A Note on Arbitrage and Stochastic Taxes

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In a recent paper Löffler and Schneider (2000) showed that introducing a tax on a financial market does not create an arbitrage opportunity. They furthermore showed that the sign of the NPV of a real investment will not change if an allowance for corporate equity (ACE) first introduced by Boadway and Bruce (1979) and Wenger (1983) exists. So far, all papers dealing with taxes assumed a deterministic tax rate. In this paper we generalize the results to a tax rate that is linear but stochastic.

Keywords: arbitrage–free valuation, equivalent martingale measure, taxes, uncertainty.

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1 Introduction

Boadway and Bruce (1984) showed conditions under which a gains tax with deductible interest payments will be neutral. Their tax system had the unpleasant restriction of a time constant tax rate and was formulated in a world without risk. Later Bond and Devereux (1995) showed that even under uncertainty this tax system remains neutral, but sticked further to a constant tax rate arguing that this is an indispensable condition for tax neutrality.[1]

2 The model

2.1 The capital market

The future has discrete time $t = 0, 1, \ldots, T$ and is uncertain. The probability space of an investor is denoted by $(\Omega, \mathcal{F}, P)$. The filtration $\mathcal{F}$ need not be finitely generated, it consists of the σ–algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T$ that describe the information set of every investor.[2] There are N tradeable financial (risky) assets that pay dividends (adapted random variables)

$$\widetilde{X}_{1,t}, \ldots, \widetilde{X}_{N,t}$$

The prices – also called values – of the risky assets at time t are adapted random variables

$$\widetilde{V}_{1,t}, \ldots, \widetilde{V}_{N,t}.$$

There is one risk–free asset, labelled $n = 0$. The prices of the risk–free asset are given by

$$V_{0,t} = \begin{cases} 1 & \text{if } t < T \\ 0 & \text{if } t = T \end{cases}$$

and the cash flows of the risk–free asset are given by

$$X_{0,t} = \begin{cases} r_f & \text{if } t < T \\ 1 + r_f & \text{if } t = T \end{cases}$$

where r_f is the risk–free interest rate.

At time $t = 0$ the investor selects a portfolio consisting of the available financial assets. This portfolio will be changed at every subsequent trading date $t = 1, \ldots, T$. The portfolio held during period t, denoted by $\widetilde{H}_{t-1}$, has a value of

$$\widetilde{H}_{t-1} \cdot \widetilde{V}_t = \sum_{n=0}^{N} \widetilde{H}_{n,t-1} \widetilde{V}_{n,t}.$$

[1] ***YT: wird die stochastische Steuer Effekt schon geprüft?***
AL: Das weiß ich nicht. Das müssen wir jetzt herausbekommen. Danach schreiben wir die Einleitung.

[2] These are standard assumptions in an uncertain economy, see Duffie (1988).

At time t the investor can withdraw the amount $\delta_t(\widetilde{H})$ given by

$$\delta_t(\widetilde{H}) = \widetilde{H}_{t-1} \cdot (\widetilde{X}_t + \widetilde{V}_t) - \widetilde{H}_t \cdot \widetilde{V}_t.$$

Note that $H_{-1} = \widetilde{H}_T = 0$ (see figure 1).

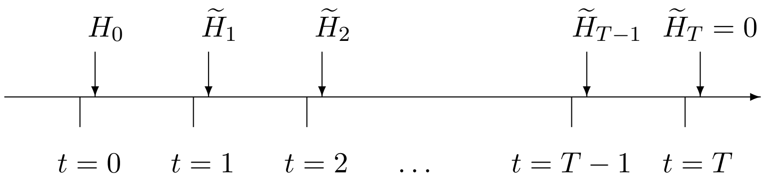

Figure 1: The time structure of the model

We use the standard definition of an arbitrage–free market.

Assumption 1 (arbitrage–free capital market) *There exists no trading strategy $\widetilde{H}$ that satisfies*

$$\delta_t(\widetilde{H}) \geq 0$$

for all t and

$$P(\delta_t(\widetilde{H}) > 0) > 0$$

for at least one t.

According to Harrison and Kreps (1979) assumption 1 implies the existence of an equivalent martingale measure Q (for a proof see Kabanov (1995)).

Proposition 1 (fundamental pricing lemma) *There exists a probability measure Q such that*

$$\widetilde{H}_t \cdot \widetilde{V}_t = \frac{E_Q[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) | \mathcal{F}_t]}{1 + r_f}. \tag{1}$$

We now introduce the tax system.

2.2 The tax system

We have to distinguish between the market value of a risky asset and the value that will be the underlying for the tax base. We denominate it the book value of a financial asset, it will be denoted by $\widetilde{B}_{nt}$ and is a random variable. We assume that the book value is an adapted random variable and that will be zero at time $t = T$. The portfolio $\widetilde{H}_{t-1}$ has the book value

$$\widetilde{H}_{t-1} \cdot \widetilde{B}_t = \sum_{n=1}^{N} \widetilde{H}_{n,t-1} \widetilde{B}_{n,t}.$$

Using the book value we define the depreciation and the gain of a portfolio as follows:

Definition 1 (depreciation, gain) *The depreciation of portfolio $\widetilde{H}_{t-1}$ in period t is given by the difference of the book values of all containing financial assets*

$$\widetilde{D}_t(\widetilde{H}_{t-1}) = -\widetilde{H}_{t-1} \cdot (\widetilde{B}_t - \widetilde{B}_{t-1}). \tag{2}$$

The gain of portfolio $\widetilde{H}_{t-1}$ is given by the difference of cash flow and depreciation in t

$$\widetilde{G}_t(\widetilde{H}_{t-1}) = \widetilde{H}_{t-1} \cdot \widetilde{X}_t - \widetilde{D}_t(\widetilde{H}_{t-1}). \tag{3}$$

The concepts of depreciation and gain are determined by the tax code and may differ from economic depreciation or profit.

Now we are able to define the tax base. We use an idea already developed by Boadway and Bruce (1979) and Wenger (1983) that the tax base will be given by the gain of a portfolio where interest on book value is tax deductible: this deduction is now termed "allowance for corporate equity" (ACE), see ?[3] Since the tax rate might vary in time we allow for a stochastic fraction of interest on book value $\widetilde{\alpha}_t$ to be deductible. This fraction $\widetilde{\alpha}_t$ is predictable (known at time $t-1$ or $\mathcal{F}_{t-1}$–measurable) since tax rates will be predictable.

Definition 2 (tax base) *The tax base $\widetilde{U}_t$ of the portfolio $\widetilde{H}_{t-1}$ in $t > 0$ is given by the difference between gain and a time–dependent, stochastic and predictable fraction $1 - \widetilde{\alpha}_t$ of interest on book value in $t-1$*

$$\widetilde{U}_t(\widetilde{H}_{t-1}) = \widetilde{G}_t(\widetilde{H}_{t-1}) - (1 - \widetilde{\alpha}_t) \cdot r_f \cdot \widetilde{H}_{t-1} \cdot \widetilde{B}_{t-1}. \tag{4}$$

where $\widetilde{\alpha}_t \in [0,1]$. $\widetilde{\alpha}_t = 0$ implies a total deduction of the interest on book value, while $\widetilde{\alpha}_t = 1$ means no deduction of the interest on book value.

If the tax base is negative, there is an immediate and full loss offset. In $t = 0$ no tax is paid.

We now turn to the main assumption of our paper. The tax rate we consider is uncertain. Nevertheless, we will assume that the tax rate is predictable. This is to say that the tax rate applicable in time $t+1$ is already known in time t. This assumption is by far satisfied in all national tax codes we know of.[4]

Assumption 2 (tax rate) *The tax rate $\widetilde{\tau}_t$ is stochastic and predictable, i.e. τ_t is already known at time $t-1$ or $\mathcal{F}_{t-1}$–measurable.*

[3] **AL: Hier wäre ein Zitat zu ACE gut.**

[4] **AL: Da müssten wir etwas zitieren. Kannst du herausbekommen (Hiwis), wo das im amerikanischen Recht steht und was da steht?**

Therefore, the tax payments in t are given by

$$\widetilde{T}_t(\widetilde{H}_{t-1}) = \widetilde{\tau}_t \cdot \widetilde{H}_{t-1} \cdot \left(\widetilde{X}_t + \widetilde{B}_t - (1 + r_f \cdot (1 - \widetilde{\alpha}_t)) \cdot \widetilde{B}_{t-1}\right) \tag{5}$$

As in our previous paper financial assets are characterized by the assumption that their book value $\widetilde{B}_{n,t}$ is equal to its value $\widetilde{V}_{n,t}$

$$\widetilde{V}_{n,t} = \widetilde{B}_{n,t}. \tag{6}$$

We now turn to the results.

2.3 Fundamental theorem and neutrality

Our first main result concerns the fact that even with a stochastic tax rate there will be no arbitrage opportunity. Furthermore, Q still stays the equivalent martingale measure and we can even prove a fundamental theorem as in the case without taxes.

Proposition 2 *There are no arbitrage opportunities under taxes. Furthermore, Q is also an equivalent martingale measure under taxes in the sense that*

$$\widetilde{H}_t \cdot \widetilde{V}_t = \frac{E_Q[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) - \widetilde{T}_{t+1}(\widetilde{H}_t)|\mathcal{F}_t]}{1 + (1 - \widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})\, r_f} \tag{7}$$

and the martingale measure under taxes is unique provided Q was unique.

Uniqueness of Q is in our model equivalent to the completeness of the market. We therefore conclude that completeness carries over if taxes are taking into account. This result will be used to value the real asset. If the real asset is just another way of realizing a cash flow, it must be possible to value it in terms of the underlying financial assets.

Assume the real asset, indexed by $n = N + 1$, pays cash flow $\widetilde{X}_{N+1,t}$ and has a book value $\widetilde{B}_{N+1,t}$ at time t. The amount of investment expenses for the real asset is denoted by $\widetilde{I}_{N+1,t}$. In contrast to our assumption concerning the financial assets we do not presuppose that this real asset has a book value equal to expenses $\widetilde{I}_{N+1,t}$. As for the financial asset the real asset has the tax base given by equations (2) to (5), the real asset is taxed in the same way as the financial assets.

If the capital market is complete, the cash flow of the real asset can be duplicated by a trading strategy with financial assets. Since the capital market is free from arbitrage, we can compute the fair value of the real asset. Let $\widetilde{V}_{N+1,t}$ denote the fair value with calculation of taxes and $\widetilde{V}^*_{N+1,t}$ the value in a world without taxes. We will not compute a relationship between $V_{N+1,t}$ and $\widetilde{V}^*_{N+1,t}$ but rather characterize the neutral tax systems directly.

In defining neutrality we follow Fane (1987) and Bond and Devereux (1995). Since we used an arbitrage argument to derive the main equation we will not consider first order conditions on consumption to define neutrality. Instead we focus on the net present value of an investment since our model is not an equilibrium model. Let $\widetilde{NPV}_t$ be the net present value at time t in a world with the tax and $\widetilde{NPV}^*_t$ the net present value without a tax:

$$\widetilde{\mathrm{NPV}}_t = \widetilde{V}_{N+1,t} - \widetilde{I}_{N+1,t} \tag{8}$$

and

$$\widetilde{\mathrm{NPV}}^*_t = \widetilde{V}^*_{N+1,t} - \widetilde{I}_{N+1,t} \tag{9}$$

We say a tax system is neutral iff the ordering of the net present value of two investment projects in a world with taxation is the same as the ordering of the net present value in a world without taxation. This criterion only makes sense if the real assets are traded at a net present value not equal to zero, i.e. the investment expenses $\widetilde{I}_{N+1,t}$ are different from the fair value $\widetilde{V}_{N+1,t}$.

The tax system is static neutral, iff $\widetilde{\mathrm{NPV}}^*_t > 0$ follows from $\widetilde{\mathrm{NPV}}_t > 0$ for the time $t = 0$, when the investment is taken. The static neutrality is satisfactory if the investment project is irreversible and not tradeable at future dates. But these conditions do not necessarily hold in general, therefore much stronger criterion for dynamic neutrality must be formulated.

Definition 3 (dynamic neutrality) *A tax system is dynamically neutral iff*

$$\widetilde{\mathrm{NPV}}_t = (1 - \widetilde{a}_t) \cdot \widetilde{\mathrm{NPV}}^*_t \tag{10}$$

is satisfied for some stochastic, but positive $a_t < 1$ *and all* $t \in [0, T-1]$.

Since the tax may increase the net present value, we do not assume that the a_t are positive. Two particular systems, which can be regarded as generalizations of well–known neutral systems, will now be discussed in detail.

Proposition 3 (neutral tax systems) *The following two tax systems are dynamically neutral:*

- *(generalization of taxation on economic rent) For all* $t \geq 0$ *the book values of the real asset equal the fair market values without taxes*

  $$\widetilde{V}^*_{N+1,t} = \widetilde{B}_{N+1,t}$$

  regardless whether there is interest deduction or not ($\widetilde{\alpha}_t$ *arbitrary). In this case* $\widetilde{a}_t$ *is equal to zero.*

- *(ACE) For all $t \geq 0$ the book values equal the investment expenses*

$$\widetilde{B}_{N+1,t} = \widetilde{I}_{N+1,t},$$

and the deductions are given by

$$\widetilde{\alpha}_1 = \frac{1+r_f}{r_f} \frac{\widetilde{\tau}_1 - a_0}{\widetilde{\tau}_1(1-a_0)}$$

and

$$\widetilde{\alpha}_{t+1} = \frac{1+r_f}{r_f} \frac{\widetilde{\tau}_{t+1} - \widetilde{\tau}_t}{\widetilde{\tau}_{t+1}(1-\widetilde{\tau}_t)} \quad \textit{for all } t > 0.$$

In this case $\widetilde{a}_t = \widetilde{\tau}_t$ for all $t > 0$.

In the first tax system our real asset "behaves" like a financial asset which is the reason for neutrality. Since we have an allowance for corporate equity our system is slightly more general then the classical tax on economic rent. The net present value remains unchanged due to taxation. The second tax system is compatible with the tax proposed Boadway and Bruce (1984) and Wenger (1983).

3 Continuous time model

In this section we will consider a continuous time setup. Our model has to be modified as follows. The value of the riskless asset evolves over time according to the differential equation (the instantaneous risk free rate is constant)

$$dV_{0,t} = rV_{0,t}dt, \tag{11}$$

and the value of the risky assets evolves according to the stochastic differential equation[5]

$$dV_t = \mu_t V_t dt - dX_t + \sigma_t V_t dW_t, \tag{12}$$

where σ_t represents the volatility and μ_t the drift. The depreciation is given by the differential $-dB_t$, and the tax payments in t are equal to

$$dT_t = \tau_t \left(dX_t + dB_t - (1-\alpha_t) r B_t dt \right). \tag{13}$$

The real asset trades at investment expenses I_t. We further assume that the tax rate τ_t is a stochastic process that is in a very strong sense predictable. In particular

Assumption 3 (predictable tax rate) *The tax rate τ is $\mathcal{F}_{t+h}$–measurable ($h > 0$).*

[5] The conditions under which this PDE is solvable are described in (Duffie 1988, p. 95ff.).

Then we can prove the following theorem which shows that the results with deterministic taxes can be carried over to the case of a stochastic tax rate.

Proposition 4 *In a continuous time setup the following two tax regimes are neutral*

- *(generalization of economic rent) The tax on economic rent is characterized by the equality of book value and value before taxes, i.e.* $B_t = V_t^*$*. Then*

$$V_t - I_t = V_t^* - I_t.$$

- *(ACE) The parameters* α_t *must satisfy the following integral differential equation*

$$\forall s \leq t \qquad \int_s^t r\alpha_u \tau_u (1 - \tau_u) du = \tau_t - \tau_s.$$

In this case we have

$$V_t - I_t = (1 - \tau_t)(V_t^* - I_t).$$

4 Conclusion

In our framework we show that, in a world with a linear but stochastic tax system with deductible interest payments, the uncertainty of the tax system do not change the ordering of the investment decisions, if an allowance for corporate equity exists or the tax system is designed as a taxation on economic rent. Since we used the martingale theory it was not necessary to assume that the investors are risk–neutral. The distributive effects of the neutral tax systems in an equilibrium model were ignored in this paper. This aspect is left for future research.

A Appendix

A.1 Proof of Proposition 2

We first prove a lemma concerning the expected tax payments due to a trading strategy.

Lemma 1 (expected tax payments) *The expected tax payments due to a trading strategy are given by*

$$E_Q[\widetilde{T}_{t+1}(\widetilde{H}_t)|\mathcal{F}_t] = \widetilde{\tau}_{t+1}\widetilde{\alpha}_{t+1} r_f \, \widetilde{H}_t \cdot \widetilde{V}_t. \tag{14}$$

Proof. We have

$$\begin{aligned}
E_Q[\widetilde{T}_{t+1}(\widetilde{H}_t)|\mathcal{F}_t] &= E_Q\left[\widetilde{\tau}_{t+1}\widetilde{H}_t \cdot \left(\widetilde{X}_{t+1} + \widetilde{B}_{t+1} - (1 + r_f(1 - \widetilde{\alpha}_{t+1}))\,\widetilde{B}_t\right)|\mathcal{F}_t\right] && \text{by (5)}\\
&= \widetilde{\tau}_{t+1}E_Q\left[\widetilde{H}_t \cdot \left(\widetilde{X}_{t+1} + \widetilde{B}_{t+1} - (1 + r_f(1 - \widetilde{\alpha}_{t+1}))\,\widetilde{B}_t\right)|\mathcal{F}_t\right] && \text{assumption 2}\\
&= \widetilde{\tau}_{t+1}E_Q\left[\widetilde{H}_t \cdot \left(\widetilde{X}_{t+1} + \widetilde{V}_{t+1} - (1 + r_f(1 - \widetilde{\alpha}_{t+1}))\,\widetilde{V}_t\right)|\mathcal{F}_t\right] && \text{by (6)}\\
&= \widetilde{\tau}_{t+1}E_Q\left[\widetilde{H}_t \cdot \left((1 + r_f)\widetilde{V}_t - (1 + r_f(1 - \widetilde{\alpha}_{t+1}))\,\widetilde{V}_t\right)|\mathcal{F}_t\right] && \text{by (1)}\\
&= \widetilde{\tau}_{t+1}\widetilde{\alpha}_{t+1}r_f\,\widetilde{H}_t \cdot \widetilde{V}_t,
\end{aligned}$$

which was to be shown, q.e.d.

We show the first part of proposition 2 (no arbitrage opportunities). Suppose, the trading strategy $\widetilde{H}$ is an arbitrage opportunity in a world with taxation. This is

$$0 \geq H_0 \cdot V_0 \tag{15}$$

for $t = 0$ and

$$\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) - \widetilde{T}_{t+1}(\widetilde{H}_t) - \widetilde{H}_{t+1} \cdot \widetilde{V}_{t+1} \geq 0 \tag{16}$$

for all $t = 0, \ldots, T-1$, and at least one inequality must be strict with positive probability. For $t = T$ we have

$$\widetilde{H}_T = 0. \tag{17}$$

After taking the conditional expectation with respect to Q and using Lemma 1 and Proposition 1 we get

$$\begin{aligned}
E_Q[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) - \widetilde{T}_{t+1}(\widetilde{H}_t)|\mathcal{F}_t] - E_Q[\widetilde{H}_{t+1} \cdot \widetilde{V}_{t+1}|\mathcal{F}_t] &\geq 0\\
E_Q[\widetilde{H}_t \cdot (1 + r_f)\widetilde{V}_t|\mathcal{F}_t] - \widetilde{\tau}_{t+1}\widetilde{\alpha}_{t+1}r_f E_Q[\widetilde{H}_t\widetilde{V}_t|\mathcal{F}_t] - E_Q[\widetilde{H}_{t+1} \cdot \widetilde{V}_{t+1}|\mathcal{F}_t] &\geq 0\\
(1 + r_f(1 - \widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1}))E_Q[\widetilde{H}_t \cdot \widetilde{V}_t|\mathcal{F}_t] &\geq E_Q[\widetilde{H}_{t+1} \cdot \widetilde{V}_{t+1}|\mathcal{F}_t].
\end{aligned}$$

Since $\widetilde{\alpha}_t \in [0, 1]$, therefore $1 + r_f(1 - \widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1}) > 0$. Together with (15) and the fact that at least one inequality must be strict with positive probability these inequalities imply by induction

$$0 \geq E_Q[\widetilde{H}_t \cdot \widetilde{V}_t|\mathcal{F}_t] \;\text{ for all }\; 0 \leq t \leq T$$

and

$$0 > E_Q[\widetilde{H}_t \cdot \widetilde{V}_t|\mathcal{F}_t] \;\text{ for all }\; t \geq t',$$

i.e. t' is the first time at which the inequality is strict. Thus

$$0 > E_Q[\widetilde{H}_T \cdot \widetilde{V}_T|\mathcal{F}_t],$$

but this contradicts (17).

Now we are able to prove the fundamental pricing lemma under taxes (7). Using equations (14) and (1) we get

$$\begin{aligned}
E_Q[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) - \widetilde{T}_{t+1}(\widetilde{H}_t)|\mathcal{F}_t] &= (1 + r_f)\widetilde{H}_t \cdot \widetilde{V}_t - \widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1}r_f\widetilde{H}_t \cdot \widetilde{V}_t\\
&= \left(1 + r_f \cdot (1 - \widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})\right)\widetilde{H}_t \cdot \widetilde{V}_t
\end{aligned}$$

which was to be shown.

The last part of proposition 2 covers the uniqueness of the martingale measure. We show that if the equivalent martingale measure in (1) is unique then there is only one martingale measure satisfying (7). Assume the contrary and consider

$$E_{Q_1}[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) - \widetilde{T}_{t+1}(\widetilde{H}_t)|\mathcal{F}_t] = E_{Q_2}[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) - \widetilde{T}_{t+1}(\widetilde{H}_t)|\mathcal{F}_t]$$

Using (14) we get

$$\begin{aligned} E_{Q_1}[(1-\widetilde{\tau}_{t+1})\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) + (1 + r_f(1-\widetilde{\alpha}_t))\widetilde{\tau}_{t+1} V_t|\mathcal{F}_t] = \\ E_{Q_2}[(1-\widetilde{\tau}_{t+1})\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1}) + (1 + r_f(1-\widetilde{\alpha}_t))\widetilde{\tau}_{t+1} V_t|\mathcal{F}_t]. \end{aligned}$$

Since V_t and $\widetilde{\tau}_{t+1}$ is $\mathcal{F}_t$–measurable the expectation is equal to the variable itself and we have for all $\widetilde{H}$

$$E_{Q_1}[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1})|\mathcal{F}_t] = E_{Q_2}[\widetilde{H}_t \cdot (\widetilde{X}_{t+1} + \widetilde{V}_{t+1})|\mathcal{F}_t] = (1 + r_f)\widetilde{H}_t \cdot V_t$$

and therefore Q_1 and Q_2 cannot be different. This completes the proof. ■

A.2 Proof of Proposition 3

We first show that equation (7) also holds for the real asset.

Lemma 2 (real asset) *If the market is complete and free of arbitrage, then the following equation holds for the real asset:*

$$\begin{aligned} \widetilde{V}_{N+1,t} = \frac{E_Q[\widetilde{X}_{N+1,t+1} + \widetilde{V}_{N+1,t+1}|\mathcal{F}_t]}{1 + (1-\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})\, r_f} - \\ - \frac{\widetilde{\tau}_t E_Q[(\widetilde{X}_{N+1,t+1} + \widetilde{B}_{N+1,t+1} - (1 + r_f(1-\widetilde{\alpha}_{t+1}))\widetilde{B}_{N+1,t})|\mathcal{F}_t]}{1 + (1-\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})\, r_f}. \end{aligned} \tag{18}$$

Proof. Due to our assumptions the after–tax cash flows of the real asset can be duplicated by a trading strategy consisting of financial assets. Then the after–tax cash flows of the real asset are equal to the dividends of the trading strategy. Therefore the value of the real asset must be equal to the value of the trading strategy. Otherwise there would exist an arbitrage opportunity. The results now stems from (7). ■

We proceed with proving Proposition 3. In the absence of taxes equation (18) has the form

$$\widetilde{V}^*_{N+1,t} = \frac{E_Q[\widetilde{X}_{N+1,t+1} + \widetilde{V}^*_{N+1,t+1}|\mathcal{F}_t]}{1 + r_f}. \tag{19}$$

Substituting the equation (19) in equation (18) and thereby eliminating $E_Q[\widetilde{X}_{N+1,t+1}|\mathcal{F}_t]$ leads to the following recursion

$$\begin{aligned}
&\Big\{\widetilde{V}_{N+1,t} - (1-\widetilde{\tau}_{t+1})\widetilde{V}^*_{N+1,t} - \widetilde{\tau}_{t+1}\widetilde{B}_{N+1,t}\Big\} = \\
&\qquad = \frac{E_Q\big[\{\widetilde{V}_{N+1,t+1} - (1-\widetilde{\tau}_{t+2})\,\widetilde{V}^*_{N+1,t+1} - \widetilde{\tau}_{t+2}\,\widetilde{B}_{N+1,t+1}\}|\mathcal{F}_t\big]}{1+r_f\cdot(1-\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})} + \\
&+\frac{r_f\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1}(1-\widetilde{\tau}_{t+1})(\widetilde{V}^*_{N+1,t}-\widetilde{B}_{N+1,t}) + E_Q[(\widetilde{\tau}_{t+1}-\widetilde{\tau}_{t+2})(\widetilde{V}^*_{N+1,t+1}-\widetilde{B}_{N+1,t+1})|\mathcal{F}_t]}{1+r_f\cdot(1-\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})}.
\end{aligned} \tag{20}$$

To prove the proposition we now consider the two cases mentioned in the theorem. We start with the taxation on economic rent.

Case 1: taxation of economic rent. In this case for the real asset book value and fair value without taxes coincide. Then (20) reduces to

$$\widetilde{V}_{N+1,t} - \widetilde{V}^*_{N+1,t} = \frac{E_Q\big[\widetilde{V}_{N+1,t+1} - \widetilde{V}^*_{N+1,t+1}|\mathcal{F}_t\big]}{1+r_f\cdot(1-\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})}.$$

Adding the investment expenses we arrive at

$$\widetilde{\mathrm{NPV}}_{N+1,t} - \widetilde{\mathrm{NPV}}^*_{N+1,t} = E_Q\left[\frac{\widetilde{\mathrm{NPV}}_{N+1,t+1} - \widetilde{\mathrm{NPV}}^*_{N+1,t+1}}{1+r_f\cdot(1-\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1})}\;|\mathcal{F}_t\right].$$

Applying this formula until $t=T$ we get

$$\widetilde{\mathrm{NPV}}_{N+1,t} - \widetilde{\mathrm{NPV}}^*_{N+1,t} = E_Q\left[\frac{\widetilde{\mathrm{NPV}}_{N+1,T} - \widetilde{\mathrm{NPV}}^*_{N+1,T}}{\prod_{s=t+1}^{T}(1+r_f\cdot(1-\widetilde{\alpha}_s\widetilde{\tau}_s))}\;|\mathcal{F}_t\right].$$

Now, since both NPV's at $t=T$ are zero (since the corresponding values and book values are zero) the claim follows.

Case 2: allowance for corporate equity. Now the $\widetilde{\alpha}_t$ satisfy a certain condition that implies

$$\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1}r_f = (1+r_f)\,\frac{\widetilde{\tau}_{t+1}-\widetilde{\tau}_t}{1-\widetilde{\tau}_t}, \qquad \text{and} \quad 1-\widetilde{\alpha}_{t+1}\widetilde{\tau}_{t+1} = \frac{\widetilde{\tau}_t - \widetilde{\tau}_{t+1} + r_f(1-\widetilde{\tau}_{t+1})}{r_f(1-\widetilde{\tau}_t)}$$

Furthermore, the investment expenses coincide with the book values. Using our definition of the NPVs (20) reduces to

$$\begin{aligned}
&\widetilde{\mathrm{NPV}}_{N+1,t} - (1-\widetilde{\tau}_{t+1})\widetilde{\mathrm{NPV}}^*_{N+1,t} = \\
&\qquad = \frac{E_Q\left[\widetilde{\mathrm{NPV}}_{N+1,t+1} - (1-\widetilde{\tau}_{t+2})\,\widetilde{\mathrm{NPV}}^*_{N+1,t+1}\middle|\mathcal{F}_t\right]}{\frac{(1-\widetilde{\tau}_{t+1})(1+r_f)}{1-\widetilde{\tau}_t}} + \\
&\qquad + \frac{(1+r_f)\,\frac{\widetilde{\tau}_{t+1}-\widetilde{\tau}_t}{1-\widetilde{\tau}_t}(1-\widetilde{\tau}_{t+1})\widetilde{\mathrm{NPV}}^*_{N+1,t} + E_Q[(\widetilde{\tau}_{t+1}-\widetilde{\tau}_{t+2})\widetilde{\mathrm{NPV}}^*_{N+1,t+1}|\mathcal{F}_t]}{\frac{(1-\widetilde{\tau}_{t+1})(1+r_f)}{1-\widetilde{\tau}_t}}.
\end{aligned}$$

After some calculations this can be simplified to

$$\widetilde{\mathrm{NPV}}_{N+1,t} - (1-\widetilde{\tau}_t)\widetilde{\mathrm{NPV}}^*_{N+1,t} = \frac{E_Q\left[\frac{1-\widetilde{\tau}_{t+1}}{1-\widetilde{\tau}_t}\left(\widetilde{\mathrm{NPV}}_{N+1,t+1} - (1-\widetilde{\tau}_{t+1})\,\widetilde{\mathrm{NPV}}^*_{N+1,t+1}\right)|\mathcal{F}_t\right]}{1+r_f}.$$

A recursion gives us

$$\widetilde{\mathrm{NPV}}_{N+1,t} - (1-\widetilde{\tau}_t)\widetilde{\mathrm{NPV}}^*_{N+1,t} = \frac{E_Q\left[\left(\widetilde{\mathrm{NPV}}_{N+1,T} - (1-\widetilde{\tau}_T)\,\widetilde{\mathrm{NPV}}^*_{N+1,T}\right)\prod_{s=t+1}^{T}\left(\frac{1-\widetilde{\tau}_s}{1-\widetilde{\tau}_{s-1}}\right)|\mathcal{F}_t\right]}{1+r_f}.$$

and since the NPVs at $t = T$ are zero we have

$$\widetilde{\mathrm{NPV}}_{N+1,t} - (1-\widetilde{\tau}_t)\widetilde{\mathrm{NPV}}^*_{N+1,t} = 0.$$

This is the desired result. ■

A.3 Proof of proposition 4

The price process under the martingale measure is given by[6]

$$dV_t = r_t V_t dt - dX_t + \sigma_t V_t dW_t^Q. \tag{21}$$

Using this equation and $B_t = V_t$ we get for the tax payments

$$\begin{aligned}
dT_t &= \tau_t dX_t + \tau_t dB_t - \tau_t(1-\alpha_t) r B_t dt \\
&= \tau_t \alpha_t r V_t dt + \tau_t \sigma_t V_t dW_t^Q.
\end{aligned}$$

For $dV_t + dX_t - dT_t$ we get

$$\begin{aligned}
dV_t &= r V_t dt - dX_t + \sigma_t V_t dW_t^Q \\
&= r(1-\alpha_t\tau_t)V_t dt - (dX_t - dT_t) + (1-\tau_t)\sigma_t V_t dW_t^Q.
\end{aligned}$$

[6] See (Duffie 1988, chapter 6).

Q is also the martingale measure under taxes. The drift reduces to $r(1-\alpha_t\tau_t)$ and the volatility to $(1-\tau_t)\sigma$.

We are now able to derive a valuation equation for the real asset. We have the two equations

$$\begin{aligned} dV_t^* &= rV_t^* dt - dX_t + \sigma V_t^* dW_t^Q, \\ dV_t &= r(1-\alpha_t\tau_t)V_t dt - (dX_t - dT_t) + (1-\tau_t)\sigma_t V_t dW_t^Q. \end{aligned} \tag{22}$$

Substituting dT_t into the second equation gives according to (13)

$$\begin{aligned} dV_t - \tau_t dB_t = {} & r(1-\alpha_t\tau_t)\left(V_t - \tau_t B_t\right) dt + \\ & + r\alpha_t\tau_t(1-\tau_t)B_t dt - (1-\tau_t)dX_t + (1-\tau_t)\sigma_t V_t dW_t^Q. \end{aligned} \tag{23}$$

Multiplying equation (22) with $-(1-\tau)$ and adding to (23) we get

$$\begin{aligned} dV_t - (1-\tau_t)dV_t^* - \tau_t dB_t = {} & r(1-\alpha_t\tau_t)\left(V_t - (1-\tau_t)V_t^* - \tau_t B_t\right) dt + \\ & + r\alpha_t\tau_t(1-\tau_t)(B_t - V_t^*)dt + (1-\tau_t)\sigma_t(V_t - V_t^*)dW_t^Q. \end{aligned} \tag{24}$$

We denote by $N_t = V_t - I_t$ and $N_t^* = V_t^* - I_T$ the corresponding net present values. Now consider the two cases mentioned in the theorem.

Case 1: taxation of economic rent. In this case (24) reduces to

$$dV_t - dV_t^* + r(1-\alpha_t\tau_t)\left(V_t - V_t^*\right) dt + (1-\tau_t)\sigma_t(V_t - V_t^*)dW_t^Q.$$

Taking investment expenses into account we get

$$\frac{d(N_t - N_t^*)}{N_t - N_t^*} = r(1-\alpha_t\tau_t)dt + (1-\tau_t)\sigma_t dW_t^Q.$$

Since

$$N_T^* = N_T = 0$$

the net present values coincide for all $t \leq T$ (see Liptser and Shiryaev (2001, theorem 4.10)).

Case 2: ACE. Using our assumption the last equation can be formulated as

$$\begin{aligned} dV_t - (1-\tau_t)dV_t^* - \tau_t dI_t = {} & r(1-\alpha_t\tau_t)(V_t - (1-\tau_t)V_t^* - \tau_t I_t)dt + \\ & + r\alpha_t\tau_t(1-\tau_t)(I_t - V_t^*) + (1-\tau_t)\sigma_t(V_t - V_t^*)dW_t^Q. \end{aligned}$$

We show in the lemma below that using our assumptions this can be simplified to

$$d(N_t - (1-\tau_t)N_t^*) = r(1-\alpha_t\tau_t)(N_t - (1-\tau_t)N_t^*) + (1-\tau_t)\sigma_t(V_t - V_t^*)dW_t^Q.$$

Using a standard result in martingale theory (see Liptser and Shiryaev (2001, theorem 4.10)) and $N_T = N_T^* = 0$ we arrive at

$$N_t = (1 - \tau_t)N_t^*.$$

Lemma. Under the assumptions made

$$-\int_0^t r\alpha_s\tau_s(1-\tau_s)N_s^* ds = \int_0^t \tau_s dN_s^* - N_t^*\tau_t + N_0^*\tau_0$$

Proof. Let t_i be a partition of $[0,t]$. We have

$$-\sum_{i=1}^n N_{t_i}^*(\tau_{t_{i+1}} - \tau_{t_i}) = \sum_{i=1}^n \tau_{t_i}(N_{t_{i+1}}^* - N_{t_i}^*) - (N_t^*\tau_t - N_0^*\tau_0) + \sum_{i=1}^n (\tau_{t_{i+1}} - \tau_{t_i})(N_{t_{i+1}}^* - N_{t_i}^*)$$

The left hand side converges with $n \to \infty$ to

$$-\int_0^t r\alpha_s\tau_s(1-\tau_s)N_s^* ds$$

given the assumption on the α. We have to show that the last sum on the right hand side converges to zero. For $n \to \infty$ it converges to the quadratic covariation $< N^*, \tau >_t$ which is an increasing process. But we furthermore know that τ_t is $\mathcal{F}_{t+\delta}$–measurable and if δ is large enough the terms $(\tau_{t_{i+1}} - \tau_{t_i})$ and $(N_{t_{i+1}}^* - N_{t_i}^*)$ are independent. Hence

$$E[\sum_{i=1}^n (\tau_{t_{i+1}} - \tau_{t_i})(N_{t_{i+1}}^* - N_{t_i}^*)] = \sum_{i=1}^n \left(E[\tau_{t_{i+1}}] - E[\tau_{t_i}]\right)\left(E[N_{t_{i+1}}^*] - E[N_{t_i}^*]\right).$$

In fact this is the quadratic covariation of $< E[\tau], E[N^*] >_t$. Since both expectations have bounded variation its quadratic covariation is zero or

$$< E[\tau], E[N^*] >_t = 0.$$

An increasing process that has expectation zero a.e. must be zero itself a.e. This proves the lemma. ■

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