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A Simple Model for Credit Contagion

ABSTRACT

We propose a simple and implementable model of credit contagion where we include macro- and microstructural dependencies among the debtors within a credit portfolio. We show that, even for diversified portfolios, moderate microstructural dependencies have already a significant impact on the tails of the loss distribution. This impact increases dramatically for less diversified microstructures. Since the inclusion of microstructural dependencies acts on the tails, the choice of an appropriate risk measure for credit risk management is a delicate task.

JEL Classification Codes: C19, C69, G18, G21.

Key Words: Credit Portfolio Risk Management, Contagion, Macroeconomic Dependence, Microstructural Dependence, Value-at-Risk, Expected Shortfall

We propose a simple model of credit contagion. We include both macro- and microstructural dependencies among the debtors within a given credit portfolio and firms, which are not necessarily part of the credit portfolio. The model is “simple”, since our approach derives in a direct and natural way from the problem formulation, is implementable and, at the same time, meets the business needs of the bank's credit department.

A major concern for measuring and managing risk of credit portfolios is a high degree of dependence of defaults. Such dependence may be caused by both macro- and microstructural channels. The macrostructure relates to the sensitivity of debtors to common factors such as, e.g., changes in macroeconomic or industry-sector specific fundamentals. For portfolios that are not well diversified in terms of their common factors, such a macrostructural dependence causes defaults to be positively correlated. Recent examples are the banking problems in Japan, the Asian crisis, and the Russian meltdown. However, pure macrostructural models can hardly accommodate the high degree of correlation found in empirical data on defaults.

The microstructure captures dependencies between debtors that go beyond their exposure to common factors and is caused by, e.g., business or legal dependencies. These microstructure dependencies can lead to so-called “contagion”. Hence, in our model we define contagion as a transmission effect that underlies a microstructure dependence. Contagion risk is then the risk that the credit deterioration of a counterparty triggers the credit deterioration of other counterparties through microstructural channels.

To account for the two dependence structures, we consistently embed two different types of data into a simple model for credit contagion. First, we calibrate our model to statistical data represented by a credit migration or transition matrix. Credit migration matrices characterize the past changes in the credit quality of debtors. They are cardinal inputs to most risk management applications, including portfolio risk assessment, pricing credit derivatives, and the calculation of economic and regulatory capital requirements. Transition matrices are either obtained by internal models or they are provided by rating agencies such as Moody's and Standard & Poor's. The matrix entries measure default as the average frequency with which debtors of the same rating have defaulted. But there are many other ways to define credit migration.

As a second input to our model, we use additional counterparty specific data. These data contain accounting figures, business data or expert knowledge based on self-assessment by customer advisors. Some of the data are private information to the bank, but often are also publicly available. E.g., large real estate companies in Switzerland explicitly state their business dependence with the five largest renters in terms of percentage of total income. We model such microstructural information using weighted graphs. The graph's nodes represent firms and the graph's edges constitute the business dependencies between the different firms. The firms can be part of the loan portfolio and, hence, be the bank's debtors or they can just have a dependence with some of the debtors. The edge's weight represents the strength of the microstructural dependence. These weights can be measured, e.g., by the value of business volume.

In historical credit data, macro- and microstructural causes for default and rating migration are interwoven. It is difficult to assess how much of the empirical default and transition probabilities can be attributed to macroeconomic causes or must have been triggered by a propagation of economic distress through a microstructural channel. Moreover, the rating processes in a financial institution incorporate very often some additional quantitative or subjective assessments of the debtor's business environment to fine-tune the statistically calibrated rating matrix. Therefore, if a financial institution estimates its own transition matrix from internal historical data, it will inevitably contain both types of causes. Since rating agencies consider large populations of global and foremost medium- or large-sized firms, we claim that their migration matrix is approximatively independent of microstructural effects. The calibration of our macrostructure model hinges upon the use of such a diversified transition matrix that is approximatively free of microstructural effects.

We contribute to the literature by consistently incorporating both micro- and macrostructural dependencies. This integration allows us to fully investigate the impact of microstructural dependencies on credit portfolios and to identify their marginal contribution to the overall risk of a credit portfolio. We find that microstructural dependence significantly increases the correlation among debtors and fattens the tails of the credit loss distribution. Even for well-diversified credit portfolios with moderate microstructural dependencies, we find striking evidence for a significant increase in rating volatility and migration speed. The high volatility can be attributed to feedback effects channelled through business dependencies, even if these are moderate. A pure macrostructural approach totally neglects all these effects and, therefore, substantially underestimates the credit portfolio's risk. Finally, since business dependencies are mostly tail effects, our findings hint at the danger of using Value-at-Risk to manage credit risk and give strong support for tail-sensitive risk measures such as expected shortfall.

Our paper is structured as follows. Section I presents some background information on previous endeavors to incorporate correlated default and contagion within a credit risk model. Section II presents our approach. Section III shows how to calibrate our model to macroeconomic data. In Section IV, we explore the microstructure by using numerical examples. Section V concludes.

I. Background

Credit risk models fall into two broad categories, the structural and the reduced-form models. The structural models, inspired by the seminal work of Black and Scholes (1973), are first studied in Merton (1974) and variants of it in, e.g., Geske (1979), Mason and Bhattacharya (1981), Shimko, Tejima and van Deventer (1993), Buffet (2000). Reduced-form models were initiated by Pye (1974) and Litterman and Iben (1991), extended and formalized in, e.g., Lando (1998) and Jarrow and Turnbull (1998). Since multiple defaults can be made independent conditional on a set of state variables, the reduced-form models lend themselves to model conditionally independent default times. Lando (1998) exploits this additional flexibility. To generate realistic default correlations, Jarrow, Lando and Yu (2003), Duffie and Singleton (1999), and recently Yu (2002), Gagliardini and Gourieroux (2003) extend the model of Lando (1998). Yu (2002) disproves the common belief that reduced-form models are incapable of generating the default correlation seen in the data (see, e.g., Hull and White (2001), Schoenbucher and Schubert (2001)).

The financial health of a firm and therefore the creditworthiness of a potential debtor is influenced in a substantial manner by the prevailing macroeconomic conditions. This is the usual starting point for the stochastic modeling of the fluctuation of default probabilities or credit ratings (see, e.g., Lando (1998), Duffie and Singleton (1999), Duffee (1999), and Driessen (2002)). A synchronized dependence on macroeconomic fundamentals can lead to a clustering of defaults around an economic recession. Indeed, we observe a high degree of cyclical default correlation in real data (see, e.g., Keenan (2000)).

However, the dependence on macroeconomic fundamentals is only one channel that leads to correlated default. The propagation of financial distress from one firm to another is, to a large part, caused by business and legal dependencies. These dependencies not only amplify economic distress, but also serve as channel for defaults triggered by other than pure macrostructural factors. Among others, Lang and Stulz (1992) show that bankruptcy filings do have a significant impact on stock returns of non-defaulted companies.

The literature on credit risk models so far has paid little attention to microstructural dependence. Only in recent years have we seen a growing literature where more refined dependence structures are introduced. To our best knowledge, the first publications which address contagion risk are Davis and Lo (1999) and (2001) proposing infectious defaults, and Jarrow and Yu (2001) generalizing existing reduced-form models to include default intensities dependent on the default of a counterparty. Yu (2002) extends the model of Jarrow and Yu (2001) by allowing for firm-specific information to model the default process. The joint distribution of default times is implied by the default intensities and, therefore, bypasses the specification of an ad-hoc copula that introduces substantial model risk (see, e.g., Frey and McNeil (2002)).

Giesecke and Weber (2002) introduce credit contagion models on the basis of interacting particle systems. Furthermore, Giesecke and Weber (2003) make a first attempt to integrate macrostructures, causing cyclical default correlations, and contagion phenomena, associated with the local interaction of debtors with their business partners.

Recently, Frey and Backhaus (2003) present an approach within a Markovian setting that incorporates interacting defaults and counterparty risk. Their approach is based on intensity-based credit models and uses results from mean-field theory. However, their model is numerically intensive and it remains unclear how it is calibrated to large and heterogenous portfolios.

We follow a more direct approach, where we incorporate microstructural data that are available in the bank's credit risk department. By using a graph representation, we construct a topological risk map of the bank's credit portfolio. Therefore, from a practical perspective, possible infections within the portfolio can be detected, traced back to a single counterparty, and hence counteracted in the most appropriate way.

II. The Model

A. Basic Assumptions

In this section, we present the basic assumptions underlying our modeling approach. As we claim in the introduction, the building blocks of our model are the transition matrices and the availability of debtor specific data on business dependencies. To unify these two blocks within a consistent framework for credit risk modeling, we require some assumptions.

The creditworthiness of a debtor is given by her rating. The rating dynamics are described by a stationary discrete-time Markov chain. The stationarity assumption can easily be removed by adding cyclical trends. Indeed, it is widely held that default rates are negatively correlated with real economic activity over the business cycle.[1] However, without loss of generality, we prefer to work under the stationarity assumption.

Since credit defaults may be transmitted through business related dependencies in a most pernicious way, the integration of both macro- and microstructure on a business level is of fundamental importance. Our modeling approach imposes the following two assumptions.

Assumption 1. *The rating dynamics depends both on macrostructural as well as microstructural variables.*

To formalize Assumption 1, we model an explicit microstructure not by relying on a statistical methodology but on a causal model based on weighted graphs. However, the integration of macro- and microstructure should be carried out with care. It is a reasonable assumption that

[1] See, e.g., Fons (1991) and Jonsson and Fridson (1996).

a financial institution is able to assess correctly the rating migration probabilities of individual debtors. Therefore, at the portfolio level, if we add macro- and microstructural dependence between debtors, we need to gauge our model to respect the above business assumption. We therefore assume:

Assumption 2. *Adding macro- and microstructure dependency at the portfolio level does not change the migration probabilities at the individual debtor level.*

Although the unconditional default probabilities for individual debtors are left unchanged, the new dependencies may alter the conditional default probabilities.

B. Rating Dynamics

Given a debtor i, we describe her creditworthiness by d different rating classes $\mathcal{X} = \{1, \ldots, d\}$ with d the absorbing default state. By assumption, the rating dynamics of a single debtor i follow a discrete-time process $X_i = (X_i(t))_{t \geq 0}$. Given a credit portfolio with N different debtors, the joint rating dynamics $X(t)$ follows the Markov chain $X(t) = (X_1(t), \ldots, X_N(t))' \in \mathcal{X}^N$. The transition probabilities between the different rating classes are summarized in a transition matrix $\mathbf{T}_{xy}$. Formally, we write for a transition from a portfolio state $x = (x_1, \ldots, x_N)$ to a state $y = (y_1, \ldots, y_N)$

$$\mathbf{T}_{xy} = \mathbb{P}\left[X(t+1) = y \mid X(t) = x\right]. \tag{1}$$

So far, expression (1) is defined on the portfolio level. This generality is not feasible in practice if, say, several ten thousands of debtors are within a portfolio. To reduce the complexity, we follow a standard approach and model the rating migrations of the debtors $i = 1, \ldots, N$ as conditionally independent with respect to some random vector $\mathcal{Z}$. This leads to the following product structure

$$\mathbf{T}_{xy} = \mathbb{E}\left[\prod_{i=1}^{N} \mathbf{T}^{(i)}_{x_i y_i} | \mathcal{Z}\right] , \tag{2}$$

where

$$\mathbf{T}^{(i)}_{x_i y_i} | \mathcal{Z} = \mathbb{P}\left[X_i(t) = x_i \mid X_i(t-1) = y_i, \mathcal{Z}\right] , \tag{3}$$

is the individual conditional transition matrix of debtor i. Consequently, equation (2) translates the dependence structure between the debtors into a dependence between the coordinates of $\mathcal{Z}$. A conditional independence structure is not only more intuitive than a generic model (1) but also computationally more attractive.

The causal model behind the rating dynamics is considered in two steps. In the first step, we model the macroeconomic causality in terms of a latent variable model. In the second step, we further decompose the idiosyncratic part of the latent variable to incorporate the microstructure.

C. Macrostructure

We have fully specified the dynamics of the rating process once the conditional independence structure and the conditional migration matrices $\mathbf{T}^{(i)}|\mathcal{Z}$ in (2) are determined. To simplify (2) further, we group the debtors into a small number K of different sectors according to some classification scheme. The resulting classification function

$$s : \{1, \ldots, N\} \to \{1, \ldots, K\} \ , \tag{4}$$

maps every debtor to one of these sectors and (2) reduces to

$$\mathbf{T}_{xy} = \mathbb{E}\left[\prod_{i=1}^{N} \mathbf{T}^{s(i)}_{x_i y_i} | \mathcal{Z}\right] \ , \tag{5}$$

To model the macrostructure effects, we assume that the independence generating variables $\mathcal{Z}$ are given by systematic sector-specific risk factors

$$\mathcal{Z} = Z = (Z_s)_{s=1,\ldots K} \sim N(0, \mathbf{\Lambda}), \qquad \mathbf{\Lambda} = (\lambda_{i,j})_{i,j=1,\ldots,K},$$

and denote by

$$Y = (Z_{s(i)})_{i=1,\ldots N} \sim N(0, \mathbf{\Gamma}), \qquad \mathbf{\Gamma} = (\lambda_{s(i),s(j)})_{i,j=1,\ldots,N},$$

the vector of systematic factors on the debtor specific level and its covariance matrix.

We express the conditional migration matrices $\mathbf{T}^s|Z$ in terms of a latent variable model, very much in the spirit of generalized linear models.[2] For each debtor i, we define a synthetic asset return A_i given by a univariate standard Gaussian latent variable

$$A_i = \sqrt{1 - w_{s(i)}^2}\, Z_{s(i)} + w_{s(i)}\epsilon_i \ , \quad A_i \sim N(0,1). \tag{6}$$

The synthetic return in (6) consists of two components, a sector-specific and an idiosyncratic debtor-specific component. The vector $\epsilon = (\epsilon_i)_{i=1,\ldots,N} \sim N(0, \mathbf{1}_N)$ collects debtor-specific factors within the credit portfolio. By construction, the ϵ_i are mutually independent. The debtor-specific factors enter the synthetic asset return with sector specific weights

$$w = (w_s)_{s=1,\ldots,K}. \tag{7}$$

The vector Z can be interpreted as the synthetic risks of the different sector classes. We assume that the vectors Z and ϵ are independent. Consequently, Z serves as a conditional independence structure for the latent variable A_i.

To model the credit migration matrix, we assume that a credit migration from a rating class x to a new class y is triggered whenever A_i exceeds some threshold value θ_{xy},

$$-\infty = \theta_{xd+1} \leq \theta_{xd} \leq \ldots \leq \theta_{x2} \leq \theta_{x1} = \infty \ . \tag{8}$$

Thus, for a debtor i the transition probability of migrating at time t from class x into a new rating class y at time $t+1$ is

$$\mathbf{T}_{xy}^{s(i)} = \mathbb{P}\left(X_i(t+1) = y \mid X_i(t) = x\right) = \mathbb{P}\left(A_i \in [\theta_{x(y+1)}, \theta_{xy})\right) \ . \tag{9}$$

The threshold values $\theta = (\theta_{xy})$, the correlation matrix $\mathbf{\Lambda}$ and the specific risk weights $w_s = (w_1, \ldots, w_K)$ are free model parameters, which we will calibrate to historical default data.

[2] See, e.g., McCulloch and Nelder (1989) and Fahrmeir and Tutz (2001).

D. Integrating Microstructure

So far, the return specification in equation (6) only allows to incorporate macrostructural dependence or sector-specific risks. However, we aim at explicitly modeling business dependencies. Therefore, we provide a different specification of the return A_i. To do so, we introduce some further notation.

Given N debtors, we write the macrostructural (or sector-specific) weights on the individual debtor level as

$$v = (w_{s(i)})_{i=1,\dots,N}. \tag{10}$$

Also on the debtor level, we indicate the strength of the business dependence between two counterparties with the matrix

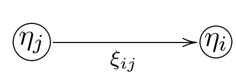
$$\mathbf{\Xi} = (\xi_{ij})_{i,j=1,\dots,N}, \tag{11}$$

which we call the "business matrix". A proxy for the weights ξ_{ij} might be the business volume of debtor j with one of its non-substitutable counterparties i. We restrict our analysis to positive business relations. For all debtors without a significant microstructural dependence, we assume that the weights are set to zero. Furthermore, since a firm does not self-affect itself, we assume $\xi_{jj} = 0$ for $j = 1, \dots, N$. The vector

$$\eta = (\eta_i)_{i=1,\dots,N} \tag{12}$$

finally attaches weights to the residual risk of each debtor. In contrast to the macrostructural setup in Section C where the parameters are statistically calibrated, the microstructural model parameters $\mathbf{\Xi}$ and η are determined by expert knowledge, reflecting the bank's credit expertise. With the notation at hand, we formalize the microstructure for a credit portfolio with N debtors.

Definition 3. *A microstructure for a collection of counterparties $\mathcal{C} = \{1, \dots, N\}$ is a directed weighted graph $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \mathbf{\Xi}, \eta)$. The nodes correspond to the counterparties. A directed weighted edge $\mathcal{E}_{ji}$*



from j to i indicates that the firm j has an counterparty risk effect on i of strength ξ_{ij} given by the edge weights. The node weight η_i represents the residual risk of debtor i.

Figure 1 presents a glimpse of where we are headed. Panel (A) illustrates the dependencies in a pure macrostructural model. The credit rating depends only on the macrostructural variables $Z_{s(i)}$ and $Z_{s(j)}$. Firm i cannot be directly influenced by the variable $Z_{s(j)}$, nor can firm j be influenced by $Z_{s(i)}$. The only dependence is generated through $\mathbf{\Gamma}$. However, by introducing a business dependence ξ_{ij}, the credit rating of firm i now becomes dependent on both the macrostructural variable $Z_{s(j)}$ and the idiosyncratic risk ϵ_j through her business relation with firm j. At the same time, the residual firm-specific idiosyncratic risk for firm i reduces to η_i. We note that our model also allows asymmetric effects. Since we work in a directed graph, we can impose a weight ξ_{ji} which differs from ξ_{ij}. Then, the financial health of firm j can be infected by $Z_{s(i)}$ through her business relation with firm i.

In general, the collection of counterparties includes all firms or individuals which affect the creditworthiness of debtors. Hence, also firms which are not the bank's debtors contribute indirectly to the bank's credit risk through business relations with other debtors.

To integrate a microstructure $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \mathbf{\Xi}, \eta)$ into a macrostructural model, we set $A = (A_i)_{i \in \mathcal{C}}$ for the vector of all synthetic asset returns. With the above notation, we write the macrostructural model in equation (6) in a more compact form as[3]

$$A = \mathbf{D}\left(\sqrt{1 - v^2}\right) Y + \mathbf{D}\left(v\right) \epsilon. \tag{13}$$

The integration has to satisfy Assumptions 1 and 2. To satisfy Assumption 1, we extend the macrostructural equation (13) by making the idiosyncratic part of A dependent on the graph structure of the portfolio. Therefore, we replace ϵ in equation (13) by a normalized function $\varepsilon(\mathcal{G}, Y, \epsilon)$, which depends on the graph, the idiosyncratic component ϵ and possibly on the macroeconomic factors as well. We satisfy Assumption 2 by imposing the following two compatibility conditions:

[3] If v is a vector, $\mathbf{D}(v)$ is the diagonal matrix with diagonal elements v. If f is a real valued function of a real variable then $f(\mathbf{D}(v)) = \mathbf{D}(f(v))$, where $f(v) = (f(v_i))$ is defined componentwise. If $\mathbf{\Gamma}$ is a square matrix, then $\mathbf{D}(\mathbf{\Gamma})$ denotes the diagonal matrix with the same diagonal as $\mathbf{\Gamma}$.

(C1) Compatibility with macrostructural specific risk. Sufficient is the condition that $\varepsilon(\mathcal{G}, Y, \epsilon)$ is normalized to standard normal.

(C2) Compatibility with exogenously prescribed transition probabilities. A sufficient condition is that the marginals of A remain standard normal.

Therefore, to consistently add microstructural dependence we replace equation (13) by

$$A = \mathbf{D}_Y \mathbf{D}\left(\sqrt{1-v^2}\right) Y + \mathbf{D}_\varepsilon \mathbf{D}(v)\, \varepsilon(\mathcal{G}, Y, \epsilon), \tag{14}$$

where the two normalizing diagonal matrices, $\mathbf{D}_Y$ and $\mathbf{D}_\varepsilon$ need to be determined such that the conditions **(C1)** and **(C2)** are fulfilled. We note that the pure macrostructural specification in (13) is just a special case of the specification in (14) with $\varepsilon(\mathcal{G}, Y, \epsilon) = \epsilon$ and $\mathbf{D}_Y = \mathbf{D}_\varepsilon = \mathbf{1}$.

D.1. Recursive Integration of Microstructural

There is no canonical way to specify the microstructural dependence in the idiosyncratic term $\varepsilon(\mathcal{G}, Y, \epsilon)$. In this paper, we specify A as having a direct linear effect on $\varepsilon(\mathcal{G}, Y, \epsilon)$, i.e., we set

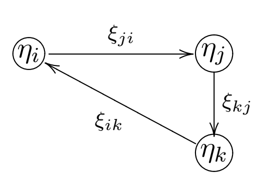
$$\varepsilon(\mathcal{G}, Y, \epsilon) = \mathbf{\Xi} A + \mathbf{D}(\eta)\, \epsilon. \tag{15}$$

Then,

$$A = \mathbf{D}_Y \mathbf{D}\left(\sqrt{1-v^2}\right) Y + \mathbf{D}_\varepsilon \mathbf{D}(v) \left(\mathbf{\Xi} A + \mathbf{D}(\eta)\, \epsilon\right), \tag{16}$$

where $\mathbf{D}_Y$ and $\mathbf{D}_\varepsilon$ need to be determined such that conditions (**C1**) and (**C2**) are fulfilled.

Representation (16), although simple, already provides a rich structure for business dependencies. It allows us to take into account both indirect and cyclic influences. A possible situation is stylized below.



The asset return of firm i may affect firm j's asset return. The business dependence strength ξ_{ji} indicates the infection intensity. Firms j and i can be members of different sectors and, hence, differ in their dependence on macroeconomic variables. Therefore, firm j may import a microstructural shock from a different sector via the microstructure channel $\mathcal{E}_{ij}$. In addition, feedback effects may occur. Consider an idiosyncratic event that hits firm i and deteriorates her financial health and possibly her rating. This deterioration affects firm j through its business relation with i. Since firm j has business relations with firm k, firm k's financial health also deteriorates. If firm k has business relations with firm i, firm i's financial health deteriorates even further. Hence, the idiosyncratic event for firm i induces a feedback.

We rewrite (16) as

$$A = \mathbf{C}_Y Y + \mathbf{C}_\epsilon \epsilon, \tag{17}$$

with

$$\begin{aligned}
\mathbf{C}_Y &= \left(\mathbf{1} - \mathbf{D}_\varepsilon \mathbf{D}\left(v\right) \mathbf{\Xi}\right)^{-1} \mathbf{D}_Y \mathbf{D}\left(\sqrt{1-v^2}\right), \\
\mathbf{C}_\epsilon &= \left(\mathbf{1} - \mathbf{D}_\varepsilon \mathbf{D}\left(v\right) \mathbf{\Xi}\right)^{-1} \mathbf{D}_\varepsilon \mathbf{D}\left(v\eta\right).
\end{aligned}$$

Note that the matrix $\left(\mathbf{1} - \mathbf{D}_\varepsilon \mathbf{D}\left(v\right) \mathbf{\Xi}\right)^{-1}$ is invertible, because the operator norm of $\mathbf{D}_\varepsilon \mathbf{D}\left(v\right) \mathbf{\Xi}$ is strictly less than one.[4] Equation (17) implies that A is still a centered Gaussian vector with covariance

$$\mathrm{Cov}(A, A) = \mathbf{C}_Y \mathbf{\Gamma} \mathbf{C}_Y^\top + \mathbf{C}_\epsilon \mathbf{C}_\epsilon^\top. \tag{18}$$

To guarantee consistency with (**C1**) and (**C2**), we have to proceed in two steps. In the first step, we need to choose $\mathbf{D}_\varepsilon$ such that the idiosyncratic term $\mathbf{D}_\varepsilon \varepsilon(\mathcal{G}, Y, \epsilon)$ in (16) has unit variance. To this end, we calculate the covariance of $\varepsilon(\mathcal{G}, Y, \epsilon)$, which we denote by $\mathbf{\Sigma}$, as

$$\mathbf{\Sigma} = \mathbf{\Xi}\left(\mathbf{C}_Y \mathbf{\Gamma} \mathbf{C}_Y^\top + \mathbf{C}_\epsilon \mathbf{C}_\epsilon^\top\right) \mathbf{\Xi}^\top + \mathbf{D}\left(\eta^2\right) + \mathbf{\Xi} \mathbf{C}_\epsilon \mathbf{D}\left(\eta\right) + \mathbf{D}\left(\eta\right) \mathbf{C}_\epsilon^\top \mathbf{\Xi}^\top. \tag{19}$$

[4] All matrix elements are positive and the row sums are strictly smaller than one.

In the second step, we must choose $\mathbf{D}_Y$ such that A in (17) has unit variance. Therefore, for the recursive integration of microstructural dependencies, the conditions (**C1**) and (**C2**) read

$$\begin{cases} (\mathbf{C1}) & : \quad \mathbf{D}_{\varepsilon}^{2}\mathbf{D}\left(\mathbf{\Sigma}\right)=1, \\ (\mathbf{C2}) & : \quad \mathbf{D}\left(\mathbf{C}_{Y}\mathbf{\Gamma}\mathbf{C}_{Y}^{\top}+\mathbf{C}_{\epsilon}\mathbf{C}_{\epsilon}^{\top}\right)=1. \end{cases} \tag{20}$$

We note that these two normalization steps are dependent, since they result in coupled equations for both $\mathbf{D}_Y$ and $\mathbf{D}_\varepsilon$. The coupling follows from the dependence of $\mathbf{\Sigma}$ on both $\mathbf{C}_Y$ and $\mathbf{C}_\epsilon$ and, hence, on $\mathbf{D}_Y$ and $\mathbf{D}_\varepsilon$ (see equation 19). So far, we are not aware of a closed form solution to (20). For large credit portfolios, finding a numerical solution for equation (20) may turn out to be tedious. Furthermore, since the matrix $\mathbf{\Xi}$ is sparse, the required inversion $\left(\mathbf{1}-\mathbf{D}_{\varepsilon}\mathbf{D}\left(v\right)\mathbf{\Xi}\right)^{-1}$ can be numerically unstable.

D.2. Approximated Recursive Integration of Microstructure

We can circumvent the inversion of $\left(\mathbf{1}-\mathbf{D}_{\varepsilon}\mathbf{D}\left(v\right)\mathbf{\Xi}\right)^{-1}$ by using an approximative procedure. Instead of recursively integrating the microstructure as in (16), we replace A on the right hand side of (16) by the pure macroeconomic effects. Writing the macroeconomic synthetic return as

$$A^{(0)}=\mathbf{D}\left(\sqrt{1-v^{2}}\right)Y+\mathbf{D}\left(v\right)\epsilon, \tag{21}$$

we define the idiosyncratic part as

$$\begin{aligned} \varepsilon^{(1)}(\mathcal{G},Y,\epsilon) &= \mathbf{\Xi}A^{(0)}+\mathbf{D}\left(\eta\right)\epsilon \\ &= \mathbf{\Xi}\mathbf{D}\left(\sqrt{1-v^{2}}\right)Y+\left(\mathbf{\Xi}\mathbf{D}\left(v\right)+\mathbf{D}\left(\eta\right)\right)\epsilon \\ &\equiv \mathbf{E}_{Y}^{(1)}Y+\mathbf{E}_{\epsilon}^{(1)}\epsilon. \end{aligned}$$

Calculating the covariance matrix of $\varepsilon^{(1)}(\mathcal{G},Y,\epsilon)$,

$$\mathbf{\Sigma}^{(1)}=\mathbf{E}_{Y}^{(1)}\mathbf{\Gamma}\mathbf{E}_{Y}^{(1)\top}+\mathbf{E}_{\epsilon}^{(1)}\mathbf{E}_{\epsilon}^{(1)\top}, \tag{22}$$

we observe that (22) is independent of the normalization factors $\mathbf{D}_Y^{(1)}$ and $\mathbf{D}_\varepsilon^{(1)}$. This property allows us to explicitly derive $\mathbf{D}_\varepsilon^{(1)}$ such that we are consistent with condition (**C1**).

In the second step, we must choose $\mathbf{D}_Y^{(1)}$ such that $A^{(1)}$ has unit variance. Given the specification of the idiosyncratic term, we can write the first-order approximation for the synthetic asset return as

$$A^{(1)} = \mathbf{D}_Y^{(1)}\mathbf{D}\left(\sqrt{1-v^2}\right)Y + \mathbf{D}_\varepsilon^{(1)}\mathbf{D}\left(v\right)\varepsilon^{(1)}(\mathcal{G}, Y, \epsilon). \tag{23}$$

Collecting terms, we can express $A^{(1)}$ as

$$A^{(1)} = \mathbf{C}_Y^{(1)}Y + \mathbf{C}_\epsilon^{(1)}\epsilon, \tag{24}$$

where

$$\begin{aligned}
\mathbf{C}_Y^{(1)} &= \mathbf{D}_Y^{(1)}\mathbf{D}\left(\sqrt{1-v^2}\right) + \mathbf{D}_\varepsilon^{(1)}\mathbf{D}\left(v\right)\mathbf{E}_Y^{(1)}, \\
\mathbf{C}_\epsilon^{(1)} &= \mathbf{D}_\varepsilon^{(1)}\mathbf{D}\left(v\right)\mathbf{E}_\epsilon^{(1)}.
\end{aligned}$$

We calculate the covariance of $A^{(1)}$ as

$$\mathrm{Cov}\left(A^{(1)}, A^{(1)}\right) = \left(\mathbf{D}_Y^{(1)}\mathbf{D}\left(\sqrt{1-v^2}\right) + \mathbf{B}^{(1)}\right)\mathbf{\Gamma}\left(\mathbf{D}_Y^{(1)}\mathbf{D}\left(\sqrt{1-v^2}\right) + \mathbf{B}^{(1)\top}\right) + \mathbf{C}_\epsilon^{(1)}\mathbf{C}_\epsilon^{(1)\top},$$

where, for notational convenience, we defined $\mathbf{B}^{(1)} = \mathbf{D}_\varepsilon^{(1)}\mathbf{D}\left(v\right)\mathbf{E}_Y^{(1)}$. We note that since $\mathbf{\Gamma}$ is symmetric, we have $\mathbf{D}\left(\mathbf{B}^{(1)}\mathbf{\Gamma}\right) = \mathbf{D}\left(\mathbf{\Gamma}\mathbf{B}^{(1)\top}\right)$. Therefore, condition (**C2**) which requires $\mathrm{Var}\left(A^{(1)}\right) = \mathbf{D}\left(\mathrm{Cov}\left(A^{(1)}, A^{(1)}\right)\right) = \mathbf{1}$ implies that $\mathbf{D}_Y^{(1)}$ is the solution to the quadratic matrix equation

$$\begin{aligned}
\mathbf{1} &= (\mathbf{D}_Y^{(1)})^2\mathbf{D}\left(\mathbf{\Gamma}\right)\mathbf{D}\left(1-v^2\right) + 2\mathbf{D}_Y^{(1)}\mathbf{D}\left(\sqrt{1-v^2}\right)\mathbf{D}\left(\mathbf{B}^{(1)}\mathbf{\Gamma}\right) + \mathbf{D}\left(\mathbf{C}_\epsilon^{(1)}\mathbf{C}_\epsilon^{(1)\top} + \mathbf{B}^{(1)}\mathbf{\Gamma}\mathbf{B}^{(1)\top}\right) \\
&\equiv \mathbf{Q}(\mathbf{D}_Y^{(1)}).
\end{aligned}$$

The above equation can explicitly be solved for $\mathbf{D}_Y^{(1)}$.

Summarizing the above analysis for the first-order approximated recursive integration model, the normalizing matrices that guarantee consistency with the conditions (**C1**) and (**C2**) read

$$\begin{cases} (\mathbf{C1}) & : \qquad (\mathbf{D}_{\varepsilon}^{(1)})^2 = \mathbf{D}\left(\mathbf{\Sigma}^{(1)}\right)^{-1}, \\ (\mathbf{C2}) & : \qquad \mathbf{1} = \mathbf{Q}\left(\mathbf{D}_{Y}^{(1)}, \mathbf{D}_{\varepsilon}^{(1)}\right). \end{cases} \tag{25}$$

Equation (23) is a first-order approximation to (16) that allows us to express the asset return by an explicit equation. However, the first-order model does not account for any reverse or back-propagation effects. To capture possible feedback effects, we need to consider at least second-order approximations. We obtain the n-th order approximations $A^{(n)}$ by recursively plugging $A^{(n-1)}$ into the definition of the idiosyncratic part $\varepsilon^{(n)}(\mathcal{G}, Y, \epsilon)$.

Proposition 4. *Consider the n-th order approximation of the return A,*

$$A^{(n)} = \mathbf{D}_{Y}^{(n)}\mathbf{D}\left(\sqrt{1-v^2}\right)Y + \mathbf{D}_{\epsilon}^{(n)}\mathbf{D}\left(v\right)\varepsilon^{(n)}(\mathcal{G}, Y, \epsilon), \tag{26}$$

where $\varepsilon^{(0)}(\mathcal{G}, Y, \epsilon) = \epsilon$ and $\mathbf{D}_{Y}^{(0)} = \mathbf{D}_{\epsilon}^{(0)} = \mathbf{1}$. The idiosyncratic part is given as

$$\varepsilon^{(n)}(\mathcal{G}, Y, \epsilon) = \mathbf{E}_{Y}^{(n)}Y + \mathbf{E}_{\epsilon}^{(n)}\epsilon, \tag{27}$$

with

$$\begin{aligned} \mathbf{E}_{Y}^{(n+1)} &= \sum_{i=0}^{n}\left(\prod_{j=0}^{n-i-1}\mathbf{\Xi}\mathbf{D}_{\varepsilon}^{(n-j)}\mathbf{D}\left(v\right)\right)\mathbf{\Xi}\mathbf{D}_{Y}^{(i)}\mathbf{D}\left(\sqrt{1-v^2}\right), \\ \mathbf{E}_{\epsilon}^{(n+1)} &= \sum_{i=0}^{n}\left(\prod_{j=0}^{n-i-1}\mathbf{\Xi}\mathbf{D}_{\varepsilon}^{(n-j)}\mathbf{D}\left(v\right)\right)\mathbf{D}\left(\eta\right) + \prod_{i=0}^{n}\mathbf{\Xi}\mathbf{D}_{\varepsilon}^{(n-i)}\mathbf{D}\left(v\right). \end{aligned}$$

The normalizing matrices $\mathbf{D}_{Y}^{(n)}$ and $\mathbf{D}_{\varepsilon}^{(n)}$ solve

$$\begin{cases} (\mathbf{C1}) & : \qquad (\mathbf{D}_{\varepsilon}^{(n)})^2 = \mathbf{D}\left(\mathbf{\Sigma}^{(n)}\right)^{-1}, \\ (\mathbf{C2}) & : \qquad \mathbf{1} = \mathbf{Q}\left(\mathbf{D}_{Y}^{(n)}, \mathbf{D}_{\varepsilon}^{(n)}\right). \end{cases} \tag{28}$$

Proof. See Appendix A. □

D.3. Autoregressive Integration of Microstructure

Another suitable specification to integrate microstructural dependence is to model specific risk in an autoregressive way, i.e.,

$$A(t) = \mathbf{D}_Y \mathbf{D}\left(\sqrt{1-v^2}\right) Y + \mathbf{D}_\varepsilon \mathbf{D}(v)\left(\mathbf{\Xi} A(t-1) + \mathbf{D}(\eta)\,\epsilon\right). \tag{29}$$

The recursion starts with an exogenous value $A(-1)$. This can be done as follows. Assume that the rating of a debtor i at time $t=-1$ is x and the rating of time $t=0$ is y. Then if the model would have generated this move, we would have had that $A_i(-1) \in [\theta_{x(y+1)}, \theta_{xy})$. Because in our model the asset returns are normally distributed, we choose each component of $A(-1)$ to be truncated normally distributed, where the truncation interval is given by the rating migration thresholds, e.g., for debtor i this interval is $[\theta_{x(y+1)}, \theta_{xy})$. Obviously, this procedure presumes a rating history of at least one time period back. Because Y and ϵ are independent of $A(t-1)$ we can calculate $\mathrm{Cov}(A(t), A(t))$ in terms of $\mathrm{Cov}(Y, Y)$ and $\mathrm{Cov}(A(t-1), A(t-1))$. Resolving the recursion, we end up with $\mathrm{Cov}(A(-1), A(-1))$. However, in the above setup we only fixed the marginal distributions of $A(-1)$ to be truncated normal, but not their correlation structure. Therefore, the autoregressive model adds another degree of freedom that further increases the sensitivity on its initialization. In what follows, we focus on the recursive integration method and not on a autoregressive specification.

III. Calibration to Macroeconomic Data

In this section we calibrate the macrostructural model to historical default rates for the different industry sectors. We call this approach top-down, since we calibrate the individual transition probabilities of every debtor i to a given migration matrix $(\hat{\mathbf{T}}_{xy})$. For the macrostructural model, we need to calibrate three sets of parameters, i.e., the thresholds (θ_{xy}), the risk weights (w_k), and the correlation matrix $\mathbf{\Lambda}$.

First, we have to fix the threshold values (8) in such a way that the resulting transition probabilities ($\mathbf{T}_{xy}$) match the empirical matrix ($\hat{\mathbf{T}}_{xy}$).[5] The standard Gaussian assumption on A_i and (9) imply the calibration condition

$$\hat{\mathbf{T}}_{xy} = \Phi\left(\theta_{xy}\right) - \Phi\left(\theta_{x(y+1)}\right) \ , \tag{30}$$

with $\Phi(\cdot)$ the cumulative distribution function of the standard normal. Therefore, we fix the default threshold value

$$\theta_{xd} = \Phi^{-1}\left(\hat{\mathbf{T}}_{xd}\right), \tag{31}$$

and calculate the remaining thresholds as

$$\theta_{xy} = \Phi^{-1}\left(\hat{\mathbf{T}}_{xy} + \Phi\left(\theta_{x(y+1)}\right)\right) \ . \tag{32}$$

Second, we identify the sector specific risk weights $w = (w_1, \ldots, w_K)$ for the K sectors by using a mean value consideration. Because the risk weights do not depend on the rating, we only take into account sector specific effects and leave aside the debtors' rating. For a representative debtor in each sector class s, we consider the synthetic asset return

$$A_s = \sqrt{1 - w_s^2}\, Z_s + w_s \epsilon \ , \tag{33}$$

where $Z_s, \epsilon \sim N(0,1)$, $Z \sim N(0, \boldsymbol{\lambda})$, and Z and ϵ are independent. A representative debtor in class s defaults, if the asset return falls below a sector-specific threshold valueϑ_s. The threshold value ϑ_s is defined by the unique solution of

$$\mathbb{E}\left[\chi_s\right] = \Phi\left(\vartheta_s\right) = \hat{m}_s \ . \tag{34}$$

where $\chi_s = \mathbb{1}_{\{A_s \leq \vartheta_s\}}$ is the default indicator and $\hat{m}_s$ is the mean value of the historical default rate of the class s. To calculate the sector specific weights w_s, we fit the variance $\mathbb{E}[\chi_s|Z_s]$ of the conditional default indicator for class s to the empirical volatility $\hat{\sigma}_s^2$ of the historical

[5] We assume the migration matrix to be estimated from internal historical rating data or adapted from rating agencies.

default rates. In other words, conditioning w.r.t. Z means averaging over the microstructure and idiosyncratic effects and the conditional expectation $\mathbb{E}[\chi_s|Z_s]$ is then the corresponding orthogonal projection on the macro variables. Hence, we calculate the variance as

$$\operatorname{Var}\left(\mathbb{E}[\chi_s|Z]\right) = \mathbb{E}\left[\Phi\left(\frac{\vartheta_s - \sqrt{1-w_s^2}\,Z_s}{w_s}\right)^2\right] - \mathbb{E}\left[\Phi\left(\frac{\vartheta_s - \sqrt{1-w_s^2}\,Z_s}{w_s}\right)\right]^2 . \tag{35}$$

Since the distribution of Z_s is explicitly known, a nonlinear condition follows

$$\operatorname{Var}\left(\mathbb{E}[\chi_s|Z]\right) = \hat{\sigma}_s^2 . \tag{36}$$

Using a Newton scheme, the solution of the nonlinear equation

$$w_s = f\left(\hat{m}_s, \hat{\sigma}_s\right), \tag{37}$$

is readily found where w_s is a function of the empirical parameters $\hat{m}_s$ $\hat{\sigma}_s$.

Finally, given the risk weights w_s and the thresholds ϑ_s, we can calibrate the correlation matrix $\mathbf{\Lambda}$. To do so, we consider two representative debtors in two sector classes s_1 and s_2. The mixed moments of the conditional default indicators is then, because of conditional independence,

$$\mathbb{E}[\mathbb{E}[\chi_{s_1}|Z]\mathbb{E}[\chi_{s_2}|Z]] = \mathbb{E}[\mathbb{P}(A_{s_1} \leq \vartheta_{s_1}|Z)\mathbb{P}(A_{s_1} \leq \vartheta_{s_1}|Z)] = \mathbb{P}(A_{s_1} \leq \vartheta_{s_1} \wedge A_{s_2} \leq \vartheta_{s_2}) . \tag{38}$$

Because $(A_1, A_2) \sim N(0, \mathbf{R})$ is a bivariate Gaussian with covariance

$$\mathbf{R} = \begin{pmatrix} 1 & \varrho_{s_1 s_2} \\ \varrho_{s_1 s_2} & 1 \end{pmatrix} , \tag{39}$$

we get a nonlinear relation

$$\begin{aligned} \Phi_2(\Phi^{-1}(\hat{m}_{s_1}), \Phi^{-1}(\hat{m}_{s_2}), \varrho_{s_1 s_2}) &= \operatorname{Cor}\left(\mathbb{E}[\chi_{s_1}|Z], \mathbb{E}[\chi_{s_2}|Z]\right)\sigma(\mathbb{E}[\chi_{s_1}|Z])\sigma(\mathbb{E}[\chi_{s_2}|Z]) \\ &\quad + \mathbb{E}\left[\chi_{s_1}|Z_{s_1}\right]\mathbb{E}\left[\chi_{s_2}|Z_{s_2}\right] . \end{aligned} \tag{40}$$

where Φ is the normal distribution function and Φ_2 is the bivariate normal distribution function. The right hand side of (40) can finally be entirely expressed in empirical variables, i.e.

$$\mathrm{Cor}\left(\mathbb{E}[\chi_{s_1}|Z],\mathbb{E}[\chi_{s_2}|Z]\right)\sigma(\mathbb{E}[\chi_{s_1}|Z])\sigma(\mathbb{E}[\chi_{s_2}|Z])+\mathbb{E}\left[\chi_{s_1}|Z_{s_1}\right]\mathbb{E}\left[\chi_{s_2}|Z_{s_2}\right]=\hat{\varrho}_{s_1s_2}\hat{\sigma}_{s_1}\hat{\sigma}_{s_2}+\hat{m}_{s_1}\hat{m}_{s_2}\,. \tag{41}$$

Again, using Newton's scheme a solution for $\varrho_{s_1s_2}$ is readily found. Performing the same procedure for all other sectors, we end up with the calibrated matrix $\mathbf{\Lambda}$. Once $\varrho_{s_1s_2}=\mathrm{Cor}\left(A_1,A_2\right)$ is calculated, the correlation $\lambda_{s_1s_2}=\mathrm{Cor}\left(Z_{s_1},Z_{s_2}\right)$ is determined by

$$\lambda_{s_1s_2}=\mathrm{Cor}\left(Z_{s_1},Z_{s_2}\right)=\frac{\varrho_{s_1s_2}}{\sqrt{1-w_{s_1}^2}\sqrt{1-w_{s_2}^2}}\,. \tag{42}$$

IV. Application

We apply our modeling approach to different credit risk portfolios and explore the resulting risk figures.

A. Data and Test Portfolios

To model the macroeconomic structure, we adopt the industry sector classification from the Swiss agency BAK[6] which uses $K=14$ sectors (see Table I). The historical default rates for the 14 industry sectors follow from a time series starting in 1980 and ending in 1997. The migration matrix has been adapted from Moody's migration matrix based on historical default data between 1970 and 2002 (see Table II). Given the default rates and the migration matrix, we obtain the thresholds (θ_{xy}), the risk weights (w_k), and the correlation matrix $\mathbf{\Lambda}$ using the calibration procedure outlined in Section III. (See Tables I, IV, and V.)

Our test portfolios are based on a collection of counterparties grouped at random into the BAK-industry sectors (see Table I). Each portfolio consists of 102 counterparties. We consider eight rating classes. Rating class 1 corresponds to the highest rating and class 8 to the defaulted class.

[6] Konjunkturforschung Basel AG.

We assume that every counterparty has an initial rating different from the default state. We analyze the credit portfolio under four different initial credit qualities, which range from high to average, low, and very low quality. The portfolio compositions corresponding to these different qualities are taken from Gordy (2000) (see Table III).

So far, we are given the macrostructure and the four credit portfolios with different initial rating quality. We next focus on possible specifications on the portfolios' microstructure. To this end, we consider three different types of microstructural dependencies. The macroeconomic model serves as benchmark.

The first microstructure represents a highly ordered hierarchical group of counterparties in a bank's portfolio (see Figure 2). We assume two large firms in the portfolio with two distinct groups of direct suppliers or direct service providers. Each of the suppliers has a business volume of 30% with one of the two large firms A and B. A and B moderately interact with 10% of their business volume. The direct suppliers of the large firms are non-substitutable counterparties for a second set of firms. The business interaction is again assumed to be 30% of total business volume.

The second microstructure is based on a random dependence pattern between the 102 counterparties. The different counterparties are linked in an unstructured way, as shown in Figure 3. Therefore, we will call this portfolio structure a "Diversified Debtor's Portfolio" (DDP), whereas we relate to the first microstructure as the "Heavy Gravity Portfolio" (HGP).

The above examples are rather stylized portfolios representing two specific microstructural topologies. As a third example, we consider a portfolio, the "Real Business Portfolio" (RBP), which tries to capture some of the typical structures in credit portfolios. The RBP has most likely a less stylized structure than the HGP or the DDP on the global level, although some subportfolios have a structure similar to a DDP or HGP.

We divide the RBP into five subportfolios: real estate, electricity, banking, retail, and weakly-dependent random debtors. The real estate subportfolio is characterized by a dependence structure similar to the HGP. In our example, two large real estate firms build the two central nodes which connect with a large number of renters. Different renters are often independent from each other. However, renters have tenancies with different real estate firms.

Electricity companies typically possess a two-level hierarchical dependence. On the top level, we have electricity holdings. Each holding has several investments in power supplying firms. These firms themselves possess investments in power plants. Similar to the real estate case, different holdings may be invested in the same firms and different firms may be invested in the same power plants.

The banking subportfolio is similar to the electricity graph, i.e., it may have two or even more levels of dependence. However, the dependence obeys no longer a hierarchical order. More likely, the dependence is horizontal between the different subsidiaries and investments of different banking firms due to large and frequent interbanking activities.

Retail graphs are similar to real estate graphs but the number of suppliers of the large retail companies is much larger than the number of renters in the real estate graph. Also, the business dependence goes in the opposite direction. Retail suppliers heavily depend on large companies demanding their goods and services. In contrast, real estate firms depend on their renters. These different dependence directions reflect different market and bargaining power and have eventually an effect on the credit risk of the bank's credit portfolio.

The random class gathers dependencies which are difficult to characterize. Basically, it has the same properties as the DDP, but debtors share only small or negligible business dependence.

In addition to the dependencies outlined above, there are also dependencies between firms of different subportfolios. Consider, e.g., a bank which is (i) a renter in a real estate group, (ii) possesses several subsidiaries in the banking sector, and (iii) is invested in retail and electricity companies. To sum up, the RBP is characterized by several different graph classes and dependencies between them.

To make the RBP comparable to the DDP and HGP, we assume the same number of nodes and edges. Furthermore, we construct the RBP such that the total of the weights in the business matrix equals the total of the DDP and HGP.

B. Simulation Results

We analyze the effect of different dependence structures on the rating distributions for the portfolios of different credit qualities ranging from "high" to "very low" using Monte Carlo simulation. Our time horizon is five years with a yearly time interval. For each specification and each time step, we use a sample size of 10^6 simulations.

B.1. Rating Volatility

By construction, the unconditional default and transition probabilities at the individual counterparty level remain unchanged by adding microstructural dependencies. Consequently, the mean number of debtors in different rating states is the same for all four dependence structures. However, the impact of possible dependence structures acts on the correlation structure of credit migration.

Figures 4 through 6 plot the additional correlation structure due to microstructural dependencies. The additional correlation is expressed as the absolute difference between the correlations in the pure macrostructural model and the recursive integration model. On the left panels, we plot the first-order approximation. On the right panels, we plot the additional correlation in the full recursive integration model. It turns out that there is still a substantial amount of correlation generated by higher order effects. Therefore, for the calculation of risk figures the first-order approximation, which neglects feedback and looping effects, may only give a very crude approximation of the true underlying risk.

A striking feature of the different figures is how the graphical structure of the debt portfolio adds characteristic patterns to the correlation matrix. Whereas the DDP adds correlation in a uniform manner, hierarchical structures such as the HGP give rise to blockwise increases in correlation. For both the HGP and the DDP, the additional correlation is substantial (up to 0.5). In the RBP, both structures, the DDP- and the HGP-structure, are present in subportfolios. Due to strong dependencies between debtors with a large number of edges, the increase in correlation reaches 0.8 for some debtors. Furthermore, as we see from the left panel of Figure 6, neglecting feedback effects in the RBP heavily underestimates the additional correlation.

Contrary to the mean, the volatility about that mean is expected to be different and dependent on the underlying microstructure. This change is caused by the change in the correlation structure. A higher correlation leads to an increase in the speed of migrating through the different categories of the transition matrix. The higher migration speed will eventually have a significant impact on the volatility and, hence, on the credit risk calculations.

Figures 7 and 8 compare the volatility surfaces implied by a pure macroeconomic model and different microstructural dependence models. The surfaces are parameterized by the time horizon and the rating classes. The upper panels of Figure 7 show the rating volatilities for an initial portfolio with a high and a very low rating quality, respectively. The lower surface represents the volatility surface in the macroeconomic model and the upper surface represents the volatility in the HGP. For a portfolio with high quality, we see that microeconomic dependencies impact the medium rating classes the most. As expected, the impact becomes more pronounced with increasing time-horizon. When we consider a portfolio with a very low rating quality, the strongest impact is on the lowest rating class, i.e., on the defaulted firms. Hence, if we exclude microeconomic dependencies, we will underestimate the volatility of the default class.

The lower panels of Figure 7 show the rating volatility for the DDP. The figure on the left assumes a high, the figure on the right a very low initial rating quality. We see that the impacts of microstructural dependencies exhibit the same patterns for the different portfolio qualities as in the HGP. However, the diversified structure of the business dependencies in the DDP dampens the impacts of adding microstructural dependencies. This finding justifies, at least partially, our calibration approach in Section III. There, we argue that the migration matrices are estimated from a large global sample of firms where microstructural effects diversify away to a large extent.

Figure 8 shows the volatility surfaces for the RBP. As we can expect from the discussion of the additional correlation in the RBP (see Figure 6), the increase in migration volatility is substantial. Again, the figures exhibit the same qualitative structure as the volatility surfaces in the DDP and the HGP. For high initial portfolio qualities, the inclusion of microstructural dependencies is most pronounced for medium ratings. For very low initial portfolio qualities, the increase in volatility materializes most prominently for the default class. Indeed, the increase almost reaches a factor of two. Such an increase will have a significant impact on the calculations of the loss distribution.

B.2. Value-at-Risk and Expected Shortfall

We explore next the effects of our previous findings concerning the rating volatility on the default risk and the loss distribution. We consider Value-at-Risk and expected shortfall as risk measures. We focus on the probability distribution of potential credit losses. Therefore, we only consider the rating counts in the rating class 8 for the macroeconomic model and the different microstructural dependencies. To calculate the risk figures, we use four different confidence levels, the 95%, 97.5%, 99%, and the 99.5% confidence level.

Tables VI, VII, and VIII collect the simulation results. All entries are percentage increases in the corresponding risk figures due to microstructural dependence. So, e.g., for Value-at-Risk we calculate $(\mathrm{VaR}^{\mathrm{macro\,+\,micro}} - \mathrm{VaR}^{\mathrm{macro}})/\mathrm{VaR}^{\mathrm{macro}}$ in percentage numbers and the same for expected shortfall.

In Table VI, the entries are the risk figures for the DDP. This portfolio exhibits, by construction, a diversified microstructural dependence with a small impact on the rating volatility. However, already a small change in the rating volatility seems to have a large effect on the resulting risk figures. Depending on the initial portfolio quality and the confidence level, the expected shortfall of the debt portfolio can be underestimated by as much as 59% compared to a pure macrostructural model. Using Value-at-Risk, this difference is slightly less accentuated.

In Table VI with a diversified microstructure, we find some evidence pointing to the danger of using Value-at-Risk as relevant risk figure. Given a high initial portfolio quality and a one year horizon, the Value-at-Risk figure at the 95% confidence level does not pick up any additional risk from microstructural dependence. This finding indicates that microstructural dependence contributes significantly to the tails of the loss distribution. For the less diversified HGP, the above results become more pronounced (see Table VII). Furthermore, Value-at-Risk again fails to account for microstructural dependence on low quantile levels.

Surprisingly, the effects discussed above for synthetically constructed portfolios are even stronger in the RBP. Table VIII shows that for our real-life example, the macroeconomic model underestimates portfolio risk by more than 193% of expected shortfall for the one-year horizon and a confidence level of 99.5%. Furthermore, for the same time-horizon the Value-at-Risk calculations for both the 95% and the 97.5% confidence level do not pick up at all any microstructural dependencies.

B.3. Importance of Feedback Effects

To clarify the impact of feedback effects, we study the difference between risk figures obtained by a first-order approximation with the risk figures from the full model. This difference captures all additional risk caused by feedback and looping amongst the firms within the graph. Table IX reports the percentage increase in Value-at-Risk and expected shortfall caused by feedback effects for different confidence levels, i.e., we calculate $(\mathrm{VaR}^{(\infty)} - \mathrm{VaR}^{(1)})/\mathrm{VaR}^{(\infty)}$ in percentage numbers for the Value-at-Risk and $(\mathrm{ES}^{(\infty)} - \mathrm{ES}^{(1)})/\mathrm{ES}^{(\infty)}$ in percentage numbers for the expected shortfall (ES).

From Table IX we note that feedback effects are tail effects, since they heavily increase the tails of the distribution. For a one year horizon and a high initial portfolio rating, the influence of feedback effects from microstructural dependencies on the Value-at-Risk is zero for all confidence levels below 97.5%. Only for a 99% confidence level, there is an increase in Value-at-Risk of around 25%. For the 95% confidence level, there is an increase in Value-at-Risk caused by feedback effects only after a three year horizon. However, if we measure risk as expected shortfall, we see an increase of more than 25% in risk already for the one year time horizon and the 95% confidence bound. This finding follows from the fact that, contrary to Value-at-Risk, expected shortfall takes into account all tail events. Therefore, expected shortfall gives a better picture on the portfolio's true tail risk.

B.4. Marginal Risk Contribution

From a practical viewpoint, it is of paramount importance to quantify the marginal risk from adding an additional counterparty to the bank's credit portfolio. This marginal risk eventually enters the pricing of loans. To analyze marginal risk contributions in the RBP, we consider the marginal contributions of five representative debtors. Debtor 1 is a large real estate firm, debtor 2 an electricity holding, debtor 3 a large bank holding, debtor 4 a large retail company, and debtor 5 is a small debtor with only small microstructural dependence. For our analysis, we assume that debtors 3 and 4 share the same sector specific risk factor.

Figure 9 plots the marginal risk contributions for a one-year and a three-year time horizon. We use a sample size of 10^9 simulations and calculate both the additional Value-at-Risk and expected shortfall that result by adding one of the debtors to the portfolio. The y-axis plots the additional risk per dollar of the credit portfolio's notional as a function of the confidence level. A negative additional risk means that by adding the corresponding debtor, the credit risk manager can exploit some diversification effect, since one dollar of the credit portfolio would have to bear less risk than before.[7] However, as we see from Figure 9, adding an additional debtor does not always result in a better diversification. Indeed, the existence of microstructural dependence may lead to anti-diversification and to an increase in the relative portfolio risk (expressed as risk per dollar). As expected, the contribution of adding a small debtor to the portfolio (debtor 5) is almost zero.

For the one-year horizon, a striking property of our calculations is that the marginal Value-at-Risk for debtors 1 through 5 does not differ significantly below the 98% confidence level. However, the marginal expected shortfall differs already at the 95% confidence level. Again, this salient feature is attributed to the fact that microstructural dependence acts on the tails, which becomes most evident for debtor 3. Debtor 3's Value-at-Risk dramatically increases beyond the 98% confidence level. This increase is anticipated in the expected shortfall with lower confidence levels. Therefore, even an expected shortfall with low confidence level already gives a reliable estimate of tail risks.

[7] This decrease in relative risk does not mean that the absolute risk level of the credit portfolio becomes lower.

Marginal Value-at-Risk and expected shortfall need not be order preserving. Consider, e.g., the 95% confidence level as the relevant confidence level. At this level, debtor 3 seems to lower the risk per dollar, if risk were to be defined as Value-at-Risk. However, if we consider expected shortfall, debtor 3 increases the portfolio's risk per dollar. For a confidence level of 97%, the opposite conclusion would hold if we analyze debtor 4. Therefore, Value-at-Risk leads to a different assessment of a debtor's marginal risk contribution compared to expected shortfall and, hence, gives a wrong guidance for loan pricing.

Considering a time horizon of three years, we see that the microstructural dependencies not only influence the very outer tails. Their influence seems to spread also across lower confidence levels such that the Value-at-Risk better picks up the loss distribution. The difference between expected shortfall and Value-at-Risk becomes less severe. However, we still observe a large bias for debtor 2, when we would only compare risk figures below the 97% confidence level. Therefore, even for longer time horizons we suggest to use expected shortfall instead of Value-at-Risk.

What would the marginal risk contributions look like, if we only used a pure macroeconomic model? Figure 10 plots the same numbers as in Figure 9, but all calculations are based on a pure macroeconomic model. Since microstructural dependencies are absent, Value-at-Risk and expected shortfall are almost the same. For the one-year time horizon, almost all debtors induce a lowering of the relative risk level and, therefore, seem to provide a potential diversification effect. However, the effects are rather small. Interestingly, comparing with the one-year Value-at-Risk in Figure 9, which includes microstructural effects, there is no significant difference below the 97% confidence level compared to the Value-at-Risk in the purely macroeconomic model. Hence, considering Value-at-Risk with low confidence level as relevant risk measure turns out to be as misleading as discarding microstructural dependencies.

For a time horizon of three years, these effects become slightly larger and, at least for some confidence levels, debtors 3 and 4 contribute positively to the risk per dollar. Since these two debtors belong to the same sector, they provide exactly the same marginal risk in the macroeconomic model. Thus, if a credit decision is based on a purely macroeconomic model, these debtors would be treated as perfect substitutes. However, if we turn back to the microstructural model, the picture changes radically. Indeed, it is only debtor 3 who contributes to an increase in the relative risk, whereas debtor 4 provides a diversification opportunity. For the three year time horizon, the neglecting macrostructural dependence weighs more heavily already for lower confidence levels. The marginal expected shortfall for debtor 3 is underestimated by the macroeconomic model by at least a factor of ten. Also, the diversification potential of debtor 4 remains undetected.

V. Conclusion

To evaluate and manage a bank's credit risk, it is not sufficient to scrutinize individual debtors. We need to identify the concentration of risk within the credit portfolio. Such concentration not only arises in portfolios which poorly diversify with respect to sectors but, more importantly, also in portfolios which exhibit microstructural dependencies. Our simple model of credit contagion shows that microstructural dependencies dramatically change the tail behavior of portfolio credit losses. This is the part of the distribution that both banks and regulators are most concerned about.

We find that using a purely macroeconomic model bears substantial model risk. Already a medium-sized credit portfolio with a well diversified microstructure exhibits a substantial increase in default risk. This underestimation increases further when we consider a hierarchical dependence structure. Even more concerning, however, is our finding for the real business portfolio. This portfolio consists of subportfolios with random and hierarchical microstructures. The very fact that large debtors link these subportfolios through their business dependencies leads to a boosting of the loss distribution's tails. This boost is, to a large part, induced by feedback effects.

Since microstructural dependencies act only on the very far tails, using Value-at-Risk to calculate credit risk, as proposed by regulators, leads to a dangerous underestimation of portfolio losses. Indeed, depending on the confidence levels, the Value-at-Risk including microstructural dependencies may not differ from a pure macrostructural model.

Finally, using Value-at-Risk is misleading when it comes to loan pricing and capital allocation. Therefore, it is important to use a risk measure which takes into account the tails of the portfolio's loss distribution. Expected shortfall is such a risk measure.

A. Proof of Proposition

Consider the nth-order approximation of the return A,

$$A^{(n)} = \mathbf{D}_Y^{(n)} \mathbf{D}\left(\sqrt{1-v^2}\right) Y + \mathbf{D}_\epsilon^{(n)} \mathbf{D}\left(v\right) \varepsilon^{(n)}(\mathcal{G}, Y, \epsilon), \tag{A.1}$$

where $\varepsilon^{(0)}(\mathcal{G}, Y, \epsilon) = \epsilon$ and $\mathbf{D}_Y^{(0)} = \mathbf{D}_\epsilon^{(0)} = \mathbf{1}$. The idiosyncratic part is given as

$$\varepsilon^{(n)}(\mathcal{G}, Y, \epsilon) = \boldsymbol{\Xi} A^{(n-1)} + \mathbf{D}\left(\eta\right) \epsilon. \tag{A.2}$$

Let

$$\varepsilon^{(n)}(\mathcal{G}, Y, \epsilon) = \mathbf{E}_Y^{(n)} Y + \mathbf{E}_\epsilon^{(n)} \epsilon. \tag{A.3}$$

The terms $\mathbf{E}_Y^{(n)}$ and $\mathbf{E}_\epsilon^{(n)}$ have to be determined recursively using (A.2). Consider, e.g., the second-order approximation $A^{(2)}$. We obtain

$$\begin{aligned}
\varepsilon^{(2)}(\mathcal{G}, Y, \epsilon) &= \boldsymbol{\Xi} A^{(1)} + \mathbf{D}\left(\eta\right) \epsilon \\
&= \boldsymbol{\Xi}\left(\mathbf{D}_Y^{(1)} \mathbf{D}\left(\sqrt{1-v^2}\right) Y + \mathbf{D}_\epsilon^{(1)} \mathbf{D}\left(v\right) \varepsilon^{(1)}(\mathcal{G}, Y, \epsilon)\right) + \mathbf{D}\left(\eta\right) \epsilon \\
&= \boldsymbol{\Xi}\left(\mathbf{D}_Y^{(1)} \mathbf{D}\left(\sqrt{1-v^2}\right) Y + \mathbf{D}_\epsilon^{(1)} \mathbf{D}\left(v\right) \left(\mathbf{E}_Y^{(1)} Y + \mathbf{E}_\epsilon^{(1)} \epsilon\right)\right) + \mathbf{D}\left(\eta\right) \epsilon.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{E}_Y^{(2)} &= \boldsymbol{\Xi}\left(\mathbf{D}_Y^{(1)} \mathbf{D}\left(\sqrt{1-v^2}\right) + \mathbf{D}_\epsilon^{(1)} \mathbf{D}\left(v\right) \mathbf{E}_Y^{(1)}\right) \\
&= \boldsymbol{\Xi}\left(\mathbf{D}_Y^{(1)} \mathbf{D}\left(\sqrt{1-v^2}\right) + \mathbf{D}_\epsilon^{(1)} \mathbf{D}\left(v\right) \boldsymbol{\Xi} \mathbf{D}_Y^{(0)} \mathbf{D}\left(\sqrt{1-v^2}\right)\right), \\
\mathbf{E}_\epsilon^{(2)} &= \boldsymbol{\Xi} \mathbf{D}_\epsilon^{(1)} \mathbf{D}\left(v\right) \mathbf{E}_\epsilon^{(1)} + \mathbf{D}\left(\eta\right) \\
&= \boldsymbol{\Xi} \mathbf{D}_\epsilon^{(1)} \mathbf{D}\left(v\right) \left(\boldsymbol{\Xi} \mathbf{D}_\epsilon^{(0)} \mathbf{D}\left(v\right) + \mathbf{D}\left(\eta\right)\right) + \mathbf{D}\left(\eta\right).
\end{aligned}$$

Therefore, the second-order return is

$$A^{(2)} = \left(\mathbf{D}_Y^{(2)} \mathbf{D}\left(\sqrt{1-v^2}\right) + \mathbf{D}_\varepsilon^{(2)} \mathbf{D}\left(v\right) \mathbf{E}_Y^{(2)}\right) Y + \mathbf{D}_\varepsilon^{(2)} \mathbf{D}\left(v\right) \mathbf{E}_\epsilon^{(2)} \epsilon. \tag{A.4}$$

We still have to determine the normalizing matrices $\mathbf{D}_Y^{(2)}$ and $\mathbf{D}_\varepsilon^{(2)}$. We can proceed as for the first-order approximation, but with the appropriate adjustments, i.e., $\mathbf{E}_Y^{(1)} \to \mathbf{E}_Y^{(2)}$ and $\mathbf{E}_\epsilon^{(1)} \to \mathbf{E}_\epsilon^{(2)}$, respectively.

Then, it is straightforward to obtain the second-order normalizing matrices that guarantee consistency with the conditions (**C1**) and (**C2**),

$$\begin{cases} (\mathbf{C1}) & : \quad (\mathbf{D}_{\varepsilon}^{(2)})^2 = \mathbf{D}\left(\mathbf{\Sigma}^{(2)}\right)^{-1}, \\ (\mathbf{C2}) & : \quad \mathbf{1} = \mathbf{Q}(\mathbf{D}_{Y}^{(2)}). \end{cases} \tag{A.5}$$

We note that, compared to the first-order approximation, the second-order approximation already allows for feedback or back-propagation effects. The third-order approximation for$\varepsilon^{(3)}(\mathcal{G}, Y, \epsilon)$ is easily obtained as

$$\begin{aligned}
\mathbf{E}_{Y}^{(3)} &= \Xi\left(\mathbf{D}_{Y}^{(2)}\mathbf{D}\left(\sqrt{1-v^2}\right) + \mathbf{D}_{\epsilon}^{(2)}\mathbf{D}(v)\,\mathbf{E}_{Y}^{(2)}\right) \\
&= \left(\Xi\mathbf{D}_{Y}^{(2)} + \Xi\mathbf{D}_{\epsilon}^{(2)}\mathbf{D}(v)\,\Xi\mathbf{D}_{Y}^{(1)} + \Xi\mathbf{D}_{\epsilon}^{(2)}\mathbf{D}(v)\,\Xi\mathbf{D}_{\epsilon}^{(1)}\mathbf{D}(v)\,\Xi\mathbf{D}_{Y}^{(0)}\right)\mathbf{D}\left(\sqrt{1-v^2}\right), \\
\mathbf{E}_{\epsilon}^{(3)} &= \Xi\mathbf{D}_{\epsilon}^{(2)}\mathbf{D}(v)\,\mathbf{E}_{\epsilon}^{(2)} + \mathbf{D}(\eta) \\
&= \Xi\mathbf{D}_{\epsilon}^{(2)}\mathbf{D}(v)\,\Xi\mathbf{D}_{\epsilon}^{(1)}\mathbf{D}(v)\,\Xi\mathbf{D}_{\epsilon}^{(0)}\mathbf{D}(v) + \Xi\mathbf{D}_{\epsilon}^{(2)}\mathbf{D}(v)\,\Xi\mathbf{D}_{\epsilon}^{(1)}\mathbf{D}(v)\,\mathbf{D}(\eta) \\
&\quad + \Xi\mathbf{D}_{\epsilon}^{(2)}\mathbf{D}(v)\,\mathbf{D}(\eta) + \mathbf{D}(\eta).
\end{aligned}$$

Higher-order approximations are obtained in the same manner by recursively replacing$\mathbf{E}_{Y}^{(n)}$ and $\mathbf{E}_{Y}^{(n)}$ in the expression for $\varepsilon^{(n)}(\mathcal{G}, Y, \epsilon)$ by[8]

$$\begin{aligned}
\mathbf{E}_{Y}^{(n+1)} &= \sum_{i=0}^{n}\left(\prod_{j=0}^{n-i-1}\Xi\mathbf{D}_{\varepsilon}^{(n-j)}\mathbf{D}(v)\right)\Xi\mathbf{D}_{Y}^{(i)}\mathbf{D}\left(\sqrt{1-v^2}\right), \\
\mathbf{E}_{\epsilon}^{(n+1)} &= \sum_{i=0}^{n}\left(\prod_{j=0}^{n-i-1}\Xi\mathbf{D}_{\varepsilon}^{(n-j)}\mathbf{D}(v)\right)\mathbf{D}(\eta) + \prod_{i=0}^{n}\Xi\mathbf{D}_{\varepsilon}^{(n-i)}\mathbf{D}(v).
\end{aligned}$$

By doing so, we obtain $\varepsilon^{(n+1)}(\mathcal{G}, Y, \epsilon)$ and, by following the same arguments as above, the normalizing matrices $\mathbf{D}_{Y}^{(n+1)}$ and $\mathbf{D}_{\varepsilon}^{(n+1)}$.

[8] Here, $\prod$ is understood as the matrix product operator.

B. Figures

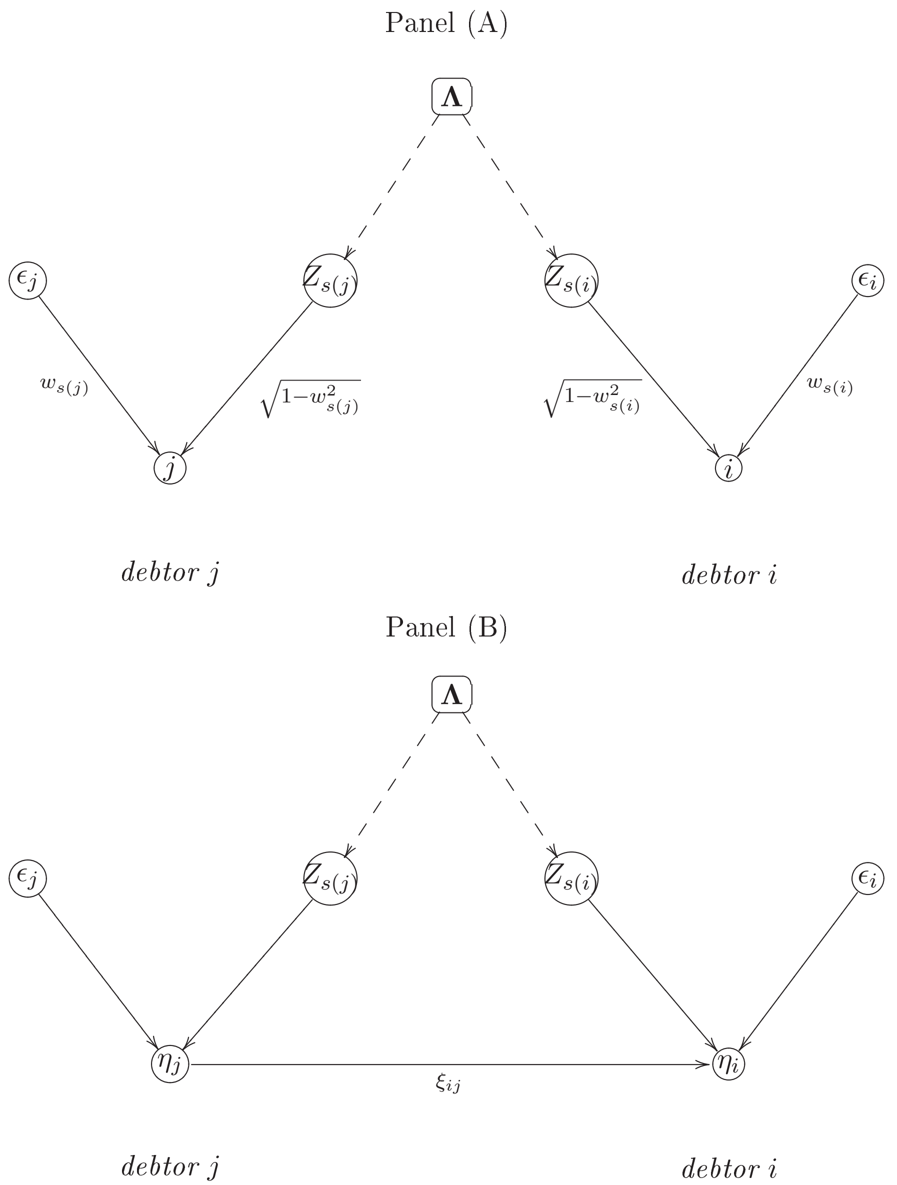


Figure 1. Integrating macro- with microstructural effects. Panel (A) illustrates the dependencies in a pure macrostructural model. Panel (B) allows for business dependencies. The macrostructural variable $Z_{s(j)}$ can now also influence debtor i through her business dependence with debtor j.

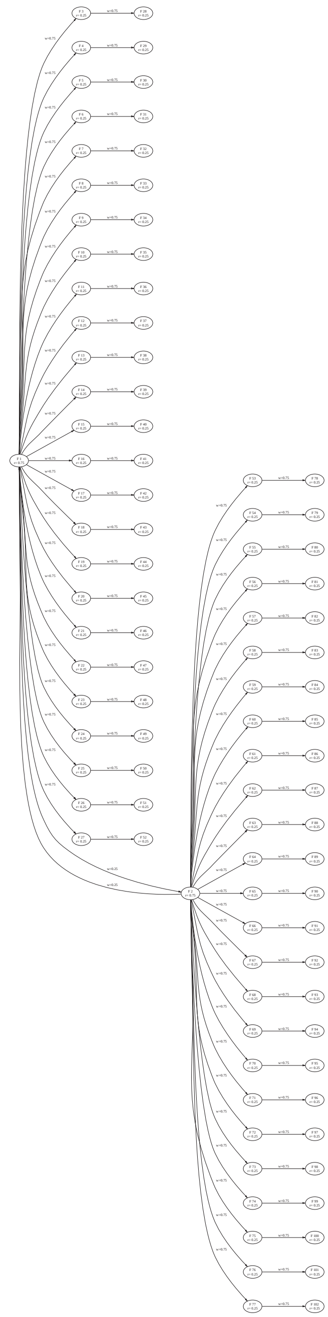


Figure 2. Heavy Gravity Portfolio (HGP). In this portfolio two firms have a large business impact on its direct suppliers and an indirect one to the second level suppliers.

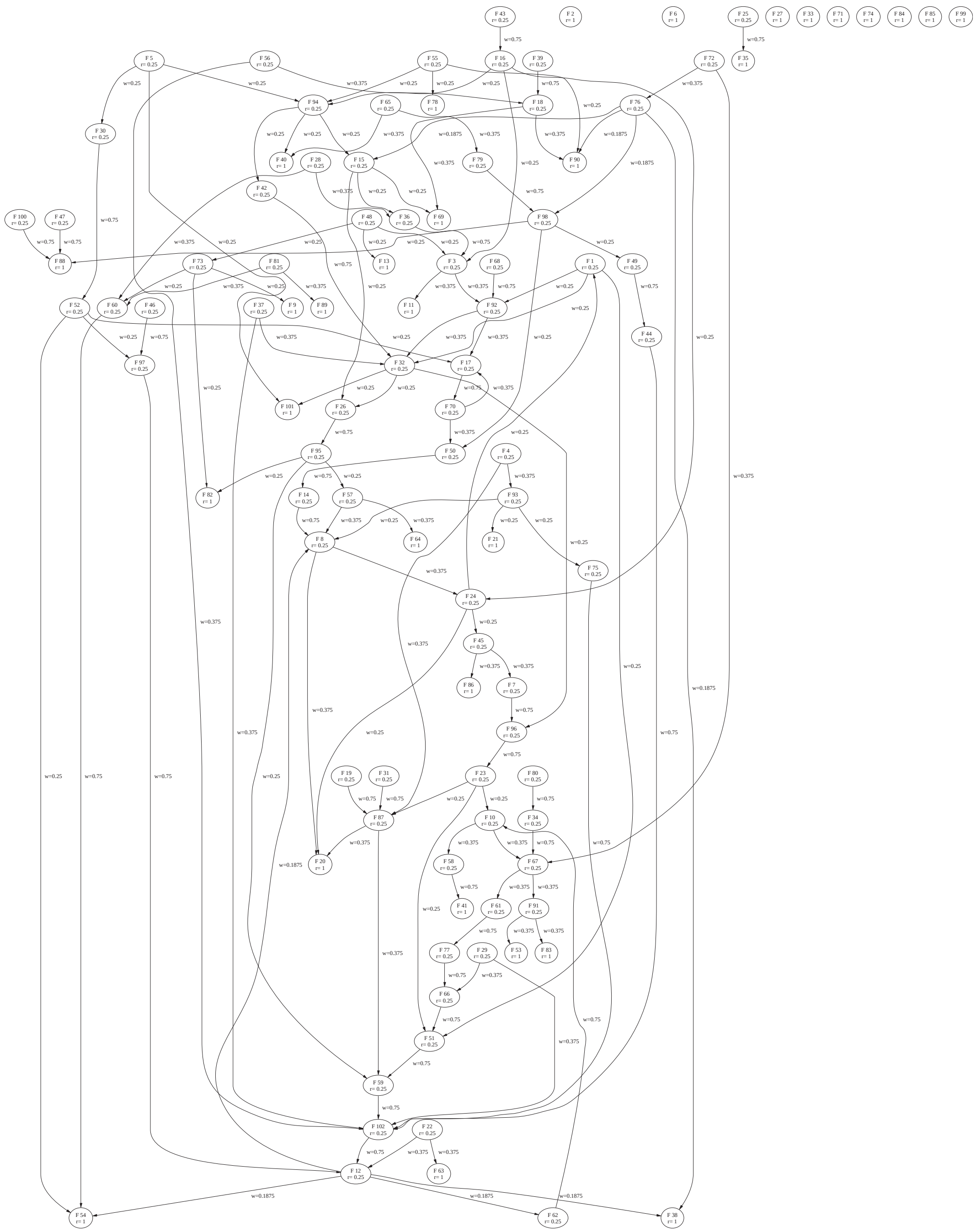


Figure 3. Diversified Debtor's Portfolio (DDP). Contrary to the HGP case, there is no firm in the portfolio which is an obvious center of business gravity. The firms on the top right side of the figure have no microstructural dependence.

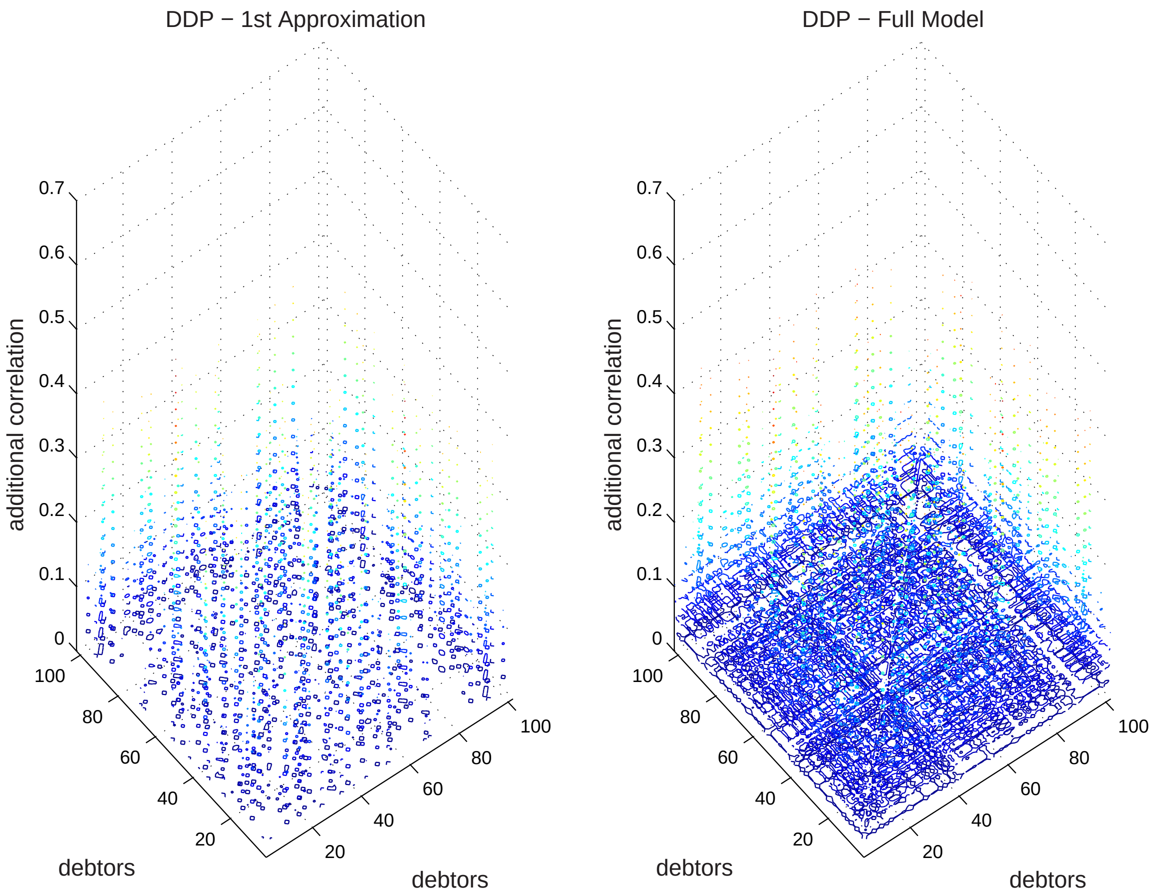


Figure 4. Additional correlation in the DDP. The figure plots the additional correlation caused by microstructural dependencies in the DDP compared to the pure macroeconomic model. The left panel plots the first-order additional correlation. The right panel illustrates the additional correlation in the full model taking feedback effects into account.

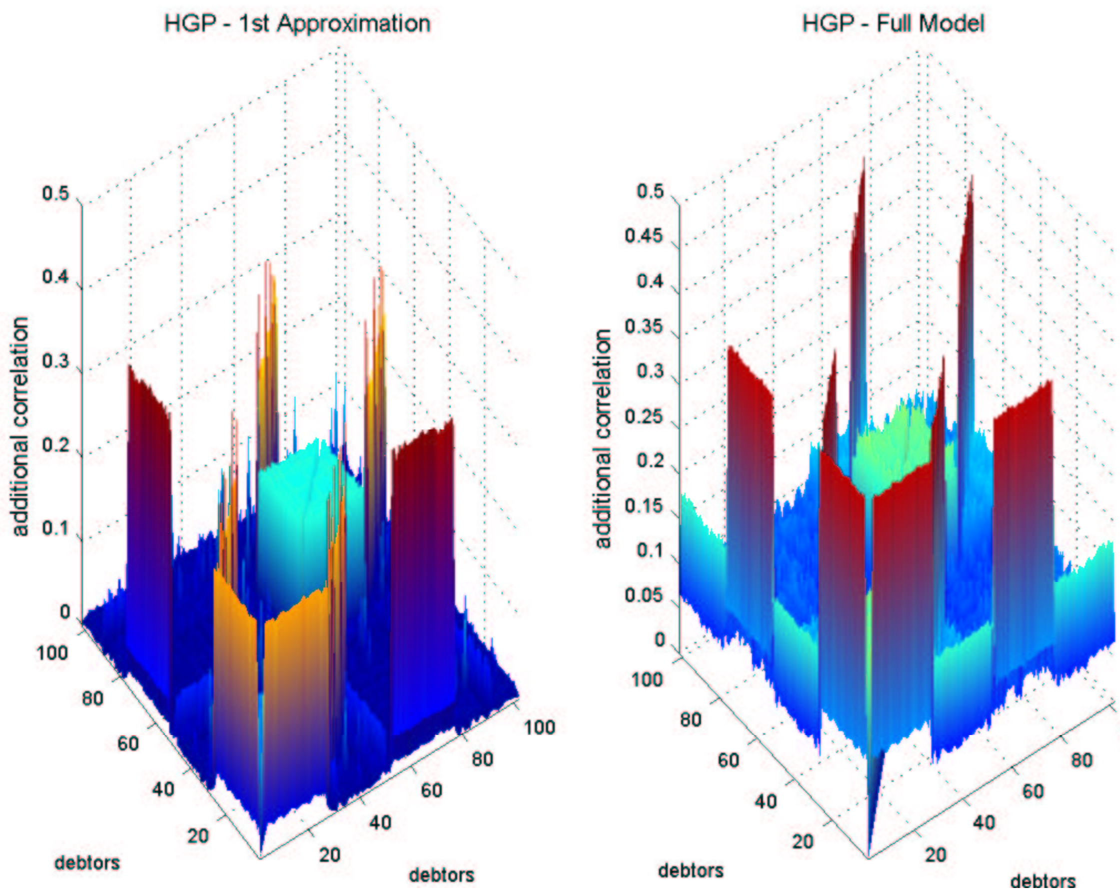


Figure 5. Additional correlation in the HGP. The figure plots the additional correlation caused by microstructural dependencies in the HGP compared to the pure macroeconomic model. The left panel plots the first-order additional correlation. The right panel illustrates the additional correlation in the full model taking feedback effects into account.

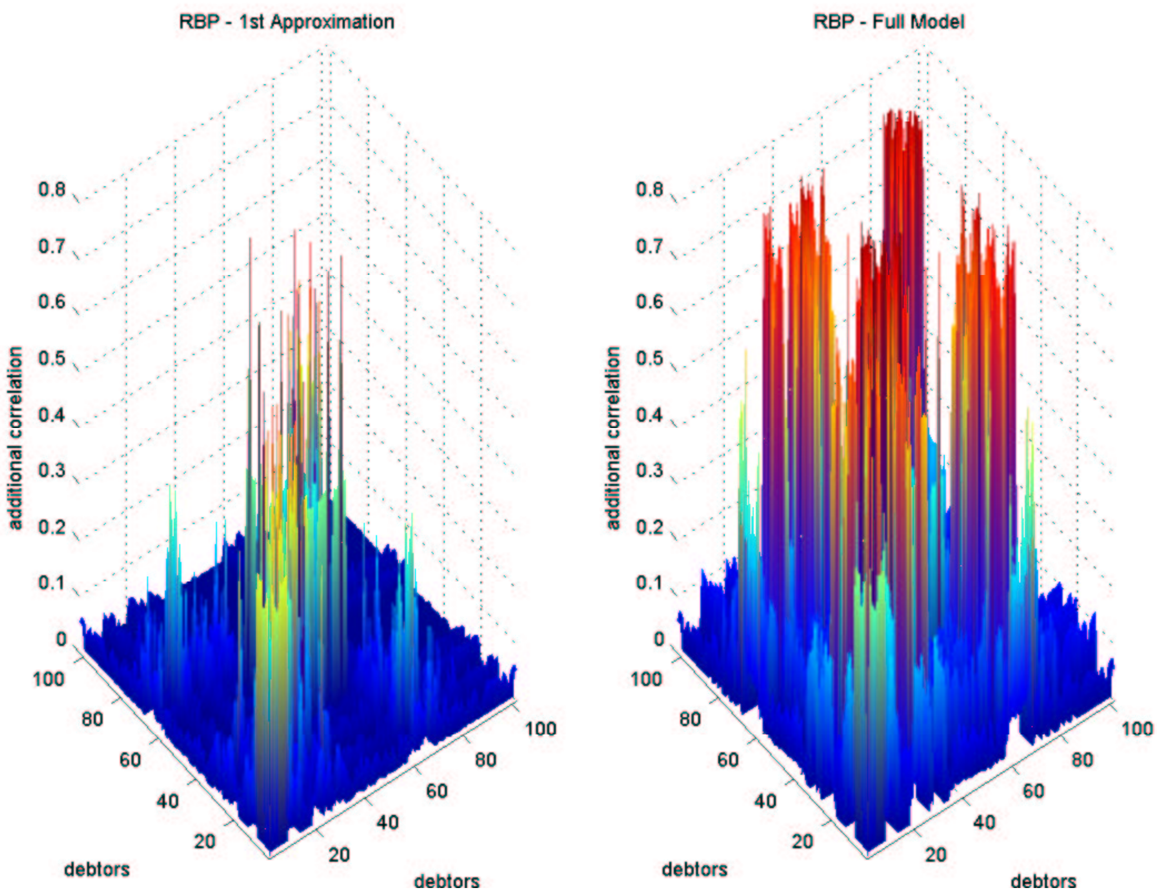


Figure 6. Additional correlation in the RBP. The figure plots the additional correlation caused by microstructural dependencies in the RBP compared to the pure macroeconomic model. The left panel plots the first-order additional correlation. The right panel illustrates the additional correlation in the full model taking feedback effects into account.

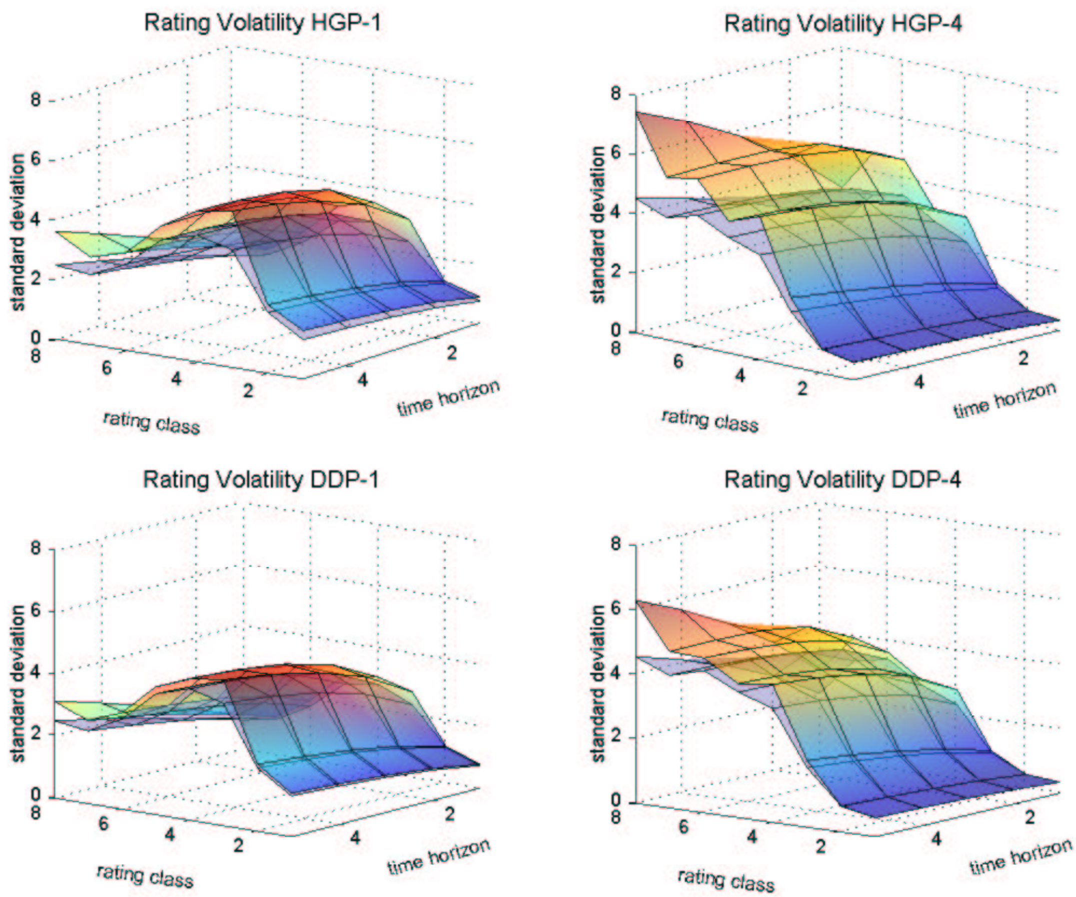


Figure 7. Rating volatilities of the DDP and HGP. The upper panels show the rating volatilities of the HGP as a function of the rating class and time horizon. The left figure assumes an initial portfolio with high rating quality and the figure on the right with a very low rating quality. The lower panels show the rating volatility for the DDP. The figure on the left assumes a high initial rating quality, the figure on the right a very low initial rating quality.

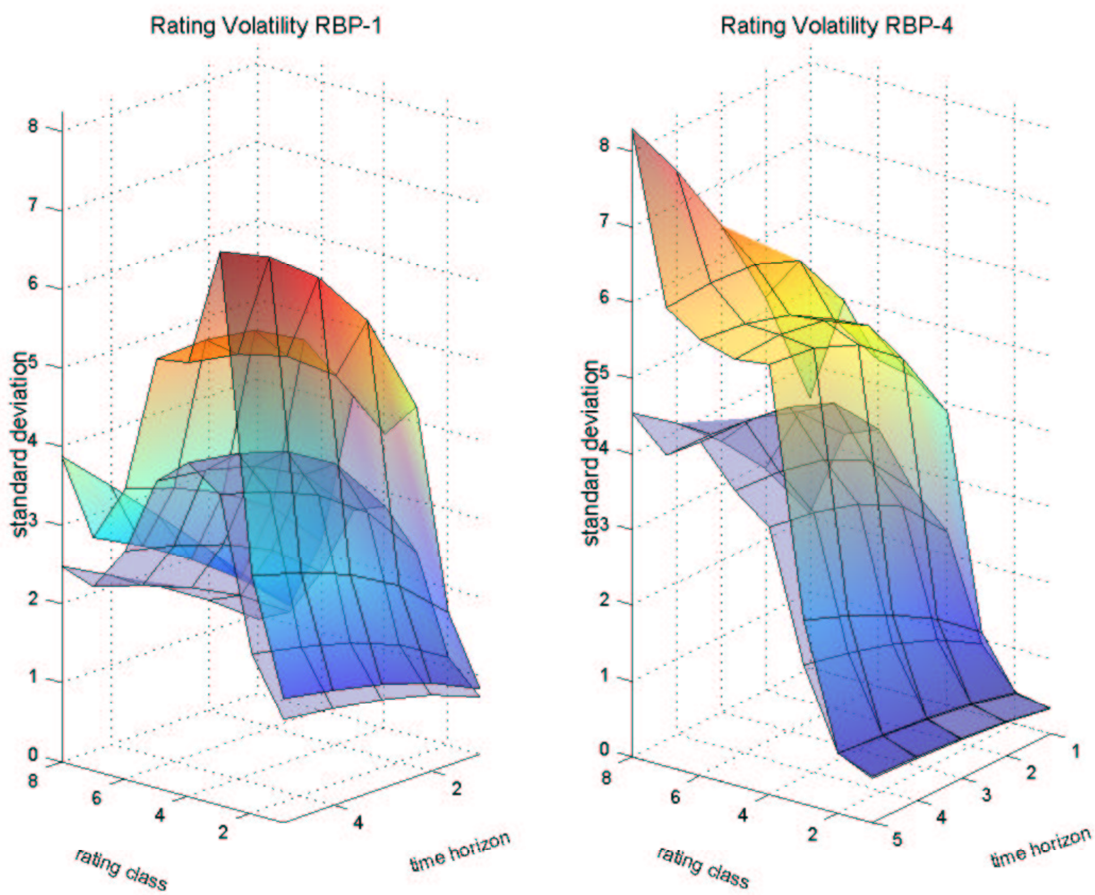


Figure 8. Rating volatilities of the RBP. The figures show the rating volatilities of the RBP as a function of rating class and time horizon. The left figure assumes an initial portfolio with high rating quality and the figure on the right with a very low rating quality.

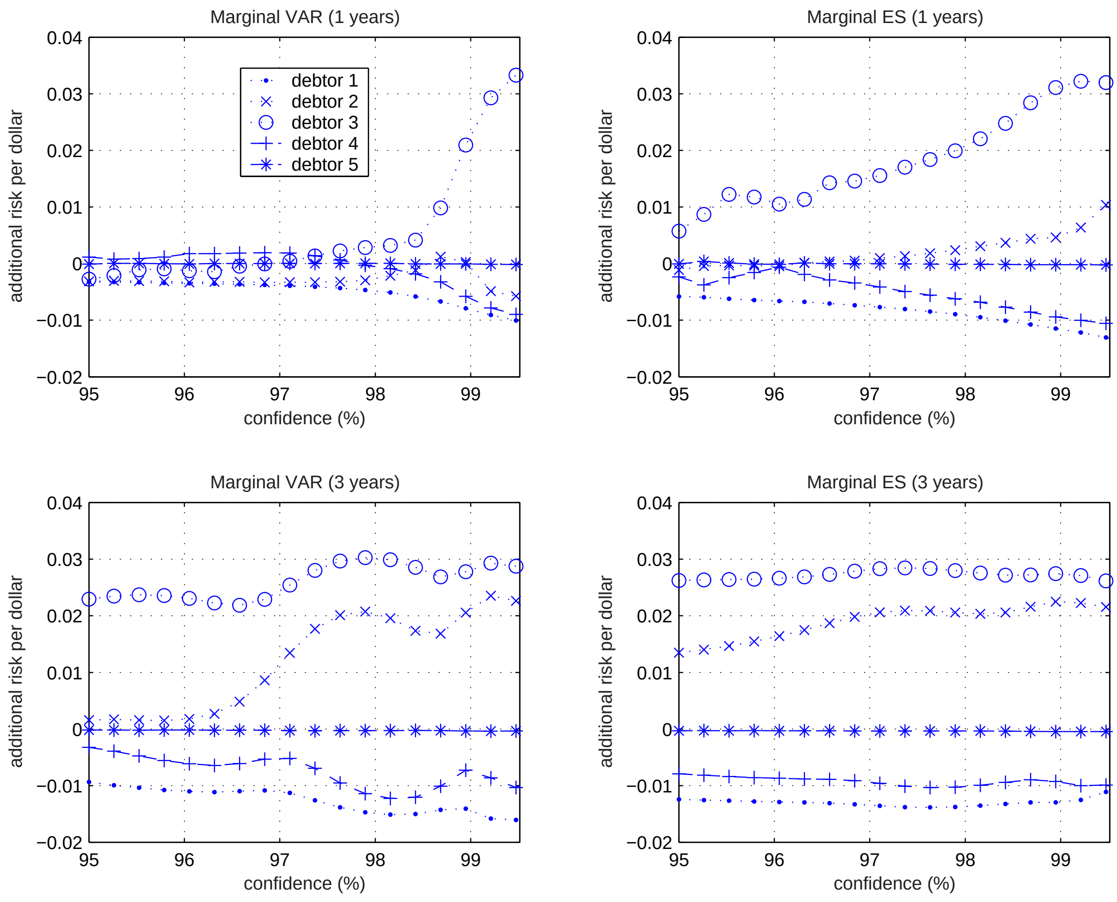


Figure 9. Marginal risk contribution in the RBP with microstructural dependencies. The figures show the marginal risk contribution of different debtors for different time horizons and different risk measures. We calculate the risk figures for different confidence levels based on a sample of 10^9 simulations.

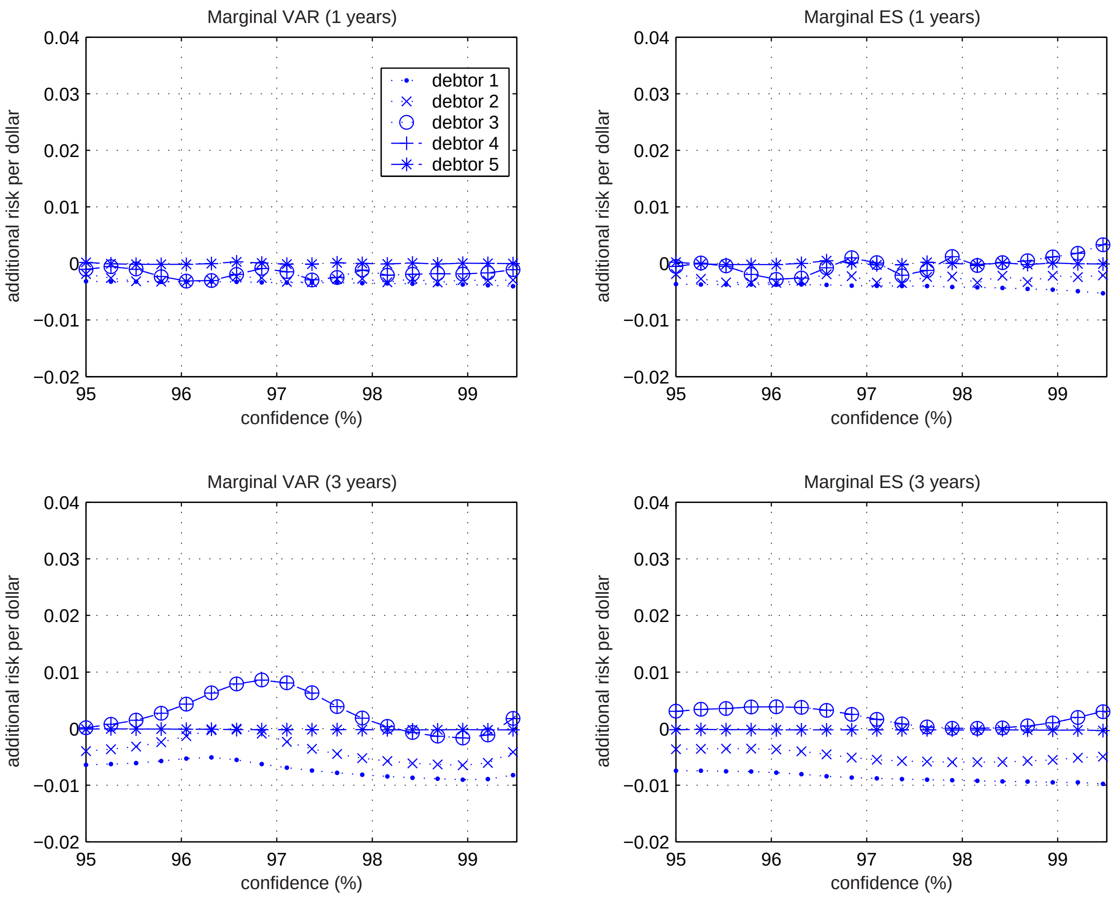


Figure 10. Marginal risk contribution in the RBP without microstructural dependencies. The figures show the marginal risk contribution of different debtors for different time horizons and different risk measures. We calculate the risk figures for different confidence levels based on a sample of 10^9 simulations.

C. Tables

Table I
Sector Classification and Weights.

The sector classification corresponds to BAK data. We calculate the sector specific weights w_s by using the calibration procedure outlined in Section III

Sector	Industry	Debtors	%	w_s
1	Food, Agriculture	8	7.84	0.9853
2	Textiles	6	5.88	0.9758
3	Paper and printing industry, wood manufacturing	7	6.86	0.9773
4	Chemical	4	3.92	0.9838
5	Plastics	5	4.90	0.9779
6	Metal and machine industry	9	8.82	0.9922
7	Electronics	8	7.84	0.9926
8	Other manufacturing, recycling	6	5.88	0.9781
9	Construction industry	12	11.76	0.9870
10	Retail, Commerce	6	5.88	0.9834
11	Hotel and restaurant industry	9	8.82	0.9603
12	Traffics, communication, energy, water	9	8.82	0.9787
13	Finance and insurance industry	7	6.86	0.9818
14	Real estate	6	5.88	0.9927

Table II
Rating migration matrix

Entries are the probabilities that a firm's rating moves from row $d = x$ to column $d = y$.

d	1	2	3	4	5	6	7	8
1	0.9181	0.0741	0.0076	0.0000	0.0000	0.0000	0.0000	0.0001
2	0.0119	0.9084	0.0759	0.0027	0.0008	0.0001	0.0000	0.0002
3	0.0005	0.0240	0.9189	0.0499	0.0051	0.0013	0.0001	0.0002
4	0.0005	0.0025	0.0533	0.8839	0.0487	0.0077	0.0016	0.0018
5	0.0001	0.0005	0.0050	0.0552	0.8517	0.0696	0.0051	0.0128
6	0.0001	0.0003	0.0014	0.0044	0.0660	0.8315	0.0288	0.0675
7	0.0000	0.0000	0.0000	0.0059	0.0178	0.0413	0.6799	0.2550
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Table III
Initial Portfolio Ratings

Entries are percentage numbers of debtors in each rating grade according to Gordy (2000).

	Portfolio Credit Quality			
Rating Grade	High	Average	Low	Very Low
1	3.82	2.92	1.00	0.80
2	5.90	5.00	1.54	1.02
3	29.26	13.38	3.70	3.16
4	37.92	31.16	16.54	13.20
5	19.08	32.44	38.06	35.60
6	2.72	11.12	32.36	37.02
7	1.30	3.98	6.80	9.50

Table IV
Threshold values Θ

The threshold values are obtained by calibrating to macroeconomic data as outlined in Section III.

x	θ_{x9}	θ_{x8}	θ_{x7}	θ_{x6}	θ_{x5}	θ_{x4}	θ_{x3}	θ_{x2}	θ_{x1}
1	$-\infty$	-3.7190	-3.7190	-3.7190	-3.7190	-3.7190	-2.4221	-1.3927	∞
2	$-\infty$	-3.5329	-3.5329	-3.4243	-3.0537	-2.6692	-1.4069	2.2598	∞
3	$-\infty$	-3.5317	-3.4230	-2.9381	-2.4725	-1.5843	1.9675	3.2816	∞
4	$-\infty$	-2.9147	-2.7114	-2.2870	-1.5566	1.5864	2.7438	3.2776	∞
5	$-\infty$	-2.2319	-2.0998	-1.3568	1.5478	2.5365	3.2178	3.7003	∞
6	$-\infty$	-1.4947	-1.3028	1.4595	2.4997	2.9075	3.3336	3.7015	∞
7	$-\infty$	-0.6588	1.5133	1.9811	2.5152	∞	∞	∞	∞
8	$-\infty$	∞	∞	∞	∞	∞	∞	∞	∞

Table V
Sector correlation matrix Λ

The correlation table is obtained by calibrating to macroeconomic data as outlined in Section III.

Sector	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1.00	0.68	0.82	0.81	0.76	0.47	0.76	0.80	0.84	0.70	0.79	0.80	0.70	0.52
2	0.68	1.00	0.85	0.35	0.93	0.47	0.74	0.92	0.87	0.97	0.56	0.77	0.96	0.60
3	0.82	0.85	1.00	0.62	0.90	0.55	0.72	0.94	0.95	0.93	0.84	0.93	0.91	0.56
4	0.81	0.35	0.62	1.00	0.49	0.36	0.69	0.61	0.69	0.41	0.78	0.73	0.47	0.39
5	0.76	0.93	0.90	0.49	1.00	0.51	0.77	0.97	0.89	0.96	0.66	0.82	0.96	0.67
6	0.47	0.47	0.55	0.36	0.51	1.00	0.33	0.54	0.63	0.46	0.46	0.60	0.44	0.44
7	0.76	0.74	0.72	0.69	0.77	0.33	1.00	0.81	0.78	0.71	0.54	0.68	0.76	0.46
8	0.80	0.92	0.94	0.61	0.97	0.54	0.81	1.00	0.94	0.95	0.73	0.88	0.96	0.63
9	0.84	0.87	0.95	0.69	0.89	0.63	0.78	0.94	1.00	0.91	0.82	0.94	0.91	0.61
10	0.70	0.97	0.93	0.41	0.96	0.46	0.71	0.95	0.91	1.00	0.68	0.84	0.98	0.59
11	0.79	0.56	0.84	0.78	0.66	0.46	0.54	0.73	0.82	0.68	1.00	0.94	0.63	0.37
12	0.80	0.77	0.93	0.73	0.82	0.60	0.68	0.88	0.94	0.84	0.94	1.00	0.81	0.47
13	0.70	0.96	0.91	0.47	0.96	0.44	0.76	0.96	0.91	0.98	0.63	0.81	1.00	0.66
14	0.52	0.60	0.56	0.39	0.67	0.44	0.46	0.63	0.61	0.59	0.37	0.47	0.66	1.00

Table VI
Additional Risk in the DDP.

Entries report the percentage increase in Value-at-Risk and expected shortfall for default caused microstructural dependencies in the DDP. We analyze different confidence levels and different time horizons. The results are based on a simulation with 10^6 samples.

Panel A: high initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	0.00	25.00	16.67	12.50	10.00	11.11	32.11	24.04	19.75	16.80
97.5%	33.33	20.00	14.29	11.11	18.18	42.47	28.16	21.98	18.47	24.17
99%	25.00	16.67	25.00	20.00	25.00	34.85	24.61	31.97	26.39	30.15
99.5%	50.00	33.33	22.22	27.27	23.08	59.24	40.38	29.29	32.98	28.18

Panel B: average initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	25.00	14.29	22.22	16.67	13.33	35.01	23.07	28.38	22.48	18.93
97.5%	20.00	12.50	30.00	23.08	18.75	30.66	21.16	35.61	28.11	23.61
99%	16.67	22.22	25.00	20.00	23.53	27.09	29.87	31.17	25.40	27.83
99.5%	33.33	30.00	23.08	25.00	27.78	42.79	36.38	29.66	29.85	31.65

Panel C: low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	16.67	20.00	21.43	16.67	13.64	28.23	27.77	27.17	21.96	18.41
97.5%	28.57	16.67	18.75	20.00	21.74	38.76	24.59	24.62	24.61	25.25
99%	37.50	30.77	29.41	22.73	24.00	46.67	37.20	33.97	26.95	27.19
99.5%	44.44	35.71	26.32	26.09	22.22	52.90	41.63	31.26	29.66	25.51

Panel D: very low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	14.29	16.67	18.75	15.00	12.50	26.45	24.63	24.46	20.12	17.24
97.5%	25.00	23.08	23.53	18.18	15.38	35.74	30.05	28.49	22.52	19.39
99%	33.33	35.71	26.32	26.09	17.86	42.97	41.15	30.89	29.34	21.50
99.5%	40.00	40.00	30.00	24.00	24.14	49.20	45.10	34.16	27.36	26.87

Table VII
Additional Risk in the HGP.

Entries report the percentage increase in Value-at-Risk and expected shortfall for default caused microstructural dependencies in the HGP. We analyze different confidence levels and different time horizons. The results are based on a simulation with 10^6 samples.

Panel A: high initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	0.00	25.00	33.33	25.00	20.00	18.78	40.50	46.09	37.12	31.65
97.5%	33.33	40.00	28.57	33.33	27.27	51.42	54.83	41.58	44.79	37.91
99%	25.00	33.33	37.50	40.00	41.67	42.59	47.90	49.80	51.09	50.97
99.5%	75.00	66.67	44.44	45.45	46.15	92.48	80.69	56.36	56.00	54.83

Panel B: average initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	25.00	28.57	33.33	25.00	20.00	45.42	43.47	44.79	35.66	29.96
97.5%	40.00	37.50	40.00	38.46	31.25	58.82	51.28	50.73	47.48	39.54
99%	50.00	44.44	41.67	40.00	41.18	67.65	57.56	51.91	48.49	48.38
99.5%	66.67	50.00	46.15	43.75	50.00	84.03	62.22	55.97	52.05	55.55

Panel C: low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	33.33	40.00	35.71	27.78	22.73	52.93	52.58	45.19	36.41	30.62
97.5%	42.86	33.33	31.25	30.00	30.43	60.90	46.21	40.88	38.03	37.16
99%	62.50	53.85	47.06	36.36	40.00	78.39	64.25	54.72	43.36	45.26
99.5%	66.67	57.14	47.37	43.48	37.04	81.59	67.04	54.71	49.61	42.67

Panel D: very low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	28.57	25.00	31.25	25.00	25.00	48.50	39.05	41.00	33.58	32.13
97.5%	50.00	38.46	41.18	31.82	26.92	67.19	50.31	49.37	39.10	33.40
99%	66.67	57.14	42.11	43.48	32.14	81.67	66.55	49.73	49.14	37.54
99.5%	70.00	60.00	50.00	40.00	37.93	83.85	68.68	56.57	45.74	42.50

Table VIII
Additional Risk in the RBP.

Entries report the percentage increase in Value-at-Risk and expected shortfall for default caused microstructural dependencies in the RBP. We analyze different confidence levels and different time horizons. The results are based on a simulation with 10^6 samples.

Panel A: high initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	0.00	25.00	16.67	12.50	20.00	32.37	66.50	57.25	49.67	53.97
97.5%	0.00	20.00	28.57	44.44	45.45	32.37	72.23	76.45	82.84	75.14
99%	25.00	66.67	75.00	80.00	75.00	98.28	125.73	114.63	107.87	95.20
99.5%	100.00	133.33	100.00	100.00	92.31	193.95	182.07	128.26	118.12	103.58

Panel B: average initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	0.00	14.29	33.33	33.33	33.33	37.84	58.61	69.28	60.99	54.59
97.5%	20.00	50.00	60.00	61.54	56.25	80.19	91.88	89.26	81.02	69.94
99%	66.67	88.89	83.33	73.33	70.59	122.74	118.75	100.88	84.63	78.82
99.5%	133.33	110.00	92.31	81.25	77.78	177.87	129.34	104.06	88.68	82.67

Panel C: low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	33.33	50.00	50.00	44.44	36.36	87.05	82.90	70.45	58.99	47.85
97.5%	57.14	58.33	56.25	50.00	52.17	108.08	83.88	71.14	60.11	58.31
99%	112.50	92.31	76.47	59.09	56.00	146.01	105.65	84.08	64.50	59.57
99.5%	133.33	100.00	73.68	65.22	55.56	154.03	108.74	78.96	68.33	58.14

Panel D: very low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	28.57	41.67	43.75	45.00	37.50	84.55	71.61	62.21	55.94	46.22
97.5%	75.00	69.23	64.71	50.00	42.31	119.15	88.75	74.65	57.01	47.81
99%	122.22	92.86	73.68	60.87	50.00	146.17	101.81	78.23	64.32	52.38
99.5%	140.00	100.00	80.00	60.00	51.72	152.10	104.86	82.19	62.11	53.47

Table IX
Importance of Feedback Effects for the RBP.

Entries report the percentage increase in Value-at-Risk and expected shortfall for default caused by feedback effect. We analyze different confidence levels and different time horizons. The results are based on a simulation with 10^6 samples.

Panel A: high initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	0.0	0.0	16.7	12.5	20.0	26.5	32.7	51.1	43.8	48.1
97.5%	0.0	20.0	28.6	44.4	33.3	26.5	65.2	69.5	75.9	56.6
99%	25.0	66.7	55.6	63.6	61.5	89.6	117.0	85.4	84.3	75.5
99.5%	60.0	100.0	80.0	83.3	78.6	129.3	135.7	99.0	95.0	84.1

Panel B: average initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	0.0	14.3	20.0	23.1	25.0	30.8	51.5	49.2	44.9	41.1
97.5%	20.0	50.0	45.5	50.0	47.1	70.7	83.6	67.9	63.7	55.6
99%	66.7	70.0	69.2	62.5	52.6	111.2	91.0	80.3	68.3	56.6
99.5%	100.0	90.9	78.6	70.6	52.4	129.7	101.7	84.0	72.4	54.1

Panel C: low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	14.3	36.4	31.3	30.0	30.4	55.1	60.9	46.5	40.3	37.5
97.5%	37.5	46.2	47.1	36.4	34.6	75.2	63.7	56.2	42.4	38.0
99%	70.0	66.7	50.0	45.8	39.3	91.6	73.6	53.9	47.6	40.2
99.5%	90.9	75.0	57.1	46.2	40.0	101.8	77.6	57.9	46.3	39.9

Panel D: very low initial portfolio rating quality

	Value-at-Risk					*expected shortfall*				
time (yrs.)	1	2	3	4	5	1	2	3	4	5
confidence										
95%	12.5	30.8	35.3	31.8	26.9	55.0	52.8	47.8	38.9	32.2
97.5%	55.6	57.1	47.4	37.5	32.1	86.5	69.1	52.6	40.8	34.5
99%	81.8	68.8	50.0	42.3	35.5	95.1	72.2	51.3	42.9	35.6
99.5%	100.0	66.7	56.5	42.9	33.3	103.6	67.2	55.6	42.1	33.3

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