s}, $s \in \{1; \ldots; k-n+1\}$, noncentrality parameters*

$$\lambda^{(j)}_s := \begin{cases} \hat{\eta}_j r(T_k) & s = k-n+1 \\ \frac{\eta_{n+s}\, b_{n+s}}{b_{n+s} + 2\mathcal{D}^{(j)}_{n+s,k}} r(T_{n+s-1}) & s \in \{1; \ldots; k-n\} \end{cases}$$

and transformation coefficients

$$b^{(j)}_{n,n+s} := \begin{cases} \hat{b}_j & s = k-n+1 \\ b_{n+s} + 2\mathcal{D}^{(j)}_{n+s,k} & s \in \{1; \ldots; k-n\} \end{cases}$$

and z^*_m is given by

$$z^*_m = b^{(j)}_{n,k+1} (g_1 - g_2)^{-1} \ln \frac{f_1}{f_2}.$$

PROOF: Again, we carry out the proof by induction, showing the first of the two analogous cases (16) and (17): For $n = k$, (16) is identical to (15). Let (16) be valid for some $0 < n \leq k$. Then we have for $n-1$:

$$
\begin{aligned}
& C\left(T_{n-1}, r(T_{n-1})\right) \\
=\ & B\left(r(T_{n-1}), T_{n-1}, T_n\right) \int_0^\infty C\left(T_n, r(T_n)\right) q_{\chi^2}\left(b_n r(T_n), \nu_n, \eta_n r(T_{n-1})\right) d(b_n r(T_n)) \\
=\ & \mathcal{A}_n(T_{n-1}, T_n) \exp\left\{-\mathcal{B}_n(T_{n-1}, T_n) r(T_{n-1})\right\} \\
& \cdot \left(\mathcal{C}^{(1)}_{n,k} \int_0^\infty \exp\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\} \int_0^{z^*_m} p\left(b^{(1)}_{n,k+1} r(t_m) = z_m\right) dz_m\, q_{\chi^2}\left(b_n r(T_n), \nu_n, \eta_n r(T_{n-1})\right) d\left(b_n r(T_n)\right)\right. \\
& \left. -\mathcal{C}^{(2)}_{n,k} \int_0^\infty \exp\{-\mathcal{D}^{(2)}_{n,k} r(T_n)\} \int_0^{z^*_m} p\left(b^{(2)}_{n,k+1} r(t_m) = z_m\right) dz_m\, q_{\chi^2}\left(b_n r(T_n), \nu_n, \eta_n r(T_{n-1})\right) d\left(b_n r(T_n)\right)\right).
\end{aligned}
$$

[5] see appendix B.3

By interchanging the order of integration and applying lemma B.1.1, we get

$$
\begin{aligned}
& \mathcal{A}_n(T_{n-1},T_n)\exp\{-\mathcal{B}_n(T_{n-1},T_n)r(T_{n-1})\} \\
& \cdot \mathcal{C}^{(j)}_{n,k}\int_0^\infty \exp\{-\mathcal{D}^{(j)}_{n,k}r(T_n)\}\int_0^{z^*_m} p\left(b^{(j)}_{n,k+1}r(t_m)=z_m\right)dz_m q_{\chi^2}(b_n r(T_n),\nu_n,\eta_n r(T_{n-1}))d(b_n r(T_n)) \\
= \; & \mathcal{A}_n(T_{n-1},T_n)\exp\{-\mathcal{B}_n(T_{n-1},T_n)r(T_{n-1})\} \\
& \cdot \mathcal{C}^{(j)}_{n,k}\int_0^{z^*_m}\int_0^\infty p\left(b^{(j)}_{n,k+1}r(t_m)=z_m\right)\exp\left\{-\frac{\mathcal{D}^{(j)}_{n,k}}{b_n+2\mathcal{D}^{(j)}_{n,k}}\eta_n r(T_{n-1})\right\}\left(\frac{b_n}{b_n+2\mathcal{D}^{(j)}_{n,k}}\right)^{\frac{1}{2}\nu_n} \\
& \cdot q_{\chi^2}\left((b_n+2\mathcal{D}^{(j)}_{n,k})r(T_n),\,\nu_n,\,\frac{\eta_n b_n}{b_n+2\mathcal{D}^{(j)}_{n,k}}r(T_{n-1})\right)d\left((b_n+2\mathcal{D}^{(j)}_{n,k})\cdot r(T_n)\right)dz_m \\
= \; & \mathcal{C}^{(j)}_{n-1,k}\exp\left\{-\mathcal{D}^{(j)}_{n-1,k}\,r(T_{n-1})\right\}\int_0^{z^*_m} p\left(b^{(j)}_{n-1,k+1}\cdot r(t_m)=z_m\right)dz_m.
\end{aligned}
$$

□

3.1.2. Remark. *Note that the $\mathcal{C}^{(j)}_{n,k}\exp\{-\mathcal{D}^{(j)}_{n,k}r(T_n)\}$ are simply the time T_n values of the underlying assets, so (16) can also be written as*

$$C(T_n,r(T_n))=V_1(T_n,r(T_n))P\left(b^{(1)}_{n,k+1}r(t_m)\leq z^*_m\right)-V_2(T_n,r(T_n))P\left(b^{(2)}_{n,k+1}r(t_m)\leq z^*_m\right)$$

3.1.3. Remark. *We can use proposition 3.1.1 to calculate the option price at any time t as a function of $r(t)$ by first setting*

$$n:=\max\{i\in\{0;\ldots;N\}|t>T_i\}$$

and then $T_n:=t$.

Proposition 3.1.1 allows us to price a wide class of contingent claims. For one, all claims which can be represented as portfolios of (European) options on zero coupon bonds are covered. This includes caps and floors, and also swaptions and options on coupon bonds, since in the one–factor model considered here bond prices are monotonic in the short rate, thus the argument of Jamshidian (1989) is applicable.

Spread options on forward LIBOR can also be priced. More generally, consider a spread option on two forward rates with actuarial compounding. Such a time T forward rate with compounding period α is given at time t by

$$r_a(t,T,\alpha)=\frac{1}{\alpha}\left(\frac{B(r(t),t,T)}{B(r(t),t,T+\alpha)}-1\right)$$

and thus the payoff on an option on the spread between two such rates is

$$C(t_m,r(t_m)):=\frac{1}{\alpha}\left[\frac{B(r(t_m),t_m,T_1)}{B(r(t_m),t_m,T_1+\alpha)}-\frac{B(r(t_m),t_m,T_2)}{B(r(t_m),t_m,T_2+\alpha)}\right]^+$$

Since the quotients of zero coupon bond prices are simple exponentials of $r(t_m)$, proposition 3.1.1 applies. Similarly, futures on simple exponentials of r remain simple exponentials in the segmented square root model, as they do in the original CIR, so proposition 3.1.1 can also be used to price options on futures on zero coupon bonds. Finally, proposition 3.1.1 can easily be extended to options on linear combinations of simple exponentials of r, in order to valuate options on actuarial yield spreads, for example.

The pricing formulae (16) and (17) call for the evaluation of multiply compound noncentral χ^2 distribution functions, which at first glance appears to be a task of high numerical

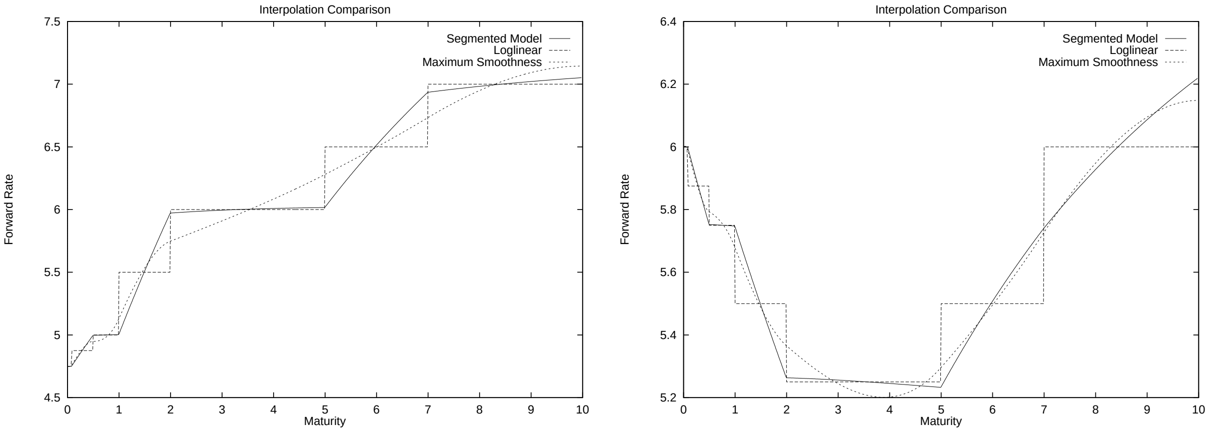

FIGURE 1. Interpolated term structures

complexity. However, as discussed in appendix B.3, the number of operations needed to calculate the value of an n–times multiply compound noncentral χ^2 distribution function only grows linearly in n. Furthermore, note that this n is determined by the number of time segments up to the expiry of the option only, since the value of the underlyings at option expiry is known explicitly as a function of r.

3.2. **Term Structure Interpolation.** Proposition 3.1.1 also provides interpolated zero coupon bond prices: If one sets $f_1(t_m) = 1$ and $g_1(t_m) = g_2(t_m) = f_2(t_m) = 0$, then $C(T_n, r(T_n)) = \mathcal{C}^{(1)}_{n,k} \exp\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\}$ is the time T_n price of a zero coupon bond maturing in t_m.

3.2.1. Proposition. *Interpolated forward rate curves in the segmented square root model are continuous.*

PROOF: At any future point in time t, we can without loss of generality set $T_j := t$, where T_{j+1} is the earliest segment boundary greater than t in the original segmentation. We have

$$r_c(r(T_j), T_j, t_m) = -\frac{\partial}{\partial t_m} \ln \mathcal{C}^{(1)}_{j,k} \exp\{-\mathcal{D}^{(1)}_{j,k} r(T_j)\} \tag{18}$$

Since $\mathcal{C}^{(1)}_{j,k}$ and $\mathcal{D}^{(1)}_{j,k}$ are smooth functions of t_m for $t_m \in]T_k; T_{k+1}[$, we only need to show continuity of (18) on the segment boundaries, i.e. for

$$r_c(r(T_j), T_j, T_k) = -\frac{\partial}{\partial T_k} \ln \mathcal{C}_{j,k} + r(T_j) \frac{\partial}{\partial T_k} \mathcal{D}_{j,k} \tag{19}$$

with $\mathcal{C}_{j,k}$ and $\mathcal{D}_{j,k}$ defined as in proposition 2.2.1. It is sufficient to show the continuity of the two terms in (19) separately. $\frac{\partial}{\partial T_k}\mathcal{D}_{j,k}$ is already given in (10). Consider

$$\begin{aligned}
\frac{\partial}{\partial T_k}\mathcal{B}_k(T_{k-1}, T_k) &= -2w_k(T_{k-1}, T_k)^2 (c_k + a_k) c_k \exp\{c_k(T_k - T_{k-1})\}(\exp\{c_k(T_k - T_{k-1})\} - 1) \\
&\quad +2w_k(T_{k-1}, T_k) c_k \exp\{c_k(T_k - T_{k-1})\} \\
&= 2w_k(T_{k-1}, T_k)^2 ((c_k + a_k) c_k \exp\{c_k(T_k - T_{k-1})\}(1 - \exp\{c_k(T_k - T_{k-1})\}) \\
&\quad +c_k \exp\{c_k(T_k - T_{k-1})\}((c_k + a_k)\exp\{c_k(T_k - T_{k-1})\} + c_k - a_k)) \\
&= 4c_k^2 w_k(T_{k-1}, T_k)^2 \exp\{c_k(T_k - T_{k-1})\} = \eta_k b_k^{-1}
\end{aligned}$$

Inserting this into (10), we get

$$\begin{aligned}\frac{\partial}{\partial T_k}\mathcal{D}_{j,k}\bigg|_{T_k=T_{k-1}} &= \prod_{i=j+1}^{k-1}\eta_i b_i(b_i+2\mathcal{D}_{i,k})^{-2}\\ &= \left(\prod_{i=j+1}^{k-2}\eta_i b_i(b_i+2\mathcal{D}_{i,k})^{-2}\right)\eta_{k-1}b_{k-1}^{-1}\\ &= \frac{\partial}{\partial T_{k-1}}\mathcal{D}_{j,k-1}\end{aligned}$$

thus showing the continuity of the second term on the segment boundary. The first term is

$$-\frac{\partial}{\partial T_k}\ln\mathcal{C}_{j,k} = -\frac{\partial}{\partial T_k}\ln\mathcal{A}_k(T_{k-1}.T_k) + \sum_{i=j+1}^{k-1}\nu_i(b_i+2\mathcal{D}_{i,k})^{-1}\frac{\partial}{\partial T_k}\mathcal{D}_{i,k} \tag{20}$$

where

$$\begin{aligned}-\frac{\partial}{\partial T_k}\ln\mathcal{A}_k(T_{k-1}.T_k) &= -\frac{1}{2}\nu_k\left(2c_k w_k(T_{k-1},T_k)\exp\left\{\frac{1}{2}(c_k+a_k)(T_k-T_{k-1})\right\}\right)^{-1}\\ &\quad\cdot\Bigg(2c_k\frac{1}{2}(c_k+a_k)w_k(T_{k-1},T_k)\exp\left\{\frac{1}{2}(c_k+a_k)(T_k-T_{k-1})\right\}\\ &\quad -2c_k w_k(T_{k-1},T_k)^2(c_k+a_k)c_k\exp\{c_k(T_k-T_{k-1})\}\\ &\quad \exp\left\{\frac{1}{2}(c_k+a_k)(T_k-T_{k-1})\right\}\Bigg)\\ &= -\frac{1}{2}\nu_k\left(\frac{1}{2}(c_k+a_k)-c_k w_k(T_{k-1},T_k)(c_k+a_k)\exp\{c_k(T_k-T_{k-1})\}\right)\\ &= \frac{1}{4}\nu_k w_k(T_{k-1},T_k)(c_k+a_k)(c_k-a_k)(\exp\{c_k(T_k-T_{k-1})\}-1)\\ &= \frac{1}{4}\nu_k\sigma_k^2\mathcal{B}_k(T_{k-1},T_k) = \nu_k b_k^{-1}\end{aligned}$$

Inserting this into (20), by the continuity of the $\frac{\partial}{\partial T_k}\mathcal{D}_{i,k}$ we get

$$\begin{aligned}-\frac{\partial}{\partial T_k}\ln\mathcal{C}_{j,k}\bigg|_{T_k=T_{k-1}} &= \sum_{i=j+1}^{k-1}\nu_i(b_i+2\mathcal{D}_{i,k})^{-1}\frac{\partial}{\partial T_k}\mathcal{D}_{i,k}\\ &= \left(\sum_{i=j+1}^{k-2}\nu_i(b_i+2\mathcal{D}_{i,k-1})^{-1}\frac{\partial}{\partial T_{k-1}}\mathcal{D}_{i,k-1}\right)+\nu_{k-1}b_{k-1}^{-1}\\ &= \frac{\partial}{\partial T_{k-1}}\ln\mathcal{C}_{j,k-1}\end{aligned}$$

□

Figure (1) shows examples of how the segmented square root model interpolates initial term structures, as compared to loglinear interpolation of zero coupon bond prices and the "maximum smoothness" approach of Adams and van Deventer (1994).

3.3. **Hedging.** Since (16) and (17) were determined using the no-arbitrage condition, there remain two conditions which a self-financing portfolio strategy duplicating the option must satisfy: The value of the portfolio must equal the value of the option at all times and the martingale part of the portfolio process must match the martingale part of the option process. For (16), the former yields[6]

$$\phi_1 = P\left(b^{(1)}_{n,k+1} r(t_m) \leq z^*_m\right) - \frac{\mathcal{C}^{(2)}_{n,k} \exp\left\{-\mathcal{D}^{(2)}_{n,k} r(T_n)\right\}}{\mathcal{C}^{(1)}_{n,k} \exp\left\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\right\}} \left(P\left(b^{(2)}_{n,k+1} r(t_m) \leq z^*_m\right) + \phi_2\right) \tag{21}$$

Let $(X)^M$ denote the (uniquely determined) martingale part of a continuous semimartingale X. By Itô's Lemma

$$d(C(T_n, r(T_n)))^M = \frac{\partial}{\partial r(T_n)} C(T_n, r(T_n)) d(r(T_n))^M$$

with

$$\begin{aligned} \frac{\partial}{\partial r(T_n)} C(T_n, r(T_n)) \quad = \quad & \sum_{j=1}^{2} (-1)^j \left(\mathcal{D}^{(j)}_{n,k} \mathcal{C}^{(j)}_{n,k} \exp\left\{-\mathcal{D}^{(j)}_{n,k} r(T_n)\right\} P\left(b^{(j)}_{n,k+1} r(t_m) \leq z^*_m\right)\right. \\ & \left. - \mathcal{C}^{(j)}_{n,k} \exp\left\{-\mathcal{D}^{(j)}_{n,k} r(T_n)\right\} \frac{\partial}{\partial r(T_n)} P\left(b^{(j)}_{n,k+1} r(t_m) \leq z^*_m\right)\right) \end{aligned} \tag{22}$$

Similarly, the martingale parts of the processes of the underlyings are

$$d\left(\mathcal{C}^{(j)}_{n,k} \exp\left\{-\mathcal{D}^{(j)}_{n,k} r(T_n)\right\}\right)^M = -\mathcal{D}^{(j)}_{n,k} \mathcal{C}^{(j)}_{n,k} \exp\left\{-\mathcal{D}^{(j)}_{n,k} r(T_n)\right\} d(r(T_n))^M \tag{23}$$

Note that in square root interest rate models, in contrast to the case of options on lognormal assets (such as in the Black and Scholes (1973) model), the derivatives of the distribution function do not cancel out.

By (22) and (23), in order to match the martingale parts of the portfolio and the option processes, we must have

$$\begin{aligned} & - \mathcal{D}^{(1)}_{n,k} \mathcal{C}^{(1)}_{n,k} \exp\left\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\right\} P\left(b^{(1)}_{n,k+1} r(t_m) \leq z^*_m\right) \\ & + \mathcal{C}^{(1)}_{n,k} \exp\left\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\right\} \frac{\partial}{\partial r(T_n)} P\left(b^{(1)}_{n,k+1} r(t_m) \leq z^*_m\right) \\ & + \mathcal{D}^{(2)}_{n,k} \mathcal{C}^{(2)}_{n,k} \exp\left\{-\mathcal{D}^{(2)}_{n,k} r(T_n)\right\} P\left(b^{(2)}_{n,k+1} r(t_m) \leq z^*_m\right) \\ & - \mathcal{C}^{(2)}_{n,k} \exp\left\{-\mathcal{D}^{(2)}_{n,k} r(T_n)\right\} \frac{\partial}{\partial r(T_n)} P\left(b^{(2)}_{n,k+1} r(t_m) \leq z^*_m\right) \\ = & - \phi_1 \mathcal{D}^{(1)}_{n,k} \mathcal{C}^{(1)}_{n,k} \exp\left\{-\mathcal{D}^{(1)}_{n,k} r(T_n)\right\} - \phi_2 \mathcal{D}^{(2)}_{n,k} \mathcal{C}^{(2)}_{n,k} \exp\left\{-\mathcal{D}^{(2)}_{n,k} r(T_n)\right\} \end{aligned}$$

[6] In order not to complicate the notation further, we apply remark 3.1.3 in what follows in this section.

and inserting (21)

$$\begin{aligned}
&\mathcal{C}^{(1)}_{n,k}\exp\left\{-\mathcal{D}^{(1)}_{n,k}r(T_n)\right\}\frac{\partial}{\partial r(T_n)}P\left(b^{(1)}_{n,k+1}r(t_m)\leq z^*_m\right)\\
&+\mathcal{D}^{(2)}_{n,k}\mathcal{C}^{(2)}_{n,k}\exp\left\{-\mathcal{D}^{(2)}_{n,k}r(T_n)\right\}P\left(b^{(2)}_{n,k+1}r(t_m)\leq z^*_m\right)\\
&-\mathcal{C}^{(2)}_{n,k}\exp\left\{-\mathcal{D}^{(2)}_{n,k}r(T_n)\right\}\frac{\partial}{\partial r(T_n)}P\left(b^{(2)}_{n,k+1}r(t_m)\leq z^*_m\right)\\
=&\mathcal{D}^{(1)}_{n,k}\mathcal{C}^{(2)}_{n,k}\exp\left\{-\mathcal{D}^{(2)}_{n,k}r(T_n)\right\}P\left(b^{(2)}_{n,k+1}r(t_m)\leq z^*_m\right)+\phi_2(\mathcal{D}^{(1)}_{n,k}-\mathcal{D}^{(2)}_{n,k})\mathcal{C}^{(2)}_{n,k}\exp\left\{-\mathcal{D}^{(2)}_{n,k}r(T_n)\right\}
\end{aligned}$$

Solving for ϕ_2,

$$\begin{aligned}
\phi_2=&(\mathcal{D}^{(1)}_{n,k}-\mathcal{D}^{(2)}_{n,k})^{-1}\left(\frac{\mathcal{C}^{(1)}_{n,k}\exp\left\{-\mathcal{D}^{(1)}_{n,k}r(T_n)\right\}}{\mathcal{C}^{(2)}_{n,k}\exp\left\{-\mathcal{D}^{(2)}_{n,k}r(T_n)\right\}}\frac{\partial}{\partial r(T_n)}P\left(b^{(1)}_{n,k+1}r(t_m)\leq z^*_m\right)\right.\\
&\left.-\frac{\partial}{\partial r(T_n)}P\left(b^{(2)}_{n,k+1}r(t_m)\leq z^*_m\right)\right)-P\left(b^{(2)}_{n,k+1}r(t_m)\leq z^*_m\right)
\end{aligned}$$

and similarly

$$\begin{aligned}
\phi_1=&P\left(b^{(1)}_{n,k+1}r(t_m)\leq z^*_m\right)-(\mathcal{D}^{(1)}_{n,k}-\mathcal{D}^{(2)}_{n,k})^{-1}\left(\frac{\partial}{\partial r(T_n)}P\left(b^{(1)}_{n,k+1}r(t_m)\leq z^*_m\right)\right.\\
&\left.-\frac{\mathcal{C}^{(2)}_{n,k}\exp\left\{-\mathcal{D}^{(2)}_{n,k}r(T_n)\right\}}{\mathcal{C}^{(1)}_{n,k}\exp\left\{-\mathcal{D}^{(1)}_{n,k}r(T_n)\right\}}\frac{\partial}{\partial r(T_n)}P\left(b^{(2)}_{n,k+1}r(t_m)\leq z^*_m\right)\right)
\end{aligned}$$

The derivative of the multiply compound noncentral χ^2 distribution function with respect to $r(T_n)$ is given in lemma B.3.4.

Of course, the duplicating portfolio can also be constructed using two instruments different from the underlying securities, either directly as above or by first duplicating the underlyings: To duplicate the option using some zero coupon bond $B(r(T_n),T_n,T_x)$ and the savings account, we use the equations ($j=1;2$):

$$\begin{aligned}
\mathcal{C}^{(j)}_{n,k}\exp\left\{-\mathcal{D}^{(j)}_{n,k}r(T_n)\right\} &= \phi^{(j)}_0+\phi^{(j)}_xB(r(T_n),T_n,T_x)\\
-\mathcal{D}^{(j)}_{n,k}\mathcal{C}^{(j)}_{n,k}\exp\left\{-\mathcal{D}^{(j)}_{n,k}r(T_n)\right\} &= -\phi^{(j)}_x\mathcal{D}_{n,x}B(r(T_n),T_n,T_x)
\end{aligned}$$

and thus

$$\begin{aligned}
\phi^{(j)}_x &= \frac{\mathcal{D}^{(j)}_{n,k}\mathcal{C}^{(j)}_{n,k}\exp\left\{-\mathcal{D}^{(j)}_{n,k}r(T_n)\right\}}{\mathcal{D}_{n,x}B(r(T_n),T_n,T_x)}\\
\phi^{(j)}_0 &= \mathcal{C}^{(j)}_{n,k}\exp\left\{-\mathcal{D}^{(j)}_{n,k}r(T_n)\right\}\left(1-\frac{\mathcal{D}^{(j)}_{n,k}}{\mathcal{D}_{n,x}}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\phi^{(1)}_x\phi_1+\phi^{(2)}_x\phi_2 &&\text{of } B(r(T_n),T_n,T_x)\\
\text{and}\quad &\phi^{(1)}_0\phi_1+\phi^{(2)}_0\phi_2 &&\text{in the savings account}
\end{aligned}$$

duplicate the option.

4. Market Example

As an example, we fit the segmented square root model to prices of futures and options on futures on three–month Euromark deposits traded at the LIFFE, taking the Frankfurt DEM overnight rate as a proxy for the short rate. Calibrating the model to observed option prices serves to illustrate its flexibility in also fitting a term structure of *implied* volatility. As discussed in section 2.3, the "term structure of volatility" has several dimensions, of which the volatility of zero coupon bonds of different maturities is only one, though one on which most of the literature focuses[7]. However, one may want a model to have the capability to fit either type of volatility input, with implied volatilities particularly relevant when pricing derivatives *relative* to a liquid market such as the LIFFE.

4.1. **Futures Valuation.** Consider a futures contract maturing in T on some asset V with mark–to–market occuring at times t_k, $k \in \{1; \dots; n-1\}$, $t_n = T$. Cox, Ingersoll jr. and Ross (1981) show that the futures price $H(t_0, T, W)$ equals the time t_0 value of an asset which pays

$$V(T)\prod_{k=0}^{n-1} B(r(t_k), t_k, t_{k+1})^{-1}$$

at time T. We have

$$V(T)\prod_{k=0}^{n-1} B(r(t_k), t_k, t_{k+1})^{-1} = V(T)\exp\left\{\sum_{k=0}^{n-1}(t_{k+1}-t_k)y(t_k, t_{k+1})\right\}, \tag{24}$$

where $y(t_k, t_{k+1})$ is the continuously compounded yield at time t_k for a riskless investment up to time t_{k+1}. Letting the mark–to–market go to the continuous limit with $n \to \infty$, (24) becomes

$$\lim_{n\to\infty} V(T)\prod_{k=0}^{n-1} B(r(t_k), t_k, t_{k+1})^{-1} = V(T)\exp\left\{\int_{t_0}^{T} r(s)ds\right\}.$$

Thus the futures price becomes

$$\begin{aligned} H(t_0, T, V) &= E\left[\exp\left\{-\int_{t_0}^{T} r(s)ds\right\} V(T)\exp\left\{\int_{t_0}^{T} r(s)ds\right\}\middle|\mathcal{F}_{t_0}\right] \\ &= E\left[V(T)|\mathcal{F}_{t_0}\right], \end{aligned} \tag{25}$$

i.e. the futures price is the expected value at delivery of the underlying asset under the risk neutral measure.

In order to calculate the expectation in (25), we need the distribution of the short rate under the risk neutral measure, given in lemma B.2.1 in the appendix. Again, consider an asset whose value at time T is a simple exponential in $r(T)$:

$$V(T) = f(T)\exp\{-g(T)r(T)\}.$$

On a constant parameter segment $[T_{j-1}; T_j]$ in our model we have

$$\begin{aligned} H(T_{j-1}, T_j, V) &= E\left[V(T_j)\,|\mathcal{F}_{T_{j-1}}\right] \\ &= \int_0^{\infty} f(T_j)\exp\{-g(T_j)r(T_j)\}\, q_{\chi^2}\left(b_j r(T_j), \nu_j, \eta_j r(T_{j-1})\right) d\left(b_j r(T_j)\right) \end{aligned} \tag{26}$$

[7] see for example Jamshidian (1995).

with

$$b_j := \frac{4a_j}{\sigma_j^2}\left(1-e^{-a_j(T_j-T_{j-1})}\right)^{-1}$$
$$\nu_j := \frac{4\theta_j}{\sigma_j^2}$$
$$\eta_j := \frac{4a_j}{\sigma_j^2}\left(e^{a_j(T_j-T_{j-1})}-1\right)^{-1}$$

Thus by lemma B.1.1

$$H(T_{j-1},T_j,V) = f(T_j)\exp\left\{-\frac{g(T_j)}{b_j+2g(T_j)}\eta_j r(T_{j-1})\right\}\left(\frac{b_j}{b_j+2g(T_j)}\right)^{\frac{1}{2}\nu_j}. \tag{27}$$

Using the martingale property of the futures price process under the risk neutral measure we can carry out an induction analogous to the proof of proposition 2.2.1, yielding

4.1.1. Proposition. *In the segmented square root model with time segments* $[T_{j-1},T_j]$, $j\in\{1;\dots;N\}$*, the time* T_{j-1} *futures price of an asset with value* $V(T_k)=f(T_k)\exp\{-g(T_k)r(T_k)\}$ *at contract maturity* T_k, $j\le k\le N$*, is given by*

$$H(T_{j-1},T_k,V) = \mathcal{G}_{j-1,k}\exp\{-\mathcal{H}_{j-1,k}\,r(T_{j-1})\} \tag{28}$$

with $\mathcal{G}_{j-1,k}$ *and* $\mathcal{H}_{j-1,k}$ *recursively defined as*

$$\mathcal{G}_{j-1,k} := \mathcal{G}_{j,k}\cdot\left(\frac{b_j}{b_j+2\mathcal{H}_{j,k}}\right)^{\frac{1}{2}\nu_j}$$
$$\mathcal{H}_{j-1,k} := \eta_j\frac{\mathcal{H}_{j,k}}{b_j+2\mathcal{H}_{j,k}}$$

where $\mathcal{G}_{k,k}:=f(T_k)$ *and* $\mathcal{H}_{k,k}:=g(T_k)$, b_j *and* η_j *defined as in (26) above.*

4.2. **Futures–Style American Options and Futures.** The options on futures traded at the LIFFE may be exercised prematurely (American style options). The option price is not paid at the time of purchase, but at exercise or expiry, whichever comes first. Option positions are marked–to–market in the same manner as futures positions[8]. For both puts and calls, we can ignore the American feature:

4.2.1. Lemma. *It is never rational to exercise futures–style American options on futures prematurely.*

Proof: Consider a futures–style European put with strike K on a futures contract $H(T,\overline{T},V)$. Its value at time t is the (conditional) expectation of the payoff at option expiry under the risk neutral measure,

$$E\left[[K-H(T,\overline{T},V)]^+\,\middle|\,\mathcal{F}_t\right]$$
$$\ge E\left[K-H(T,\overline{T},V)\,|\mathcal{F}_t\right] = K-H(t,\overline{T},V)$$

with the last equality due to the martingale property of the futures price process. Early exercise results in the assignment of a short position in the futures contract at the strike price. This position is immediately marked–to–market, resulting in the payoff $K-H(t,\overline{T},V)$. Since this is always less than the value of the European option, early exercise is never rational and the early exercise premium is zero. Analogously, this is also true for a call option.

[8]For a detailed contract description, see for example Inglis-Taylor (1995).

□

Thus we only need a pricing formula for futures–style European options. By conducting the calculations under the risk neutral instead of the forward measure and dropping the discounting by the zero coupon bond price $B(r(T_{n-1}), T_{n-1}, T_n)$ on each segment, we can restate proposition 3.1.1 for futures–style options.

4.2.2. Proposition. *In the segmented square root model with time segments* $[T_{n-1}, T_n]$, $n \in \{1; \ldots ; N\}$, *consider a futures–style exchange option on assets whose values at option expiry* t_m *are simple exponentials in* $r(t_m)$:

$$V_j(t_m) = f_j(t_m)\exp\{-g_j(t_m)r(t_m)\}$$

with payoff

$$C(t_m, r(t_m)) := [V_1(t_m, r(t_m)) - V_2(t_m, r(t_m))]^+$$

and let

$$k := \max\{n \in \{0; \ldots ; N\} \,|T_n < t_m\}.$$

For $n \leq k$, *the time* T_n *price of the option is given by*

$$C(T_n, r(T_n)) = H(T_n, T_m, V_1)P\left(b^{(1)}_{n,k+1}r(t_m) \leq z^*_m\right) - H(T_n, T_m, V_2)P\left(b^{(2)}_{n,k+1}r(t_m) \leq z^*_m\right), \tag{29}$$

if $V_1(t_m, r(t_m)) > V_2(t_m, r(t_m))$ *for* $r(t_m) \in]0; r^*[$, *and*

$$C(T_n, r(T_n)) = H(T_n, T_m, V_1)\left(1 - P\left(b^{(1)}_{n,k+1}r(t_m) \leq z^*_m\right)\right) - H(T_n, T_m, V_2)\left(1 - P\left(b^{(2)}_{n,k+1}r(t_m) \leq z^*_m\right)\right), \tag{30}$$

if $V_1(t_m, r(t_m)) > V_2(t_m, r(t_m))$ *for* $r(t_m) \in]r^*; \infty[$. *The* $(k-n+1)$*–times multiply compound noncentral* χ^2 *distributions have degrees of freedom* $\nu_{n+s} := 4\theta_{n+s}\sigma^{-2}_{n+s}$, $s \in \{1; \ldots ; k-n+1\}$, *noncentrality parameters*

$$\lambda^{(j)}_s := \frac{b_{n+s}\eta_{n+s}}{b_{n+s} + 2\mathcal{H}^{(j)}_{n+s,k+1}} r(T_{n+s-1})$$

and transformation coefficients

$$b^{(j)}_{n,n+s} := b_{n+s} + 2\mathcal{H}^{(j)}_{n+s,k+1}$$

with b_{n+s} *and* η_{n+s} *defined as in (26) and* $\mathcal{H}^{(j)}_{n+s,k+1}$ *given in proposition 4.1.1,*

$$z^*_m = b^{(j)}_{n,k+1}(g_1 - g_2)^{-1}\ln\frac{f_1}{f_2}.$$

4.3. **Fitting the Model to LIFFE Data.** On the LIFFE, options on futures on three–month Euromark deposits are quoted with times to expiry of up to twelve months; the longest futures contract matures in up to four years. For the first year, we want to simultaneously match futures prices and at–the–money call options. The Frankfurt DEM overnight rate will be our proxy for the short rate.

The traditional delivery months for the futures are March, June, September and December. In addition, there are two so-called "serial" futures contracts at the short end, so that there are always contracts with deliveries in each of the next three months available. The

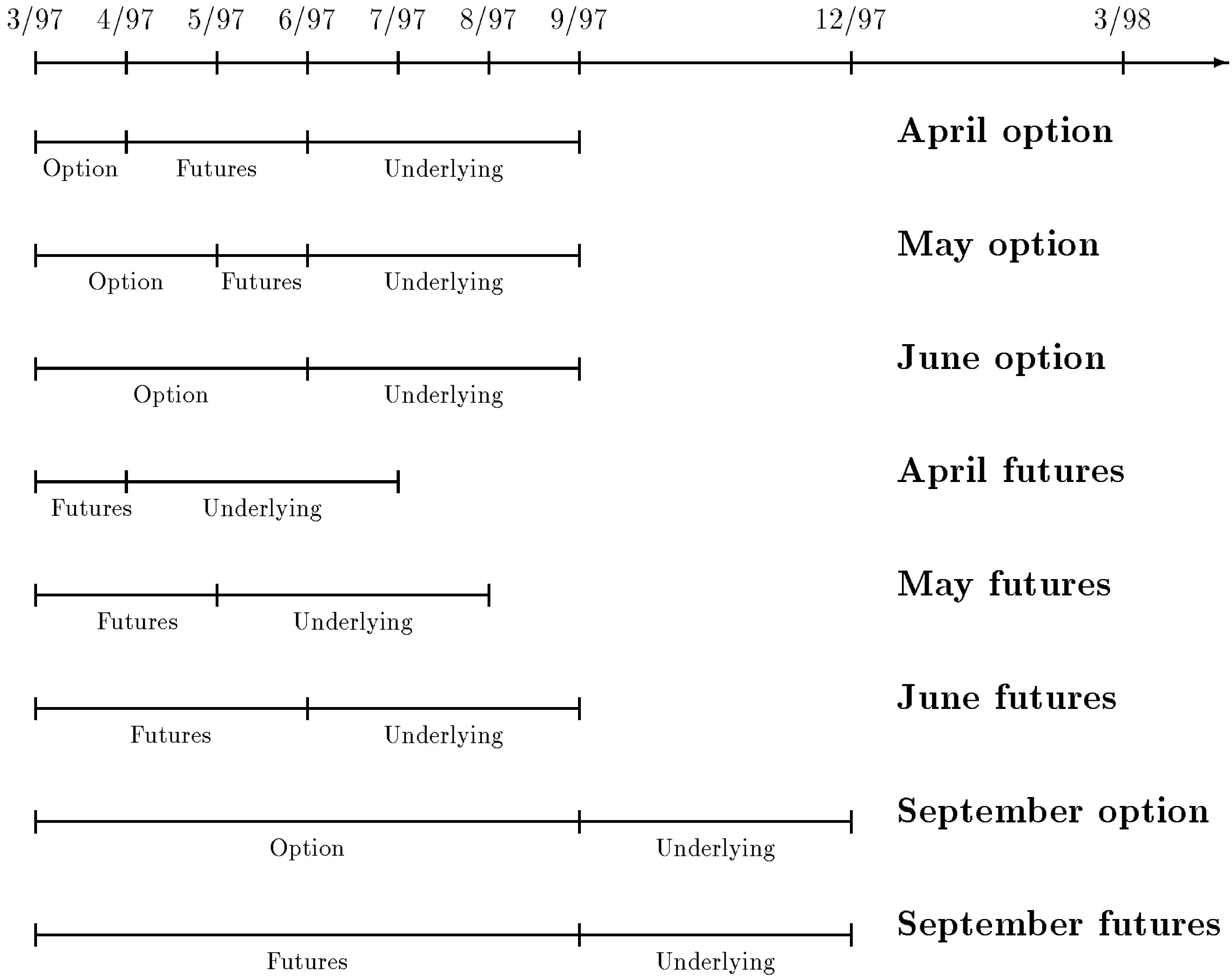

FIGURE 2. LIFFE 3–month Euromark instruments for the first 9 months

options also expire in March, June, September and December, and additionally there are two "serial" expiries so that there are options maturing in each of the next three months. The "serial" options' underlyings are not the serial futures, but the futures contract for the nearest subsequent quarterly delivery. Last trading day for futures and options alike is two business days before the third Wednesday of the delivery/expiry month; delivery is one business day after that. So on March 25, 1997, we had the following contracts trading: Futures delivering in April, May, June and from then on quarterly through March 2001; options expiring in April, May, June, September, December 1997 and March 1998. The April and May options were written on the June futures contract and the rest on contracts with deliveries matching option expiry.

In order to be able to fit the segmented model to futures and at–the–money call prices, we divide the time line into monthly segments for the first half year and into quarterly segments from then on (see figure 2). For the first three months, our choice of monthly segments is stems from the fact that we want to fit option prices by the calibrating the short volatility parameter σ up to option expiry accordingly. The first three futures contracts are written on three month Euromark deposits, maturing four, five and six months hence, respectively. Thus we take three more monthly segments to fit futures prices by adjusting the drift parameter θ.

The futures valuation formula (28) can be solved explicitly for θ, i.e. we have θ_k as a function of the observed futures price $H(T_0, T_k, V)$, the mean reversion parameters a_j and the volatility parameters σ_j for all $j \geq k$, as well as all previous θ_j, $j < k$. Substituting this into the option pricing formula (29), we get the σ calibrating the model to an option price as the solution of a one-dimensional fixed point problem, which can be found by a grid search. Note, however, that when fitting the model to the price of the April option, all parameters for the segments up to the maturity of the underlying in September enter the equation, in particular also the volatility parameter for the May segment, which is to be used to fit the price of the May option. Due to this overlap, it seems theoretically necessary to consider all options simultaneously in a multidimensional fixed point problem, which is not practicable. Instead, for each option we assume constant σ for the segments up to the maturity of the underlying. As we successively fit to options of increasing time to expiry, these σ change and the previous calibrations are no longer exactly valid. However, since the option price reacts much less to changes in σ after option expiry than before — this being one of the reasons we chose the σ *before* option expiry as the parameter with which to fit observed option prices — deviations due to these recalibrations as a rule are much smaller than tick size. In cases where they are not, one can run this calibration process iteratively, taking the parameters for the longer maturities as given from the previous iteration when recalibrating the shorter maturities. Again, due the relative sensitivities of the option prices to changes in parameter values before and after option expiry, this iterative procedure quickly converges.

5. Multifactor Extensions

Our result can also be extended to a multifactor square root model with independent factors. As in Chen and Scott (1995), let the short rate be given by the sum of independent state variables

$$r(t) = \sum_{j=1}^{J} z_j(t)$$

where the state variable dynamics are of the CIR type

$$dz_j(t) = \left(\theta^{(j)} - a^{(j)} z_j(t)\right) dt + \sigma^{(j)} \sqrt{z_j(t)} dB_t^{(j)}. \tag{31}$$

In our segmented version of the multifactor model, we replace (31) with the dynamics given in equation (2). For each segment $[T_{n-1}; T_n]$ we now have parameter vectors $\theta_n^{(\cdot)}, a_n^{(\cdot)}, \sigma_n^{(\cdot)}$, and the vector valued functions $\mathcal{A}_n^{(\cdot)}(T_{n-1}; T_n), \mathcal{B}_n^{(\cdot)}(T_{n-1}; T_n), \eta_n^{(\cdot)}, b_n^{(\cdot)}$ and $\nu_n^{(\cdot)}$ are defined element–wise as in the one–factor case. On a constant parameter segment, zero coupon bond prices are[9]

$$B(z_\bullet(T_{n-1}), T_{n-1}, T_n) = \left(\prod_{j=1}^{J} \mathcal{A}_n^{(j)}(T_{n-1}, T_n) \right) \exp\left\{ - \sum_{j=1}^{J} \mathcal{B}_n^{(j)}(T_{n-1}, T_n) z_j(T_{n-1}) \right\}.$$

The transformed factors $b_n^{(j)} z_j(T_n)$ are noncentral χ^2 distributed (conditioned on $z_j(T_{n-1})$) under the T_n forward measure, with $\nu_n^{(j)}$ degrees of freedom and noncentrality parameter $\eta_n^{(j)} z_j(T_{n-1})$. Since the factors are independently distributed, we can apply lemma B.1.1

[9] See Chen and Scott (1995).

for each factor separately and carry out the same induction as in the proof of proposition 2.2.1 to yield

5.0.1. Proposition. *In the multifactor segmented square root model with time segments* $[T_{n-1};T_n]$, $n\in\{1;\dots;N\}$, *the prices of zero coupon bonds at time* T_{n-1} *with maturity* T_k, $n\le k\le N$ *are given by*

$$B(z_\bullet(T_{n-1}),T_{n-1},T_k)=\left(\prod_{j=1}^{J}\mathcal{C}^{(j)}_{n-1,k}\right)\exp\left\{-\sum_{j=1}^{J}\mathcal{D}^{(j)}_{n-1,k}z_j(T_{n-1})\right\}$$

with $\mathcal{C}^{(j)}_{n-1,k}$ *and* $\mathcal{D}^{(j)}_{n-1,k}$ *recursively defined as*

$$\begin{aligned}\mathcal{C}^{(j)}_{n-1,k}:=&\mathcal{C}^{(j)}_{n,k}\mathcal{A}^{(j)}_n(T_{n-1},T_n)\left(\frac{b^{(j)}_n}{b^{(j)}_n+2\mathcal{D}^{(j)}_{n,k}}\right)^{\frac{1}{2}\nu^{(j)}_n}\\ \mathcal{D}^{(j)}_{n-1,k}:=&\mathcal{B}^{(j)}_n(T_{n-1},T_n)+\eta^{(j)}_n\frac{\mathcal{D}^{(j)}_{n,k}}{b^{(j)}_n+2\mathcal{D}^{(j)}_{n,k}}\end{aligned}$$

where

$$\mathcal{C}^{(j)}_{k,k}:=1\qquad\textit{and}\qquad\mathcal{D}^{(j)}_{k,k}:=0.$$

Note that when fitting the model to an initial term structure in the multifactor case, on has more parameters which can be adjusted. Thus in a two–factor model one could choose to fit two initial zero coupon bond prices on each segment, reducing the number of segments.

5.1. **Option Pricing.** As in section 3, consider a European exchange option on two securities whose terminal values are exponential affine functions of the factors. Since the factors are independent, we can write the time T_k price of the option analogously to equation (14) as[10]

(32)

$$\begin{aligned}&C(T_k,z_\bullet(T_k))=B(z_\bullet(T_k),T_k,t_m)\\&\quad\cdot\left(\int_Z f_1(t_m)\prod_{j=1}^{J}\exp\left\{-g^{(j)}_1(t_m)z_j(t_m)\right\}q_{\chi^2}\left(\tilde{b}^{(j)}z_j(t_m),\nu^{(j)}_{k+1},\tilde{\eta}^{(j)}z_j(T_k)\right)d\left(\tilde{b}^{(j)}z_j(t_m)\right)\right.\\&\quad\left.-\int_Z f_2(t_m)\prod_{j=1}^{J}\exp\left\{-g^{(j)}_2(t_m)z_j(t_m)\right\}q_{\chi^2}\left(\tilde{b}^{(j)}z_j(t_m),\nu^{(j)}_{k+1},\tilde{\eta}^{(j)}z_j(T_k)\right)d\left(\tilde{b}^{(j)}z_j(t_m)\right)\right)\end{aligned}$$

with

$$\begin{aligned}Z:=&\left\{z_\bullet(t_m)\in\mathbb{R}^J_{++}\;\middle|\;f_1(t_m)\exp\left\{-\sum_{j=1}^{J}g^{(j)}_1(t_m)z_j(t_m)\right\}>f_2(t_m)\exp\left\{-\sum_{j=1}^{J}g^{(j)}_2(t_m)z_j(t_m)\right\}\right\}\\=&\left\{z_\bullet(t_m)\in\mathbb{R}^J_{++}\;\middle|\;\sum_{j=1}^{J}\left(g^{(j)}_2(t_m)-g^{(j)}_1(t_m)\right)z_j(t_m)>\ln\frac{f_2(t_m)}{f_1(t_m)}\right\}.\end{aligned}$$

[10] Unless otherwise stated, the notation in this section is defined as in section 3.

Again we can apply lemma B.1.1 and write (32) as

$$
\begin{aligned}
&C(T_k, z_\bullet(T_k)) = B(z_\bullet(T_k), T_k, t_m) \\
&\cdot \left(f_1(t_m) \left(\prod_{j=1}^{J} \exp\left\{ -\frac{g_1^{(j)}(t_m)}{\tilde{b}^{(j)} + 2g_1^{(j)}(t_m)} \tilde{\eta}^{(j)} z_j(T_k) \right\} \left(\frac{\tilde{b}^{(j)}}{\tilde{b}^{(j)} + 2g_1^{(j)}(t_m)} \right)^{\frac{1}{2}\nu_{k+1}^{(j)}} \right) \right. \\
&\cdot \int_Z \left(\prod_{j=1}^{J} q_{\chi^2} \left(\left(\tilde{b}^{(j)} + g_1^{(j)}(t_m) \right) z_j(t_m), \nu_{k+1}^{(j)}, \frac{\tilde{b}^{(j)}}{\tilde{b}^{(j)} + 2g_1^{(j)}(t_m)} \tilde{\eta}^{(j)} z_j(T_k) \right) d\left(\left(\tilde{b}^{(j)} + 2g_1^{(j)}(t_m) \right) z_j(t_m) \right) \right) \\
&- f_2(t_m) \left(\prod_{j=1}^{J} \exp\left\{ -\frac{g_2^{(j)}(t_m)}{\tilde{b}^{(j)} + 2g_2^{(j)}(t_m)} \tilde{\eta}^{(j)} z_j(T_k) \right\} \left(\frac{\tilde{b}^{(j)}}{\tilde{b}^{(j)} + 2g_2^{(j)}(t_m)} \right)^{\frac{1}{2}\nu_{k+1}^{(j)}} \right) \\
&\left. \cdot \int_Z \left(\prod_{j=1}^{J} q_{\chi^2} \left(\left(\tilde{b}^{(j)} + g_2^{(j)}(t_m) \right) z_j(t_m), \nu_{k+1}^{(j)}, \frac{\tilde{b}^{(j)}}{\tilde{b}^{(j)} + 2g_2^{(j)}(t_m)} \tilde{\eta}^{(j)} z_j(T_k) \right) d\left(\left(\tilde{b}^{(j)} + 2g_2^{(j)}(t_m) \right) z_j(t_m) \right) \right) \right).
\end{aligned}
$$

This is the exchange option formula for a multifactor CIR model with independent factors and constant parameters. In order to calculate the option price for some arbitrary time T_n in the segmented model and derive the multifactor version of proposition 3.1.1, we carry out the same induction steps as in the proof of 3.1.1. Given the option price $C(T_n, z_\bullet(T_n))$ at the segment boundary T_n as a function of the factor realizations $z_\bullet(T_n)$, the time T_{n-1} price of the option can be calculated as the discounted expectation under the time T_n forward measure Q^{T_n}:

$$
C(T_{n-1}, z_\bullet(T_{n-1})) = B(z_\bullet(T_{n-1}), T_{n-1}, T_n) \int_{\mathbb{R}_{++}^J} C(T_n, z_\bullet(T_n)) Q^{T_n}(dz_\bullet(T_n)).
$$

The joint density of the factors $z_\bullet(T_n)$ is given by the product of the factor densities because of independence. Noting the multiplicative structure of (32), which is retained in each induction step, we see that we can carry out the induction for each factor separately, yielding

5.1.1. Proposition. *In the multifactor segmented square root model with time segments $[T_{n-1}; T_n]$, $n \in \{1; \ldots; N\}$ and J factors z_j, consider an exchange option on two assets whose values at option expiry t_m are exponential affine functions of the factors:*

$$
V_{1,2}(t_m, z_\bullet(t_m)) = f_{1,2}(t_m) \exp\left\{ -\sum_{j=1}^{J} g_{1,2}^{(j)}(t_m) z_j(t_m) \right\}.
$$

Define k as

$$
k := \max\{n \in \{0; \ldots; N\} | T_n < t_m\}.
$$

For $n \leq k$, the time T_n price of the option is given by

$$
\begin{aligned}
C(T_n, z_\bullet(T_n)) = & \left(\prod_{j=1}^{J} \mathcal{C}_{n,k}^{(1,j)} \exp\left\{ -\mathcal{D}_{n,k}^{(1,j)} z_j(T_n) \right\} \right) \\
& \cdot P_1 \left(\sum_{j=1}^{J} \left(g_1^{(j)}(t_m) - g_2^{(j)}(t_m) \right) z_j(t_m) < \ln \frac{f_1(t_m)}{f_2(t_m)} \right) \\
& - \left(\prod_{j=1}^{J} \mathcal{C}_{n,k}^{(2,j)} \exp\left\{ -\mathcal{D}_{n,k}^{(2,j)} z_j(T_n) \right\} \right) \\
& \cdot P_2 \left(\sum_{j=1}^{J} \left(g_1^{(j)}(t_m) - g_2^{(j)}(t_m) \right) z_j(t_m) < \ln \frac{f_1(t_m)}{f_2(t_m)} \right)
\end{aligned}
$$

with $\mathcal{C}_{n,k}^{(h,j)}$ and $\mathcal{D}_{n,k}^{(h,j)}$ recursively defined as in proposition (5.0.1), however with

$$
\begin{aligned}
\mathcal{C}_{k,k}^{(h,j)} :=& \mathcal{A}_{k+1}^{(j)}(T_k, t_m) f_h(t_m) \left(\frac{\tilde{b}^{(j)}}{\tilde{b}^{(j)} + 2g_h^{(j)}(t_m)} \right)^{\frac{1}{2}\nu_{k+1}^{(j)}} \\
\mathcal{D}_{k,k}^{(h,j)} :=& \mathcal{B}_{k+1}^{(j)}(T_k, t_m) + \frac{g_h^{(j)}(t_m)\tilde{\eta}^{(j)}}{\tilde{b}^{(j)} + 2g_h^{(j)}(t_m)}.
\end{aligned}
$$

$P_h \left(\sum_{j=1}^{J} \left(g_1^{(j)}(t_m) - g_2^{(j)}(t_m) \right) z_j(t_m) < \ln \frac{f_1(t_m)}{f_2(t_m)} \right)$ *is the distribution function of a weighted sum of independent $(k-n+1)$–times multiply compound noncentral χ^2 distributed factors with degrees of freedom $\nu_{n+s}^{(j)}$, $s \in \{1; \dots ; k-n+1\}$, noncentrality parameters*

$$
\lambda_s^{(h,j)} := \begin{cases} \frac{\tilde{b}^{(j)}}{\tilde{b}^{(j)} + 2g_h^{(j)}(t_m)} \tilde{\eta}^{(j)} z_j(T_k) & s = k-n+1 \\ \frac{\eta_{n+s}^{(j)} b_{n+s}^{(j)}}{b_{n+s}^{(j)} + 2\mathcal{D}_{n+s,k}^{(h,j)}} z_j(T_{n+s-1}) & s \in \{1; \dots ; k-n\} \end{cases}
$$

and transformation coefficients

$$
b_{n,n+s}^{(h,j)} := \begin{cases} \tilde{b}^{(j)} + 2g_h^{(j)}(t_m) & s = k-n+1 \\ b_{n+s}^{(j)} + 2\mathcal{D}_{n+s,k}^{(h,j)} & s \in \{1; \dots ; k-n\} \end{cases}
$$

The distribution function can be evaluated using the technique described by Chen and Scott (1995); the characteristic function of the multiply compound noncentral χ^2 distribution is given by proposition B.3.5 in the appendix. Note that by representing the value of the distribution function as an integral of a product of the characteristic functions of independent factors, this technique reduces the dimension of the numerical integration to one for any number of factors and any number of segments. Thus for a large number of segments before option expiry it may be efficient to employ this technique even in the one–factor case.

6. Conclusion

In the present paper we have constructed the one– and the multifactor versions of a term structure model with non-negative interest rates which fits an initial yield curve while retaining analytical tractability for fast solutions for derivative pricing and hedging. The factor stochastic differential equations are Cox/Ingersoll/Ross (CIR) type "square root" diffusions with piecewise constant parameters, where the constant parameter segments are determined by the initial term structure data, i.e. by the maturities for which zero coupon bond prices are given. Prices of European options on linear combinations of securities whose value at option expiry is an exponential affine function of the model factors can be expressed in terms of a "multiply compound" noncentral chi-square distribution function, i.e. a noncentral chi-square distribution whose noncentrality parameter is again (multiply compound) chi-square distributed.

In the one-factor case the number of segments n up to option expiry determines the numerical complexity of the problem of calculating this distribution function; the number of operations necessary grows only linearly in n. In the multifactor case and for a large number of segments in the one-factor case, the (explicitly derived) characteristic function can be used to calculate the value of the multiply compound noncentral chi-square distribution function by a one-dimensional numerical integration.

Thus we have arguably closed form solutions for a large class of fixed income derivatives, including caps, floors, yield spreads, options on interest rate futures and, in the one-factor case, swaptions.

Our approach to fitting an initial term structure does not require that we exogenously specify zero coupon bond prices for the continuum of maturities. Instead, the model interpolates endogenously in a manner consistent with the short rate dynamics. However, exogenous interpolation schemes such as splines can be approximated by a sufficiently large number of segments should one choose to do so.

Appendix A. On the Non-Explosion of Solutions of an SDE

Let $\mathcal{U} \subset \mathbb{R}^d$ be open. We denote by $\widehat{\mathcal{U}} := \mathcal{U} \cup \{\Delta\}$ the Alexandrov–Compactification of $\mathcal{U}$. Then $\widehat{\mathcal{U}}$ is compact and its topology has a countable basis, therefore $\widehat{\mathcal{U}}$ is Polish. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis and $X = (X_t)_{t\in\mathbb{R}_+}$ be a continuous $\mathbb{F}$–adapted process with values in $\widehat{\mathcal{U}}$. We define the explosion time e_X of X as follows:

$$e_X := \inf\{t \in \mathbb{R}_+ \mid X_t = \Delta\}.$$

Then e_X is an $\mathbb{F}$–stopping time. If X_0 only assumes values in $\mathcal{U}$, then we have $e_X > 0$ and furthermore, e_X is predictable.

We denote by $M_{d,n}(\mathbb{R})$ the set of all $d \times n$ matrices with real entries. Let $\alpha : \mathbb{R}_+ \times \mathcal{U} \to M_{d,n}(\mathbb{R})$ and $\beta : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}^d$ be two continuous functions; these will be the coefficients of the SDE we wish to consider. All stochastic bases $(\Omega, \mathcal{F}, \mathbb{F}, P)$ will fulfill the usual hypotheses.

A.0.2. Definition. (Solution): Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis on which an n–dimensional, standard Brownian motion $B = (B_t)_{t\in\mathbb{R}_+}$ is defined. A continuous, $\mathbb{F}$–adapted process X taking values in $\widehat{\mathcal{U}}$ is a solution of

$$dZ_t = \alpha(t, Z_t)dB_t + \beta(t, Z_t)dt \tag{33}$$

iff the following conditions are met:

1. X_0 only assumes values in $\mathcal{U}$.
2. For P–almost all $\omega \in \{e_X < +\infty\}$ we have

$$\forall t \geq e_X(\omega) : X_t(\omega) = \Delta.$$

(This condition is obviously fulfilled iff the two processes X and X^{e_X} are indistinguishable.)

3. If τ is an $\mathbb{F}$–stopping time with $[\![0,\tau]\!] \subset [\![0,e_X[\![$, then for every $i \in \{1,\dots,d\}$ and every $t \in \mathbb{R}_+$ we have the following:

$$X_t^{(i)^\tau} = X_0^{(i)} + \sum_{j=1}^{n} \int_0^t \alpha_{ij}\left(s, X_s^\tau\right) dB_s^{(j)^\tau} + \int_0^{t\wedge\tau} \beta_i\left(s, X_s^\tau\right) ds \quad P\text{–a.s.}$$

The solution X does not explode iff $P[e_X = +\infty] = 1$.

A.0.3. Lemma. *Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis carrying a normal n–dimensional Brownian motion W and let X be a continuous, $\mathbb{F}$–adapted process in $\widehat{\mathcal{U}}$ so that X_0 only assumes values in $\mathcal{U}$ and the two processes X and X^{ex} are indistinguishable. Let $(\tau_k)_{k\in\mathbb{N}}$ be an announcing sequence for e_X. Then the following two statements are equivalent:*

1. *The process X is a solution of*

$$dZ_t = \alpha(t, Z_t)dW_t + \beta(t, Z_t)dt.$$

2. *For each $i \in \{1,\dots,d\}$, $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we have*

$$X_t^{(i)^{\tau_k}} = X_0^{(i)} + \sum_{j=1}^{n} \int_0^t \alpha_{ij}\left(s, X_s^{\tau_k}\right) dW_s^{(j)^{\tau_k}} + \int_0^{t\wedge\tau_k} \beta_i\left(s, X_s^{\tau_k}\right) ds \quad P\text{–}a.s.$$

Proof: Obviously, we only need to show that 2. implies 1. Let τ be any stopping time with $[\![0,\tau]\!] \subset [\![0,e_X[\![$. Fixing $i \in \{1,\dots,d\}$ and $t \in \mathbb{R}_+$ we must show:

$$X_t^{(i)^\tau} = X_0^{(i)} + \sum_{j=1}^{n} \int_0^t \alpha_{ij}\left(s, X_s^\tau\right) dW_s^{(j)^\tau} + \int_0^{t\wedge\tau} \beta_i\left(s, X_s^\tau\right) ds \quad P\text{–a.s.}$$

By the stopping rules for stochastic integrals and the assumption, we have for every $k \in \mathbb{N}$,

$$\begin{aligned} X_t^{(i)^{\tau_k\wedge\tau}} &= X_0^{(i)} + \sum_{j=1}^{n} \int_0^{t\wedge\tau} \alpha_{ij}\left(s, X_s^{\tau_k}\right) dW_s^{(j)^{\tau_k}} + \int_0^{t\wedge\tau_k\wedge\tau} \beta_i\left(s, X_s^{\tau_k}\right) ds \\ &= X_0^{(i)} + \sum_{j=1}^{n} \int_0^{t\wedge\tau\wedge\tau_k} \alpha_{ij}\left(s, X_s^{\tau}\right) dW_s^{(j)^{\tau}} + \int_0^{t\wedge\tau\wedge\tau_k} \beta_i\left(s, X_s^{\tau}\right) ds \quad P\text{–a.s.} \end{aligned}$$

We can therefore find a null set N in $(\Omega, \mathcal{F}, P)$, so that for $\omega \in N^c$ the following holds:

$$\begin{aligned} \forall k \in \mathbb{N}: \quad X_t^{(i)^{\tau_k\wedge\tau}}(\omega) = X_0^{(i)}(\omega) \quad &+ \left(\sum_{j=1}^{n} \int_0^{t\wedge\tau\wedge\tau_k} \alpha_{ij}\left(s, X_s^\tau\right) dW_s^{(j)^\tau}\right)(\omega) \\ &+ \int_0^{t\wedge\tau(\omega)\wedge\tau_k(\omega)} \beta_i\left(s, X_s^\tau(\omega)\right) ds. \end{aligned}$$

Now fix an arbitrary $\omega \in N^c$. Since $t \wedge \tau(\omega) < e_X(\omega)$, there is a $k_0 \in \mathbb{N}$ fulfilling $t \wedge \tau(\omega) \le \tau_{k_0}(\omega)$. We have

$$\begin{aligned} {X^{(i)}}^\tau_t(\omega) &= X_t^{(i)^{\tau_{k_0}\wedge\tau}}(\omega) \\ &= X_0^{(i)}(\omega) + \left(\sum_{j=1}^{n} \int_0^{t\wedge\tau\wedge\tau_{k_0}} \alpha_{ij}\left(s, X_s^\tau\right) dW_s^{(j)^\tau}\right)(\omega) + \int_0^{t\wedge\tau(\omega)\wedge\tau_{k_0}(\omega)} \beta_i\left(s, X_s^\tau(\omega)\right) ds \\ &= X_0^{(i)}(\omega) + \left(\sum_{j=1}^{n} \int_0^{t} \alpha_{ij}\left(s, X_s^\tau\right) dW_s^{(j)^\tau}\right)(\omega) + \int_0^{t\wedge\tau(\omega)} \beta_i\left(s, X_s^\tau(\omega)\right) ds. \qquad \square \end{aligned}$$

The result we use in the main body of the paper is the following

A.0.4. Theorem. *Suppose that any solution of the SDE determined by α and β satisfying a deterministic initial condition does not explode. Then non–explosion also holds for solutions with random initial conditions.*

For the convenience of the reader, we will provide a complete proof of this intuitive result. Our approach follows that of Yeh (1995), § 18. In particular, we make use of regular conditional probabilities. To be assured of their existence, we must first transport our solution onto a suitably nice probability space. This is done in the next section.

A.1. **Function Space Representation of Solutions.** We first introduce a suitable analogy of Wiener space. As already mentioned, $\widehat{\mathcal{U}}$ is a Polish space, therefore the space $C(\mathbb{R}_+,\widehat{\mathcal{U}})$ of all continuous functions from $\mathbb{R}_+$ to $\widehat{\mathcal{U}}$ endowed with the topology of uniform convergence on compacts is Polish (cf. Bauer (1990), Theorem 31.6). We set

$$\tilde{C}(\mathbb{R}_+,\widehat{\mathcal{U}}) := \left\{ w \in C(\mathbb{R}_+,\widehat{\mathcal{U}}) \middle| w(0) \in \mathcal{U} \right\}.$$

Now $\tilde{C}(\mathbb{R}_+,\widehat{\mathcal{U}})$ is an open subset of $C(\mathbb{R}_+,\widehat{\mathcal{U}})$ and therefore also Polish. Finally, we define

$$\widehat{\mathbf{W}} = \tilde{C}(\mathbb{R}_+,\widehat{\mathcal{U}}) \times C(\mathbb{R}_+,\mathbb{R}^n)$$

and endow $\widehat{\mathbf{W}}$ with the product topology, making $\widehat{\mathbf{W}}$ into a Polish space also.
For every $t \in \mathbb{R}_+$ we have the canonical projection mappings

$$\begin{aligned} p_t : \quad & \widehat{\mathbf{W}} \to \widehat{\mathcal{U}}, \quad (\mathtt{w},\mathtt{w}') \mapsto \mathtt{w}(t) \\ q_t : \quad & \widehat{\mathbf{W}} \to \mathbb{R}^n, \quad (\mathtt{w},\mathtt{w}') \mapsto \mathtt{w}'(t). \end{aligned}$$

If we denote the Borel–σ–Algebra of $\widehat{\mathbf{W}}$ by $\mathfrak{W}$, we have

$$\mathfrak{W} = \sigma\left(p_s, q_s; s \in \mathbb{R}_+\right).$$

The canonical filtration $\mathbb{W} = \{\mathfrak{W}_t\}_{t\in\mathbb{R}_+}$ on $(\widehat{\mathbf{W}},\mathfrak{W})$ is given by $\mathfrak{W}_t = \sigma(p_s,q_s; s \in [0,t])$ for every $t \in \mathbb{R}_+$. We also have two canonical stochastic processes $Y = \{Y_t\}_{t\in\mathbb{R}_+}$ and $W = \{W_t\}_{t\in\mathbb{R}_+}$ on $(\widehat{\mathbf{W}},\mathbb{W})$ given by

$$\begin{aligned} Y_t(w,w') &:= p_t(\mathtt{w},\mathtt{w}') = \mathtt{w}(t) \\ W_t(w,w') &:= q_t(\mathtt{w},\mathtt{w}') = \mathtt{w}'(t) \end{aligned}$$

for every $t \in \mathbb{R}_+$. The processes Y and W are obviously continuous and $\mathbb{W}$–adapted, the explosion time e_Y of Y is a $\mathbb{W}$–stopping time with $e_Y > 0$.
Now let $(\Omega,\mathcal{F},\mathbb{F},P)$ be a stochastic basis on which we have a normal n–dimensional Brownian motion B and let X be a solution of the SDE

$$dZ_t = \alpha(t,Z_t)dB_t + \beta(t,Z_t)dt. \tag{34}$$

This solution induces a canonical map $(X,B) : \Omega \to \widehat{\mathbf{W}}$ defined by

$$(X,B)(\omega) := \left(X_\bullet(\omega), B_\bullet(\omega)\right).$$

The mapping (X,B) is $\mathcal{F}$–$\mathfrak{W}$–measurable and also $\mathcal{F}_t$–$\mathfrak{W}_t$–measurable for every $t \in \mathbb{R}_+$. Let $P_{(B,X)}$ denote the image of P under (X,B). We denote by $(\widehat{\mathbf{W}},\overline{\mathfrak{W}},\mathbb{W}^* := \{\mathfrak{W}_t^*\}_{t\in\mathbb{R}_+}, P_{(X,B)})$ the usual augmentation of the stochastic basis $(\widehat{\mathbf{W}},\mathfrak{W},\mathbb{W},P_{(X,B)})$. Observe that, as $(\Omega,\mathcal{F},\mathbb{F},P)$ fulfills the usual hypotheses by assumption, the mapping (X,B) is in fact $\mathcal{F}$–$\overline{\mathfrak{W}}$–measurable and $\mathcal{F}_t$–$\mathfrak{W}^*_t$–measurable for every $t \in \mathbb{R}_+$. It is trivial but useful to note that for every $t \in \mathbb{R}_+$ we have

$$Y_t \circ (X, B) = X_t, \quad W_t \circ (X, B) = B_t.$$

In particular, $e_Y \circ (X, B) = e_X$, and therefore:

$$P_{(X,B)}\left[e_Y = +\infty\right] = P\left[(X, B)^{-1}(\{e_Y = +\infty\})\right] = P\left[e_X = +\infty\right].$$

A.1.1. Theorem. *The process W is a normal, n–dimensional $(P_{(X,B)}, \mathbb{W}^*)$–Brownian motion and the process Y is a solution of*

$$dZ_t = \alpha(t, Z_t)dW_t + \beta(t, Z_t)dt. \tag{35}$$

PROOF: 1. The paths of W are obviously continuous, W is $\mathbb{W}^*$–adapted. Let $s, t \in \mathbb{R}_+$ with $s < t$. Then we have

$$(W_t - W_s) \circ (X, B) = B_t - B_s.$$

Therefore the distribution of $W_t - W_s$ under $P_{(X,B)}$ is just the distribution of $B_t - B_s$ under P. For the same reason we have

$$P_{(X,B)}\left[W_0 = 0\right] = P\left[B_0 = 0\right] = 1.$$

To prove that W is a Brownian motion, it only remains to show that $W_t - W_s$ is independent of $\mathfrak{W}^*_s$. We denote the Borel σ–algebra of $\mathbb{R}^n$ by $\mathcal{B}^n$. Suppose that $C \in \mathcal{B}^n$ and $A \in \mathfrak{W}^*_s$. Since $(X, B)^{-1}(A) \in \mathcal{F}_s$, we have

$$\begin{aligned} & P_{(X,B)}\left[A \cap (W_t - W_s)^{-1}(C)\right] \\ = \; & P\left[(X, B)^{-1}(A) \cap (B_t - B_s)^{-1}(C)\right] \\ = \; & P\left[(X, B)^{-1}(A)\right] P\left[(B_t - B_s)^{-1}(C)\right] \\ = \; & P_{(X,B)}[A] \cdot P_{(X,B)}\left[(W_t - W_s)^{-1}(C)\right]. \end{aligned}$$

2. We must now show that the process Y is indeed a solution of (35). By definition of $\widehat{\mathbf{W}}$, Y_0 assumes only values in $\mathcal{U}$. The processes Y and Y^{e_Y} are continuous and $\mathbb{W}$–adapted, so that $\{Y = Y^{e_Y}\} \in \mathfrak{W}$. Furthermore, the following holds:

$$\forall t \in \mathbb{R}_+ : \quad Y_t^{e_Y} \circ (X, B) = X_t^{e_X}.$$

Therefore

$$P_{(X,B)}\left[Y = Y^{e_Y}\right] = P\left[(X, B)^{-1}(\{Y = Y^{e_Y}\})\right] = P\left[X = X^{e_X}\right] = 1.$$

Let τ be a $\mathbb{W}^*$–stopping time with $[\![0, \tau]\!] \subset [\![0, e_Y[\![$, fix $i \in \{1, \dots, d\}$ and $t \in \mathbb{R}_+$. We must show:

$$Y_t^{(i)^\tau} = Y_0^{(i)} + \sum_{j=1}^{n} \int_0^t \alpha_{ij}\left(s, Y_s^\tau\right) dW_s^{(j)^\tau} + \int_0^{t\wedge\tau} \beta_i\left(s, Y_s^\tau\right) ds \quad P_{(X,B)}\text{–a.s.}$$

We define $\tilde{\tau} : \Omega \to \overline{\mathbb{R}}_+$ by $\tilde{\tau} := \tau \circ (X, B)$. One immediately sees that $\tilde{\tau}$ is an $\mathbb{F}$–stopping time with $[\![0, \tilde{\tau}]\!] \subset [\![0, e_X[\![$. Since X is a solution of (34), we know that

$$X_t^{(i)^{\tilde{\tau}}} = X_0^{(i)} + \sum_{j=1}^{n} \int_0^t \alpha_{ij}\left(s, X_s^{\tilde{\tau}}\right) dB_s^{(j)^{\tilde{\tau}}} + \int_0^{t\wedge\tilde{\tau}} \beta_i\left(s, X_s^{\tilde{\tau}}\right) ds \quad P\text{–a.s.} \tag{36}$$

We define two real random variables Ψ and Φ as follows:

$$\begin{aligned}\Psi &:= \sum_{j=1}^{n}\int_0^t \alpha_{ij}\left(s,Y_s^\tau\right)dW_s^{(j)^\tau} + \int_0^{t\wedge\tau}\beta_i\left(s,Y_s^\tau\right)ds + Y_0^{(i)} - Y_t^{(i)^\tau}\\ \Phi &:= \sum_{j=1}^{n}\int_0^t \alpha_{ij}\left(s,X_s^{\tilde\tau}\right)dB_s^{(j)^{\tilde\tau}} + \int_0^{t\wedge\tilde\tau}\beta_i\left(s,X_s^{\tilde\tau}\right)ds + X_0^{(i)} - X_t^{(i)^{\tilde\tau}}\end{aligned}$$

Now (36) is equivalent to the fact that the distribution of Φ is δ_0, the Dirac measure at the origin. It is clearly sufficient to prove that Φ and Ψ are identically distributed.
To this end, we choose a sequence $(\mathcal{Z}_m)_{m\in\mathbb{N}}$ of partitions $\mathcal{Z}_m : 0 = t_0^m < \ldots < t_{k_m}^m = t$ of $[0,t]$, so that $|\mathcal{Z}_m| \to 0$. Fix $\mathtt{w} \in \widehat{\mathbf{W}}$. For every $m \in \mathbb{N}$ we define $\beta_i^{(m)} : [0,t] \to \mathbb{R}$ by

$$\beta_i^{(m)} := \beta_i\left(0,Y_0(\mathtt{w})\right)\chi_{\{0\}} + \sum_{\nu=1}^{k_m}\beta_i\left(t_{\nu-1}^m, Y_{t_{\nu-1}^m}^\tau(\mathtt{w})\right)\chi_{]t_{\nu-1}^m,t_\nu^m]}.$$

The continuous function mapping $[0,t]$ to $\mathbb{R}$ by $s \mapsto \beta_i(s,Y_s^\tau(\mathtt{w}))$ is the pointwise limit of the sequence $(\beta_i^{(m)})_{m\in\mathbb{N}}$. The sequence $(\beta_i^{(m)})_{m\in\mathbb{N}}$ is uniformly bounded by $\sup_{s\in[0,t]}|\beta_i(s,Y_s^\tau(\mathtt{w}))| < +\infty$ on $[0,t]$. Therefore, by the dominated convergence theorem we have

$$\begin{aligned}&\int_0^{t\wedge\tau(\mathtt{w})}\beta_i\left(s,Y_s^\tau(\mathtt{w})\right)ds\\ = &\lim_{m\to\infty}\int_0^{t\wedge\tau(\mathtt{w})}\beta_i^{(m)}(s)ds\\ = &\lim_{m\to\infty}\sum_{\nu=1}^{k_m}\beta_i\left(t_{\nu-1}^m, Y_{t_{\nu-1}^m}^\tau(\mathtt{w})\right)\left(t_\nu^m\wedge\tau(\mathtt{w}) - t_{\nu-1}^m\wedge\tau(\mathtt{w})\right).\end{aligned}$$

As $\mathtt{w} \in \widehat{\mathbf{W}}$ was fixed arbitrarily, the following equation holds in the sense of pointwise convergence on $\widehat{\mathbf{W}}$:

$$\int_0^{t\wedge\tau}\beta_i\left(s,Y_s^\tau\right)ds = \lim_{m\to\infty}\sum_{\nu=1}^{k_m}\beta_i\left(t_{\nu-1}^m, Y_{t_{\nu-1}^m}^\tau\right)\left(t_\nu^m\wedge\tau - t_{\nu-1}^m\wedge\tau\right).$$

By exactly the same reasoning, we obtain

$$\int_0^{t\wedge\tilde\tau}\beta_i\left(s,X_s^{\tilde\tau}\right)ds = \lim_{m\to\infty}\sum_{\nu=1}^{k_m}\beta_i\left(t_{\nu-1}^m, X_{t_{\nu-1}^m}^{\tilde\tau}\right)\left(t_\nu^m\wedge\tilde\tau - t_{\nu-1}^m\wedge\tilde\tau\right).$$

In this case convergence is pointwise on Ω.
We now also fix $j \in \{1,\ldots,n\}$. Since the process $(s,\mathtt{w}) \mapsto \alpha_{ij}(s,Y_s^\tau(\mathtt{w}))$ is in particular left continuous, we can approximate the stochastic integral by Stieltjes sums. Denoting stochastic convergence on $(\widehat{\mathbf{W}},\overline{\mathfrak{W}},P_{(X,B)})$ by $P_{(X,B)}$–lim, we have:

$$\int_0^t\alpha_{ij}(s,Y_s^\tau)dW_s^{(j)^\tau} = P_{(X,B)}\text{–}\lim_{m\to\infty}\sum_{\nu=1}^{k_m}\alpha_{ij}\left(t_{\nu-1}^m, Y_{t_{\nu-1}^m}^\tau\right)\left(W_{t_\nu^m}^{(j)^\tau} - W_{t_{\nu-1}^m}^{(j)^\tau}\right).$$

Analogously, we obtain:

$$\int_0^t\alpha_{ij}(s,X_s^{\tilde\tau})dB_s^{(j)^{\tilde\tau}} = P\text{–}\lim_{m\to\infty}\sum_{\nu=1}^{k_m}\alpha_{ij}\left(t_{\nu-1}^m, X_{t_{\nu-1}^m}^{\tilde\tau}\right)\left(B_{t_\nu^m}^{(j)^{\tilde\tau}} - B_{t_{\nu-1}^m}^{(j)^{\tilde\tau}}\right).$$

For every $m \in \mathbb{N}$, we now define random variables $\Psi^{(m)}$ and $\Phi^{(m)}$ as follows:

$$\Psi^{(m)} := \sum_{j=1}^{n}\sum_{\nu=1}^{k_m} \alpha_{ij}\left(t^m_{\nu-1}, Y^{\tau}_{t^m_{\nu-1}}\right)\left(W^{(j)^{\tau}}_{t^m_{\nu}} - W^{(j)^{\tau}}_{t^m_{\nu-1}}\right) + \sum_{\nu=1}^{k_m}\beta_i\left(t^m_{\nu-1}, Y^{\tau}_{t^m_{\nu-1}}\right)\left(t^m_\nu \wedge \tau - t^m_{\nu-1}\wedge\tau\right)$$

$$\Phi^{(m)} := \sum_{j=1}^{n}\sum_{\nu=1}^{k_m} \alpha_{ij}\left(t^m_{\nu-1}, X^{\tilde{\tau}}_{t^m_{\nu-1}}\right)\left(B^{(j)^{\tilde{\tau}}}_{t^m_{\nu}} - B^{(j)^{\tilde{\tau}}}_{t^m_{\nu-1}}\right) + \sum_{\nu=1}^{k_m}\beta_i\left(t^m_{\nu-1}, X^{\tilde{\tau}}_{t^m_{\nu-1}}\right)\left(t^m_\nu \wedge \tilde{\tau} - t^m_{\nu-1}\wedge\tilde{\tau}\right).$$

From the arguments offered above, it follows that

$$\Psi = P_{(X,B)}\text{-}\lim_{m\to\infty}\Psi^{(m)}$$
$$\Phi = P\text{-}\lim_{m\to\infty}\Phi^{(m)}.$$

For $m \in \mathbb{N}$, we let μ_m denote the distribution of $\Psi^{(m)}$ under $P_{(X,B)}$ and denote the distribution of Ψ under $P_{(X,B)}$ by μ. The stochastic convergence of $(\Psi^{(m)})_{m\in\mathbb{N}}$ to Ψ implies the weak convergence of $(\mu_m)_{m\in\mathbb{N}}$ to μ. From the definition of Y, W and $\tilde{\tau}$ it follows that

$$\forall m \in \mathbb{N}: \quad \Psi^{(m)} \circ (X,B) = \Phi^{(m)}.$$

Therefore, for every $m \in \mathbb{N}$ the distribution of $\Phi^{(m)}$ under P is just μ_m. This shows that the sequence $(\mu_m)_{m\in\mathbb{N}}$ converges to δ_0 , the distribution of Φ under P. As the limit of a weakly convergent sequence of probability measures is uniquely determined, we conclude that $\mu = \delta_0$.

□

A.2. Constructing Solutions with Deterministic Initial Conditions from (Y, W).

A.2.1. Definition. (Regular factorized conditional probability): Let $(\Omega, \mathcal{F}, P)$ be a probability space and Z be a random variable with values in a measure space (S, Σ). A regular factorized conditional probability for P given Z is a Markov kernel K from (S, Σ) to $(\Omega, \mathcal{F})$ so that for every $A \in \mathcal{F}$ we have:

$$P[A|Z] = K(\cdot, A)\circ Z \quad P\text{–a.s.}$$

The following theorem ensures the existence of regular factorized conditional probabilities (cf Bauer (1991)).

A.2.2. Theorem. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, so that Ω is a Polish space and $\mathcal{F}$ is the Borel σ–algebra of Ω. Let Z be a random variable with values in a measure space (S, Σ). Then there exists a regular factorized conditional probability for P given Z.*

A.2.3. Remark. *1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and Z a random variable with values in a measure space (S, Σ). Suppose a regular factorized conditional probability for P given Z exists, then we denote it by $\{P^x\}_{x\in S}$, so that $\{P^x\}_{x\in S}$ is a family of probability measures on $(\Omega, \mathcal{F})$ and the Markov kernel is in fact given by the mapping $(x, A) \mapsto P^x[A]$. Let μ denote the distribution of Z under P. For every $A \in \mathcal{F}$ we have*

$$P[A] = \int_\Omega P[A|Z]dP = \int_\Omega P^{Z(\omega)}[A]\,P(d\omega) = \int_S P^x[A]\mu(dx).$$

In particular, if $P[A] = 0$, there is a null set Λ in (S, Σ, μ), so that the following is true:

$$\forall x \in \Lambda^c: \quad P^x[A] = 0.$$

2. If the σ–algebra Σ is countably generated, one can show the existence of a null set Λ in (S, Σ, μ), so that for every $x \in \Lambda^c$ the following holds:

$$\forall C \in \Sigma : \quad P^x[Z \in C] = \chi_C(x).$$

In particular, if $x \in \Lambda^c$ and $\{x\} \in \Sigma$ we have

$$Z = x \quad P^x\text{–a.s.}$$

We now return to the space $\widehat{\mathbf{W}}$ and our solution (Y, W) of the SDE determined by α and β. Since $(\widehat{\mathbf{W}}, \mathfrak{W})$ is a Polish space, we can find a regular factorized conditional probability for $P_{(X,B)}$ on $(\widehat{\mathbf{W}}, \mathfrak{W})$ given Y_0, which we denote by $\{P^x_{(X,B)}\}_{x \in \mathcal{U}}$. For each $x \in \mathcal{U}$, we denote by $(\widehat{\mathbf{W}}, \mathfrak{W}^x, \mathbb{W}^{*,x} := \{\mathfrak{W}^{*,x}_t\}_{t \in \mathbb{R}_+}, P^x_{(X,B)})$ the usual augmentation of $(\widehat{\mathbf{W}}, \mathfrak{W}, \mathbb{W}, P^x_{(X,B)})$. We let μ denote the distribution of Y_0 on $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$ under $P_{(X,B)}$.
The remainder of this section is devoted to proving the following result:

A.2.4. Theorem. *There is a null set Λ in $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$ so that for every $x \in \Lambda^c$ the process W is a normal, n–dimensional $(P^x_{(X,B)}, \mathbb{W}^{*,x})$–Brownian motion and furthermore the process Y is a solution of*

$$dZ_t = \alpha(t, Z_t)dW_t + \beta(t, Z_t)dt$$

on the stochastic basis $(\widehat{\mathbf{W}}, \overline{\mathfrak{W}}^x, \mathbb{W}^{*,x}, P^x_{(X,B)})$ fulfilling the initial condition

$$Y_0 = x \quad P^x_{(X,B)}\text{–a.s.}$$

We first obtain several partial results.

A.2.5. Lemma. *There is a null set Λ_1 in $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$, so that for every $x \in \Lambda_1^c$ the process W is a normal, n–dimensional $(P^x_{(X,B)}, \mathbb{W}^{*,x})$–Brownian motion.*

Proof: It suffices to find a null set Λ_1 in $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$, so that W is a $(P^x_{(X,B)}, \mathbb{W})$–Brownian motion. Since W is a $(P_{(X,B)}, \mathbb{W})$–Brownian motion, we have for every $y \in \mathbb{R}^n$ and all $s, t \in \mathbb{R}_+$ with $s < t$:

$$E\left[e^{i\langle y, W_t - W_s\rangle} \,\middle|\, \mathfrak{W}_s\right] = e^{-\frac{1}{2}\|y\|^2(t-s)} \; P_{(X,B)}\text{–a.s.}$$

This implies that for every $A \in \mathfrak{W}_s$ and every $C \in \mathcal{B}(\mathcal{U})$ we have:

$$\int_{A \cap \{Y_0 \in C\}} e^{i\langle y, W_t - W_s\rangle} dP_{(X,B)} = e^{-\frac{1}{2}\|y\|^2(t-s)} P_{(X,B)}\left[A \cap Y_0^{-1}(C)\right].$$

Using the regular factorized conditional probability $\{P^x_{(X,B)}\}_{x \in \mathcal{U}}$ we can rewrite this equation as follows:

$$\int_C \int_{\widehat{\mathbf{W}}} \chi_A \; e^{i\langle y, W_t - W_s\rangle} dP^x_{(X,B)} \mu(dx) = e^{-\frac{1}{2}\|y\|^2(t-s)} \int_C P^x_{(X,B)}(A)\mu(dx).$$

Since C was an arbitrary set in $\mathcal{B}(\mathcal{U})$, there is a null set $\Lambda_{y,s,t,A}$ in $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$, so that for $x \in \Lambda^c_{y,s,t,A}$ we have:

$$\int_{\widehat{\mathbf{W}}} \chi_A e^{i\langle y, W_t - W_s\rangle} dP^x_{(X,B)} = e^{-\frac{1}{2}\|y\|^2(t-s)} P^x_{(X,B)}(A).$$

The σ–algebra $\mathfrak{W}_s$ possesses a countable generator $\mathcal{E}_s$, by replacing $\mathcal{E}_s$ with the algebra it generates, we can assume that $\widehat{\mathbf{W}} \in \mathcal{E}_s$ and that $\mathcal{E}_s$ is stable under intersections. We define

$$\Lambda_{y,s,t} := \bigcup_{E \in \mathcal{E}_s} \Lambda_{y,s,t,E}.$$

Obviously $\mu(\Lambda_{y,s,t}) = 0$. Now fix $x \in \Lambda_{y,s,t}^c$. The system

$$\mathcal{D} := \left\{ A \in \mathfrak{W}_s \,\middle|\, \int_{\widehat{\mathbf{W}}} \chi_A e^{i\langle y, W_t - W_s \rangle} dP^x_{(X,B)} = e^{-\frac{1}{2}\|y\|^2 (t-s)} P^x_{(X,B)}(A) \right\}$$

is a Dynkin–system containing $\mathcal{E}_s$, therefore $\mathcal{D} = \mathfrak{W}_s$. To recapitulate, we have shown:

$$\forall x \in \Lambda_{y,s,t}^c \quad \forall A \in \mathfrak{W}_s : \quad \int_{\widehat{\mathbf{W}}} \chi_A e^{i\langle y, W_t - W_s \rangle} dP^x_{(X,B)} = e^{-\frac{1}{2}\|y\|^2 (t-s)} P^x_{(X,B)}(A).$$

We set

$$\tilde{\Lambda}_1 := \bigcup_{\substack{u,v \in \mathbb{Q}_+, u<v \\ p \in \mathbb{Q}^n}} \Lambda_{p,u,v}.$$

Suppose that $s,t \in \mathbb{R}_+$, $s < t$, $y \in \mathbb{R}^n$ are given. We choose sequences (u_k), (v_k) in $\mathbb{Q}_+$ and (p_k) in $\mathbb{Q}^n$, so that $(u_k) \downarrow s$, $(v_k) \uparrow t$, $(p_k) \to y$ and for every $k \in \mathbb{N}$ we have $u_k < v_k$. Let $x \in \tilde{\Lambda}_1^c$ and $A \in \mathfrak{W}_s$ be arbitrary. By the construction of $\tilde{\Lambda}_1$, the following equation holds for every $k \in \mathbb{N}$:

$$\int_{\widehat{\mathbf{W}}} \chi_A e^{i\langle p_k, W_{v_k} - W_{u_k} \rangle} dP^x_{(X,B)} = e^{-\frac{1}{2}\|p_k\|^2 (v_k - u_k)} P^x_{(X,B)}(A).$$

By the dominated convergence theorem it follows that

$$\int_{\widehat{\mathbf{W}}} \chi_A e^{i\langle y, W_t - W_s \rangle} dP^x_{(X,B)} = \lim_{k\to\infty} e^{-\frac{1}{2}\|p_k\|^2 (v_k - u_k)} P^x_{(X,B)}(A) = e^{-\frac{1}{2}\|y\|^2 (t-s)} P^x_{(X,B)}(A).$$

Since A was arbitrary we have shown:

$$\forall x \in \tilde{\Lambda}_1^c \quad \forall s,t \in \mathbb{R}_+, s<t, y \in \mathbb{R}^n : \quad E\left[e^{i\langle y, W_t - W_s \rangle} \middle| \mathfrak{W}_s \right] = e^{-\frac{1}{2}\|y\|^2 (t-s)} \; P^x_{(X,B)}\text{–a.s.}$$

This proves that for every $x \in \tilde{\Lambda}_1^c$, the process W is a $(P^x_{(X,B)}, \mathbb{W})$–Brownian motion. Using remark A.2.3, it is now trivial to enlarge $\tilde{\Lambda}_1$ slightly so as to ensure that $W_0 = 0$ $P^x_{(X,B)}$–a.s. □

To prove theorem A.2.4, we will use lemma A.0.3. To this end we need to fix an announcing sequence for the explosion time e_Y of Y. A natural choice is the following one. We can find a sequence $(\mathcal{U}_k)$ of open subsets of $\mathcal{U}$ with $(\mathcal{U}_k) \uparrow \mathcal{U}$, and so that for every $k \in \mathbb{N}$, $\mathcal{U}_k$ is relatively compact with $\overline{\mathcal{U}}_k \subset \mathcal{U}_{k+1}$. We define for every $k \in \mathbb{N}$:

$$\sigma_k := \inf \left\{ t \in \mathbb{R}_+ \,\middle|\, Y_t \in \widehat{\mathcal{U}} \setminus \mathcal{U}_k \right\}.$$

Since Y is continuous and $\mathbb{W}$–adapted, and $\widehat{\mathcal{U}} \setminus \mathcal{U}_k$ is a closed subset of the metric space $\widehat{\mathcal{U}}$, σ_k is a $\mathbb{W}$–stopping time for each $k \in \mathbb{N}$. One easily checks that $(\sigma_k)_{k\in\mathbb{N}}$ is indeed an announcing sequence for e_Y. Since σ_k is a $\mathbb{W}$–stopping time, it is automatically a $\mathbb{W}^{*,x}$–stopping time for every $x \in \mathcal{U}$.

A.2.6. Lemma. *Fix $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$. Let $I = (I_t)_{t\in\mathbb{R}_+}$ be a fixed version of the stochastic integral $(\int_0^t \alpha_{ij}(s, Y_s^{\sigma_k}) dW_s^{(j)^{\sigma_k}})_{t\in\mathbb{R}_+}$, where the stochastic integral refers to the stochastic basis $(\widehat{\mathbf{W}}, \mathfrak{W}, \mathbb{W}^*, P_{(X,B)})$. For each $x \in \Lambda_1^c$ let $I^x = (I_t^x)_{t\in\mathbb{R}_+}$ be a fixed version of the same integral, now taken with respect to the stochastic basis $(\widehat{\mathbf{W}}, \overline{\mathfrak{W}}^x, \mathbb{W}^{*,x}, P^x_{(X,B)})$. Then there exists a null set Λ_2 in $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$, so that for every $x \in (\Lambda_1 \cup \Lambda_2)^c$ the two processes I and I^x are $P^x_{(X,B)}$–indistinguishable.*

PROOF: Since we are dealing with continuous processes, it suffices to show that for every $t \in \mathbb{R}_+$ there exists a null set $\Lambda_{2,t}$ in $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$, so that the following holds:

$$\forall x \in (\Lambda_1 \cup \Lambda_{2,t})^c : \quad I_t = I_t^x \quad P^x_{(X,B)}\text{–a.s.}$$

We fix $t \in \mathbb{R}_+$ and choose a sequence $(\mathcal{Z}_m)_{m \in \mathbb{N}}$ of partitions $\mathcal{Z}_m : 0 = t_0^m < \ldots < t_{l_m}^m = t$ of $[0,t]$ with $|\mathcal{Z}_m| \to 0$. For each $m \in \mathbb{N}$ we define

$$\Phi_m := \sum_{\nu=1}^{l_m} \alpha_{ij} \left(t_{\nu-1}^m, Y^{\sigma_k}_{t^m_{\nu-1}} \right) \left(W^{(j)^{\sigma_k}}_{t^m_\nu} - W^{(j)^{\sigma_k}}_{t^m_{\nu-1}} \right).$$

The sequence $(\Phi_m)_{m \in \mathbb{N}}$ converges $P_{(X,B)}$–stochastically to I_t, this is just the approximation of the stochastic integral by Stieltjes sums. Choosing a subsequence if necessary, we can assume that the sequence $(\Phi_m)_{m \in \mathbb{N}}$ converges $P_{(X,B)}$–a.s. to I_t . In particular, there is a set $N \in \mathfrak{W}$ with $P_{(X,B)}[N] = 0$, so that

$$\forall \omega \in N^c : \quad I_t(\omega) = \lim_{m \to \infty} \Phi_m(\omega).$$

Again by remark A.2.3, there is a null set $\Lambda_{2,t}$ in $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$, so that

$$\forall x \in \Lambda_{2,t}^c : \quad P^x_{(X,B)}[N] = 0.$$

This shows that for $x \in \Lambda^c_{2,t}$ the sequence (Φ_m) converges $P^x_{(X,B)}$–almost surely to I_t, but for $x \in \Lambda_1^c$ we already know that it converges $P^x_{(X,B)}$–stochastically to I_t^x. Therefore we have

$$\forall x \in (\Lambda_1 \cup \Lambda_{2,t})^c : \quad I_t = I_t^x \quad P^x_{(X,B)}\text{–a.s.}$$

□

Proof of Theorem A.2.4: By remark A.2.3, there is a μ–null set Λ_0, so that for $x \in \Lambda_0^c$ the two processes Y and Y^{e_Y} are $P^x_{(X,B)}$–indistinguishable and $Y_0 = x$ $P^x_{(X,B)}$–a.s. For $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$, let $I^{(i,j,k)} = (I_t^{(i,j,k)})_{t \in \mathbb{R}_+}$ be a fixed version of the stochastic integral $(\int_0^t \alpha_{ij}(s, Y^{\sigma_k}) dW_s^{(j)^{\sigma_K}})_{t \in \mathbb{R}_+}$ computed with respect to $(\widehat{\mathbf{W}}, \overline{\mathfrak{W}}, \mathbb{W}^*, P_{(X,B)})$. For each $x \in \Lambda_1^c$, let $I^{(i,j,k,x)}$ be a fixed version of the same integral computed with respect to $(\widehat{\mathbf{W}}, \overline{\mathfrak{W}}^x, \mathbb{W}^{*,x}, P^x_{(X,B)})$. As Y is a solution of

$$dZ_t = \alpha(t, Y_t) dW_t$$

on $(\widehat{\mathbf{W}}, \overline{\mathfrak{W}}, \mathbb{W}^*, P_{(X,B)})$, there is a $P_{(X,B)}$–null set $N \in \mathfrak{W}$, so that for every $i \in \{1, \ldots, d\}$, $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we have:

$$\forall \omega \in N^c : \quad Y_t^{(i)^{\sigma_k}}(\omega) = Y_0^{(i)}(\omega) + \sum_{j=1}^n I_t^{(i,j,k)}(\omega) + \int_0^{t \wedge \sigma_k(\omega)} \beta_i \left(s, Y_s^{\sigma_k}(\omega)\right) ds.$$

There is a μ–null set Λ_3, so that

$$\forall x \in \Lambda_3^c : \quad P^x_{(X,B)}[N] = 0.$$

We therefore have

$$\forall x \in \Lambda_3^c \quad \forall i \in \{1, \ldots, d\}, k \in \mathbb{N}, t \in \mathbb{R}_+ :$$
$$Y_t^{(i)^{\sigma_k}} = Y_0^{(i)} + \sum_{j=1}^n I_t^{(i,j,k)} + \int_0^{t \wedge \sigma_k} \beta_i \left(s, Y_s^{\sigma_k}\right) ds \quad P^x_{(X,B)}\text{–a.s.}$$

By Lemma A.2.6, there is a μ–null set $\widehat{\Lambda}_2$, so that for $x \in (\Lambda_1 \cup \widehat{\Lambda}_2)^c$ and any choice of $i \in \{1, \dots, d\}$, $j \in \{1, \dots, n\}$, $k \in \mathbb{N}$ the two processes $I^{(i,j,k)}$ and $I^{(i,j,k,x)}$ are indistinguishable. This implies

$$\forall x \in \left(\Lambda_1 \cup \widehat{\Lambda}_2 \cup \Lambda_3\right)^c \quad \forall i \in \{1, \dots, d\}, k \in \mathbb{N}, t \in \mathbb{R}_+ :$$
$$Y_t^{(i)\sigma_k} = Y_0^{(i)} + \sum_{j=1}^{n} I_t^{(i,j,k,x)} + \int_0^{t\wedge\sigma_k} \beta_i\left(s, Y_s^{\sigma_k}\right) ds \quad P^x_{(X,B)}\text{–a.s}$$

Setting $\Lambda := \Lambda_0 \cup \Lambda_1 \cup \widehat{\Lambda}_2 \cup \Lambda_3$ and using Lemma A.0.3, we see that for $x \in \Lambda^c$ the process Y is a solution of $dZ_t = \alpha(t, Z_t)dW_t + \beta(t, Z_t)dt$ on the stochastic basis $(\widehat{\mathbf{W}}, \overline{\mathfrak{W}}^x, \mathbb{W}^{*,x}, P^x_{(X,B)})$ with the deterministic initial condition $Y_0 = x$ $P^x_{(X,B)}$–a.s.

□

A.3. **Concluding Proof of Theorem A.0.4:** We return to our original solution (X, B) on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. As we have already shown

$$P\left[e_X < +\infty\right] = P_{(X,B)}\left[e_Y < +\infty\right].$$

Therefore we need only show that $P_{(X,B)}[e_Y < +\infty] = 0$. By theorem A.2.4 there exists a μ–null set Λ, so that for $x \in \Lambda^c$ the process W is a standard n–dimensional Brownian motion on $(\widehat{\mathbf{W}}, \overline{\mathfrak{W}}^x, \mathbb{W}^{*,x}, P^x_{(X,B)})$ and Y is a solution of our SDE with $Y_0 = x$ $P_{(X,B)}$–a.s. By assumption, solutions with a deterministic initial condition do not explode, therefore

$$\forall x \in \Lambda^c : \quad P^x_{(X,B)}\left[e_Y < +\infty\right] = 0.$$

Now we have

$$P_{(X,B)}\left[e_Y < +\infty\right] = \int_{\mathcal{U}} P^x_{(X,B)}\left[e_Y < +\infty\right] \mu(dx) = 0.$$

With this, the desired result is finally proved. □

Appendix B. Lemmata for Asset Pricing in the Segmented Square Root Model

B.1. **Simple Exponentials of Non-Central Chi-Square Distributed Random Variables.** Following Jamshidian (1987) we can state the following

B.1.1. Lemma. *Denote the density function of a noncentral chi–square distribution with ν degrees of freedom and noncentrality parameter λ by $q_{\chi^2}(\cdot, \nu, \lambda)$. Then for $b, r > 0$ and an arbitrary constant L the following holds:*

$$e^{-rL} q_{\chi^2}(b\,r, \nu, \lambda) = \exp\left\{-\frac{L}{b + 2L}\lambda\right\} \left(\frac{b}{b + 2L}\right)^{\frac{1}{2}\nu - 1} q_{\chi^2}\left((b + 2L)r, \nu, \frac{\lambda b}{b + 2L}\right)$$

PROOF: Substituting for q_{χ^2} its infinite sum expression (see Johnson and Kotz (1970b), Chapter 28, eq. 3), we get

$$\begin{aligned} & e^{-rL} q_{\chi^2}(b\, r, \nu, \lambda) \\ = \;& e^{-rL} 2^{-\frac{1}{2}\nu} \exp\left\{-\frac{1}{2}(b\, r+\lambda)\right\} \sum_{j=0}^{\infty} \frac{(b\, r)^{\frac{1}{2}\nu+j-1}\lambda^j}{\Gamma(\frac{1}{2}\nu+j)2^{2j}j!} \\ = \;& 2^{-\frac{1}{2}\nu} \exp\left\{-\frac{1}{2}\left(r(b+2L)+\frac{\lambda b}{b+2L}\right)\right\} \exp\left\{\frac{1}{2}\left(\frac{\lambda b}{b+2L}-\lambda\right)\right\} \left(\frac{b}{b+2L}\right)^{\frac{1}{2}\nu-1} \\ & \cdot \sum_{j=0}^{\infty} \frac{((b+2L)r)^{\frac{1}{2}\nu+j-1}\left(\frac{\lambda b}{b+2L}\right)^j}{\Gamma(\frac{1}{2}\nu+j)2^{2j}j!} \\ = \;& \exp\left\{\frac{1}{2}\left(\frac{\lambda b}{b+2L}-\lambda\right)\right\} \left(\frac{b}{b+2L}\right)^{\frac{1}{2}\nu-1} q_{\chi^2}\left((b+2L)r, \nu, \frac{\lambda b}{b+2L}\right) \end{aligned}$$

□

B.1.2. Remark. *If one interprets q_{χ^2} not as a density, but as a function with complex arguments, then it is easy to see that lemma B.1.1 is also valid for complex L.*

B.2. **Distribution of the Short Rate in a Constant Parameter CIR Model.**

B.2.1. Lemma. *Let the short rate dynamics be given by*

$$dr(t) = (\theta - ar(t))dt + \sigma\sqrt{r(t)}dW(t),$$

and let

$$b := \frac{4a}{\sigma^2}\left(1-e^{-a(T-t)}\right)^{-1}.$$

Then under the risk neutral measure the random variable $br(T)$ conditioned on $r(t)$ is noncentral χ^2 distributed with $\nu := 4\theta\sigma^{-2}$ degrees of freedom and noncentrality parameter $\lambda := 4a\sigma^{-2}\left(e^{a(T-t)}-1\right)^{-1} r(t)$.

PROOF: Following Jamshidian (1987), consider the moment generating function for the distribution of $r(T)$,

$$f(r(t), t, k) = E\left[e^{kr(T)}\middle|\mathcal{F}_t\right].$$

Then f is the solution of the Kolmogorov backward equation of r,

$$f_t + \frac{1}{2}\sigma^2 r f_{rr} + (\theta - ar)f_r \;=\; 0 \tag{37}$$

subject to the terminal condition $f(r(T), T, k) = \exp\{kr(T)\}$.
Setting

$$f(r(t), t, k) = \exp\{g(t,T) + h(t,T)r(t)\}$$

we get

$$f_r = hf \qquad f_{rr} = h^2 f \qquad f_t = (g_t + h_t r(t))f.$$

Inserting this into (37),

$$g_t + h\theta + (h_t + \frac{1}{2}\sigma^2 h^2 - ah)r(t) = 0.$$

Since this must be valid for all $r(t) \in [0;\infty)$, we must have

$$g_t + h\theta = 0 \tag{38}$$

$$h_t + \frac{1}{2}\sigma^2 h^2 - ah = 0, \tag{39}$$

and the terminal conditions for these differential equations are $g(T,T)=0$ and $h(T,T)=k$. Setting $y := h^{-1}$ and dividing by h^2, we can write (39) as a linear differential equation:

$$\begin{aligned}
y_t + ay &= \frac{1}{2}\sigma^2 \\
\Leftrightarrow\quad e^{at}y_t + ae^{at}y &= \frac{1}{2}\sigma^2 e^{at} \\
\Leftrightarrow\quad [e^{at}y]' &= \frac{1}{2}\sigma^2 e^{at} \\
\Leftrightarrow\quad e^{at}y &= -\frac{1}{2}\sigma^2 \int_t^T e^{au}du + c \\
\Leftrightarrow\quad y &= \frac{\sigma^2}{2a}\left(1 - e^{a(T-t)}\right) + ce^{-at}
\end{aligned}$$

Using the terminal condition

$$\begin{aligned}
h(T,T) &= k \\
\Leftrightarrow\quad c &= e^{aT}k^{-1}
\end{aligned}$$

we get

$$h(t,T) = \left(\frac{\sigma^2}{2a}\left(1 - e^{a(T-t)}\right) + e^{a(T-t)}k^{-1}\right)^{-1}.$$

Equation (38) can then be written as

$$\begin{aligned}
g(t,T) &= \theta \int_t^T \left(\frac{\sigma^2}{2a}\left(1 - e^{a(T-u)}\right) + e^{a(T-u)}k^{-1}\right)^{-1} du \quad + \quad d \\
&= 2\sigma^{-2}\theta \int_t^T e^{-a(T-u)} \left(\frac{1}{a}\left(e^{-a(T-u)} - 1\right) + 2\sigma^{-2}k^{-1}\right)^{-1} du \quad + \quad d \\
&= 2\sigma^{-2}\theta \left(\ln 2\sigma^{-2}k^{-1} - \ln\left(\frac{1}{a}\left(e^{-a(T-t)} - 1\right) + 2\sigma^{-2}k^{-1}\right)\right) \quad + \quad d \\
&= -\frac{2\theta}{\sigma^2}\ln\left(\frac{1}{2a}\sigma^2 k\left(e^{-a(T-t)} - 1\right) + 1\right) \quad + \quad d.
\end{aligned}$$

Using the terminal condition $g(T,T)=0$ we get $d=0$. Therefore

$$\begin{aligned}
f(r(t),t,k) &= \exp\{g(t,T) + h(t,T)r(t)\} \\
&= \left(\frac{\sigma^2}{2a}k\left(e^{-a(T-t)} - 1\right) + 1\right)^{-\frac{2\theta}{\sigma^2}} \exp\left\{\left(\frac{\sigma^2}{2a}\left(1 - e^{a(T-t)}\right) + e^{a(T-t)}k^{-1}\right)^{-1} r(t)\right\}.
\end{aligned}$$

Defining b and ν as above, this can be written as

$$f(r(t),t,k) = (1 - 2b^{-1}k)^{-\frac{1}{2}\nu} \exp\left\{\frac{ke^{-a(T-t)}b^{-1}}{1 - 2b^{-1}k} br(t)\right\}. \tag{40}$$

Comparing (40) with the formula for the moment generating function of the noncentral χ^2 distribution[11], we can verify the assertion of the lemma.

□

B.3. **Multiply Compound χ^2 Distribution.** Let $b_r r$ be χ^2 distributed with ν_r degrees of freedom and noncentrality parameter $\eta_r x$, with $b_x x$ χ^2 distributed with ν_x degrees of freedom and noncentrality parameter λ. The joint distribution of $b_r r$ and $b_x x$ is given by the probability density function

$$p(b_r r = z_r,\ b_x x = z_x) = p(b_r r = z_r \mid b_x x = z_x) \cdot p(b_x x = z_x).$$

We are interested in the marginal distribution of $b_r r$, which is given by integrating over z_x:

$$p(b_r r = z_r) = \int_0^\infty p(b_r r = z_r | b_x x = z_x) p(b_x x = z_x) dz_x.$$

Both $p(b_r r = z_r | b_x x = z_x)$ and $p(b_x x = z_x)$ are noncentral χ^2 density functions. Writing these as a mixture of central χ^2 probability density functions, we have

$$\begin{aligned} p(b_r r = z_r) &= \int_0^\infty \left(\sum_{j=0}^\infty \frac{\left(\frac{1}{2}\eta_r b_x^{-1} z_x\right)^j}{j!} \exp\left\{ -\frac{1}{2}\eta_r b_x^{-1} z_x \right\} p_{\chi^2_{\nu_r+2j}}(z_r) \right) \\ &\quad \cdot \left(\sum_{j=0}^\infty \frac{\left(\frac{1}{2}\lambda\right)^j}{j!} \exp\left\{ -\frac{1}{2}\lambda \right\} p_{\chi^2_{\nu_x+2j}}(z_x) \right) dz_x \\ &= \int_0^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{\left(\frac{1}{2}\eta_r b_x^{-1}\right)^j}{j!} \frac{\left(\frac{1}{2}\lambda\right)^k}{k!} \exp\left\{ -\frac{1}{2}\lambda \right\} p_{\chi^2_{\nu_r+2j}}(z_r) \\ &\quad z_x^j \exp\left\{ -\frac{1}{2}\eta_r b_x^{-1} z_x \right\} p_{\chi^2_{\nu_x+2k}}(z_x) dz_x \,. \end{aligned} \tag{41}$$

Applying lemma B.1.1 to (41) and interchanging integration and addition, we get

$$\begin{aligned} p(b_r r = z_r) &= \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{\left(\frac{1}{2}\eta_r b_x^{-1}\right)^j}{j!} \frac{\left(\frac{1}{2}\lambda\right)^k}{k!} \exp\left\{ -\frac{1}{2}\lambda \right\} p_{\chi^2_{\nu_r+2j}}(z_r) \\ &\quad \int_0^\infty z_x^j \left(\frac{b_x}{b_x + \eta_r} \right)^{\frac{1}{2}(\nu_x+2k)} p_{\chi^2_{\nu_x+2k}} \left(\frac{b_x + \eta_r}{b_x} z_x \right) d\left(\frac{b_x + \eta_r}{b_x} z_x \right). \end{aligned} \tag{42}$$

The integral (42) can be expressed in terms of the j-th moment about zero of a central χ^2 distribution with $\nu_x + 2k$ degrees of freedom[12]:

$$\left(\frac{b_x}{b_x + \eta_r} \right)^{\frac{1}{2}\nu_x + k + j} 2^j \frac{\Gamma(\frac{1}{2}(\nu_x + 2k) + j)}{\Gamma(\frac{1}{2}(\nu_x + 2k))}.$$

B.3.1. Definition. Let strictly positive random variables $r_1, \ldots, r_n$ as well as strictly positive constants $r_0,\ b_1, \ldots, b_n,\ \eta_1, \ldots, \eta_n,\ \nu_1, \ldots, \nu_n$ be given. Suppose that for each $j \in \{1; \ldots; n\}$ the random variable $(b_j r_j)$ conditioned on r_{j-1} is noncentral χ^2 distributed with ν_j degrees of freedom and noncentrality parameter $\eta_j r_{j-1}$. Then we call r_n *n times multiply compound noncentral χ^2 distributed with transformation coefficients b_j.*

[11] See Johnson and Kotz (1970b), chapter 28, equation (11).

[12] See Johnson and Kotz (1970a), ch. 17, p. 168.

B.3.2. Lemma. *The probability density function $p(b_n \cdot r_n = z_n)$ of the multiply compound noncentral χ^2 distributed random variable $(b_n r_n)$ is*

$$p(b_n \cdot r_n = z_n) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_n=0}^{\infty} \frac{\left(\frac{1}{2}\eta_1 r_0\right)^{j_1}}{j_1!} \exp\left\{-\frac{1}{2}\eta_1 r_0\right\} p_{\chi^2_{\nu_n+2j_n}}(z_n)$$
$$\prod_{k=2}^{n} \frac{\left(\frac{1}{2}\eta_k b_{k-1}^{-1}\right)^{j_k}}{j_k!} \left(\frac{b_{k-1}}{b_{k-1}+\eta_k}\right)^{\frac{1}{2}(\nu_{k-1}+2j_{k-1})+j_k} 2^{j_k} \frac{\Gamma(\frac{1}{2}(\nu_{k-1}+2j_{k-1})+j_k)}{\Gamma(\frac{1}{2}(\nu_{k-1}+2j_{k-1}))}. \tag{43}$$

PROOF: We prove the lemma by induction: For $n = 2$ we get (42). Now let (43) be valid for some n. For $n + 1$ we have

$$\begin{aligned}
&p(b_{n+1} \cdot r_{n+1} = z_{n+1}) \\
=& \int_0^{\infty} p(b_{n+1} \cdot r_{n+1} = z_{n+1} \mid b_n r_n = z_n) p(b_n r_n = z_n) dz_n \\
=& \int_0^{\infty} \left(\sum_{j_{n+1}=0}^{\infty} \frac{\left(\frac{1}{2}\eta_{n+1} b_n^{-1} z_n\right)^{j_{n+1}}}{j_{n+1}!} \exp\left\{-\frac{1}{2}\eta_{n+1} b_n^{-1} z_n\right\} p_{\chi^2_{\nu_{n+1}+2j_{n+1}}}(z_{n+1}) \right) \\
&\cdot \left(\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_n=0}^{\infty} \frac{\left(\frac{1}{2}\eta_1 r_0\right)^{j_1}}{j_1!} \exp\left\{-\frac{1}{2}\eta_1 r_0\right\} p_{\chi^2_{\nu_n+2j_n}}(z_n) \right. \\
&\left. \prod_{k=2}^{n} \frac{\left(\frac{1}{2}\eta_k b_{k-1}^{-1}\right)^{j_k}}{j_k!} \left(\frac{b_{k-1}}{b_{k-1}+\eta_k}\right)^{\frac{1}{2}(\nu_{k-1}+2j_{k-1})+j_k} 2^{j_k} \frac{\Gamma(\frac{1}{2}(\nu_{k-1}+2j_{k-1})+j_k)}{\Gamma(\frac{1}{2}(\nu_{k-1}+2j_{k-1}))} \right) dz_n
\end{aligned}$$

and analogously to ((41) $\Leftrightarrow$ (42)) we get (43) for $n + 1$.

□

B.3.3. Remark. *Since only the values of the central χ^2 densities in (43) depend on z_n, we can interchange addition and integration and write the multiply compound noncentral χ^2 distribution function as*

$$P(b_n \cdot r_n \leq z_n) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_n=0}^{\infty} \frac{\left(\frac{1}{2}\eta_1 r_0\right)^{j_1}}{j_1!} \exp\left\{-\frac{1}{2}\eta_1 r_0\right\} P_{\chi^2_{\nu_n+2j_n}}(z_n)$$
$$\prod_{k=2}^{n} \frac{\left(\frac{1}{2}\eta_k b_{k-1}^{-1}\right)^{j_k}}{j_k!} \left(\frac{b_{k-1}}{b_{k-1}+\eta_k}\right)^{\frac{1}{2}(\nu_{k-1}+2j_{k-1})+j_k} 2^{j_k} \frac{\Gamma(\frac{1}{2}(\nu_{k-1}+2j_{k-1})+j_k)}{\Gamma(\frac{1}{2}(\nu_{k-1}+2j_{k-1}))}. \tag{44}$$

Due to the nested infinite sums, calculating (44) appears to be of exponential complexity in n. Fortunately, this is not the case. In fact, were it not for the term $\Gamma(\frac{1}{2}\nu_{k-1} + j_{k-1} + j_k)$, it would be possible to separate the terms and calculate (44) as a product of n one–dimensional sums, obviously a problem of linear complexity in n. As it is, the number of operations necessary to determine $P(b_n \cdot r_n \leq z_n)$ still only increases linearly in n.
We start by calculating the terms

$$\frac{\left(\frac{1}{2}\eta_1 r_0\right)^{j_1}}{j_1!} \exp\left\{-\frac{1}{2}\eta_1 r_0\right\} \tag{45}$$

for all $j_1 \in J_1 := \{\underline{j_1}; \ldots ; \overline{j_1}\}$. Note that (45) is unimodal in j_1, therefore $\underline{j_1}$ and $\overline{j_1}$ can be chosen in such a manner that (45) is smaller than $\epsilon > 0$ for any $j_1 \notin J_1$. For each $j_1 \in J_1$

we then calculate

$$\frac{\left(\frac{1}{2}\eta_2 b_1^{-1}\right)^{j_2}}{j_2!}\left(\frac{b_1}{b_1+\eta_2}\right)^{\frac{1}{2}(\nu_1+2j_1)+j_2} 2^{j_2}\,\frac{\Gamma(\frac{1}{2}(\nu_1+2j_1)+j_2)}{\Gamma(\frac{1}{2}(\nu_1+2j_1))} \tag{46}$$

for all $j_2 \in J_2 := \{\underline{j_2};\ldots;\overline{j_2}\}$, where (46) is again unimodal in j_2. We multiply (46) with (45) for each $(j_1;j_2) \in J_1 \times J_2$ and then sum over j_1 for each j_2, reducing the index dimension to 1 again. Substituting the result for (45) and defining (46) analogously for $(j_2;j_3)$, we iterate until we reach j_n, yielding a value for each $j_n \in J_n$, which we multiply with the respective value of $P_{\chi^2_{\nu_n+2j_n}}(z_n)$ and sum one last time to get $P(b_n \cdot r_n \le z_n)$.

B.3.4. Lemma. *The derivative of (44) with respect to r_0 is*

$$\frac{\partial}{\partial r_0}P(b_n r_n \le z_n) = -\eta_n\left(\prod_{j=1}^{n-1}\eta_j b_j^{-1}\right)\tilde{p}(b_n r_n = z_n) \tag{47}$$

where $\tilde{p}(b_n r_n = z_n)$ is defined as $p(b_n r_n = z_n)$, however with $\tilde{\nu}_j := \nu_j + 2\ \forall\ 1 \le j \le n$, all other coefficients identical.

Proof: For $n = 1$, we have

$$\frac{\partial}{\partial r_0}p(b_n r_n = z_n) \;= \frac{\partial}{\partial r_0}q_{\chi^2}(z,\nu_1,\eta_1 r_0) = \eta_1\frac{\partial}{\partial(\eta_1 r_0)}q_{\chi^2}(z,\nu_1,\eta_1 r_0)$$

which is equal to[13]

$$= -\,\eta_1\frac{\partial}{\partial z}q_{\chi^2}(z,\nu_1+2,\eta_1 r_0)$$

Now let

$$\frac{\partial}{\partial r_0}p(b_n r_n = z_n) = -\eta_n\left(\prod_{j=1}^{n-1}\eta_j b_j^{-1}\right)\frac{\partial}{\partial z_n}\tilde{p}(b_n r_n = z_n) \tag{48}$$

for some n. Then for $n+1$

$$\frac{\partial}{\partial r_0}p(b_{n+1}r_{n+1} = z_{n+1}) = \frac{\partial}{\partial r_0}\int_0^\infty q_{\chi^2}(z_{n+1},\nu_{n+1},\eta_{n+1}r_n)p(b_n r_n = z_n)dz_n$$

$$= \int_0^\infty q_{\chi^2}(z_{n+1},\nu_{n+1},\eta_{n+1}b_n^{-1}z_n)\left(-\eta_n\left(\prod_{j=1}^{n-1}\eta_j b_j^{-1}\right)\frac{\partial}{\partial z_n}\tilde{p}(b_n r_n = z_n)\right)dz_n$$

[13] see Jamshidian (1995), p. 69

and doing integration by parts:

$$
\begin{aligned}
&= \eta_n \left(\prod_{j=1}^{n-1} \eta_j b_j^{-1} \right) \Bigg(\underbrace{\left[-q_{\chi^2}(z_{n+1}, \nu_{n+1}, \eta_{n+1} b_n^{-1} z_n) \tilde{p}(b_n r_n = z_n) \right]_0^\infty}_{=0} \\
&\quad + \int_0^\infty \left(\frac{\partial}{\partial z_n} q_{\chi^2}(z_{n+1}, \nu_{n+1}, \eta_{n+1} b_n^{-1} z_n) \right) \tilde{p}(b_n r_n = z_n) dz_n \Bigg) \\
&= \eta_n \left(\prod_{j=1}^{n-1} \eta_j b_j^{-1} \right) \int_0^\infty -\eta_{n+1} b_n^{-1} \left(\frac{\partial}{\partial z_{n+1}} q_{\chi^2}(z_{n+1}, \nu_{n+1} + 2, \eta_{n+1} b_n^{-1} z_n) \right) \tilde{p}(b_n r_n = z_n) dz_n \\
&= -\eta_{n+1} \left(\prod_{j=1}^{n} \eta_j b_j^{-1} \right) \frac{\partial}{\partial z_{n+1}} \tilde{p}(b_{n+1} r_{n+1} = z_{n+1})
\end{aligned}
$$

We have thus shown (48) for all n by induction, and integrating with respect to z_n yields (47).

□

In order to implement the multifactor version of the segmented square root model along the lines of Chen and Scott (1995), we need the following

B.3.5. Proposition. *The characteristic function of the $(k-n)$ times multiply compound noncentral χ^2 distribution is given by*

$$
(49) \qquad \Psi_{n+1,k}(x) = \left(\prod_{j=n+1}^{k} (1 + 2\mathcal{L}_{j,k}(x))^{-\frac{1}{2}\nu_j} \right) \exp\left\{ -\frac{\mathcal{L}_{n+1,k}(x)}{1 + 2\mathcal{L}_{n+1,k}(x)} \eta_{n+1} r_n \right\}
$$

with $\mathcal{L}_{j,k}(x)$ recursively defined as

$$
(50) \qquad \mathcal{L}_{j-1,k}(x) := \frac{\mathcal{L}_{j,k}(x) \eta_j b_{j-1}^{-1}}{1 + 2\mathcal{L}_{j,k}(x)} \qquad \textit{and} \qquad \mathcal{L}_{k,k}(x) := -ix
$$

PROOF: For $k - n = 1$ we have

$$
\Psi_{k,k}(x) = (1 - 2ix)^{-\frac{1}{2}\nu_k} \exp\left\{ ix \eta_k r_{k-1} (1 - 2ix)^{-1} \right\}
$$

which is the characteristic function of the noncentral χ^2 distribution with ν_k degrees of freedom and noncentrality parameter $\eta_k r_{k-1}$[14]. Let (49) be valid for some $0 < n < k$. Then we have for $n - 1$:

$$
\begin{aligned}
\Psi_{n,k}(x) &= E\left[\exp\{ ix b_k r_k \} \right] \\
&= \int_0^\infty e^{ixz_k} p_{n-1}(b_k r_k = z_k) dz_k \\
&= \int_0^\infty e^{ixz_k} \int_0^\infty p_n(b_k r_k = z_k \left| \eta_{n+1} b_n^{-1} z_n \right.) q_{\chi^2}(z_n, \nu_n, \eta_n r_{n-1}) dz_n dz_k
\end{aligned}
$$

interchanging the order of integration yields

$$
= \int_0^\infty \Psi_{n+1,k}\left(x \left| \eta_{n+1} b_n^{-1} z_n \right. \right) q_{\chi^2}(z_n, \nu_n, \eta_n r_{n-1}) dz_n
$$

[14]See for example Johnson and Kotz (1970b).

and inserting (49)

$$= \left(\prod_{j=n+1}^{k} (1+2\mathcal{L}_{j,k}(x))^{-\frac{1}{2}\nu_j} \right) \cdot \int_0^\infty \exp\left\{ -\frac{\mathcal{L}_{n+1,k}(x)}{1+2\mathcal{L}_{n+1,k}(x)} \eta_{n+1} b_n^{-1} z_n \right\} q_{\chi^2}(z_n, \nu_n, \eta_n r_{n-1}) dz_n$$

and applying (50) and remark B.1.2

$$= \left(\prod_{j=n}^{k} (1+2\mathcal{L}_{j,k}(x))^{-\frac{1}{2}\nu_j} \right) \exp\left\{ -\frac{\mathcal{L}_{n,k}(x)}{1+2\mathcal{L}_{n,k}(x)} \eta_n r_{n-1} \right\} \cdot \underbrace{(1+2\mathcal{L}_{n,k}(x)) \int_0^\infty q_{\chi^2} \left((1+2\mathcal{L}_{n,k}(x))\, z_n, \nu_n, \overbrace{(1+2\mathcal{L}_{n,k}(x))^{-1} \eta_n r_{n-1}}^{=:\lambda} \right) dz_n}_{=:\xi(x)}.$$

Now if $\xi(x) = 1$ for all x, then the proposition is proven. Inserting for q_{χ^2} its infinite sum expression[15]

$$\begin{aligned}
\xi(x) =& (1+2\mathcal{L}_{n,k}(x)) \int_0^\infty \sum_{j=0}^\infty \frac{\left(\frac{1}{2}\lambda\right)^j}{j!} e^{-\frac{1}{2}\lambda} p_{\chi^2_{\nu_n+2j}} \left((1+2\mathcal{L}_{n,k}(x))\, z_n \right) dz_n \\
=& \sum_{j=0}^\infty \frac{\left(\frac{1}{2}\lambda\right)^j}{j!} e^{-\frac{1}{2}\lambda} (1+2\mathcal{L}_{n,k}(x)) \int_0^\infty p_{\chi^2_{\nu_n+2j}} \left((1+2\mathcal{L}_{n,k}(x))\, z_n \right) dz_n.
\end{aligned}$$

This equals 1 if $(1+2\mathcal{L}_{n,k}(x)) \int_0^\infty p_{\chi^2_{\nu_n+2j}} \left((1+2\mathcal{L}_{n,k}(x))\, z_n \right) dz_n = 1 \ \forall j \in \mathbb{N}$. We have

$$\begin{aligned}
& (1+2\mathcal{L}_{n,k}(x)) \int_0^\infty p_{\chi^2_{\nu_n+2j}} \left((1+2\mathcal{L}_{n,k}(x))\, z_n \right) dz_n \\
=& (1+2\mathcal{L}_{n,k}(x)) \int_0^\infty 2^{-\frac{1}{2}\nu_n - j} \Gamma\left(\frac{1}{2}\nu_n + j \right)^{-1} \left((1+2\mathcal{L}_{n,k}(x))\, z_n \right)^{\frac{1}{2}\nu_n + j - 1} \\
& \cdot \exp\left\{ -\frac{1}{2}(1+2\mathcal{L}_{n,k}(x)) z_n \right\} dz_n \\
=& \Gamma\left(\frac{1}{2}\nu_n + j \right)^{-1} \left(\frac{1}{2} (1+2\mathcal{L}_{n,k}(x)) \right)^{\frac{1}{2}\nu_n + j} \int_0^\infty z_n^{\frac{1}{2}\nu_n + j - 1} \exp\left\{ -\frac{1}{2}(1+2\mathcal{L}_{n,k}(x)) z_n \right\} dz_n
\end{aligned}$$

which by formula 6.1.1 in Abramowitz and Stegun (1964) equals

$$= \Gamma\left(\frac{1}{2}\nu_n + j \right)^{-1} \Gamma\left(\frac{1}{2}\nu_n + j \right) = 1$$

□

[15]Note that q_{χ^2} and p_{χ^2} must now be interpreted as function with complex arguments, not densities.

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