

National Centre of Competence in Research  
Financial Valuation and Risk Management

Working Paper No. 286

## A Behavioral Foundation of Reward-Risk Portfolio Selection and the Asset Allocation Puzzle

Enrico De Giorgi

Thorsten Hens

János Mayer

First version: June 2005  
Current version: May 2006

This research has been carried out within the NCCR FINRISK project on  
“Evolution and Foundations of Financial Markets”

# A Behavioral Foundation of Reward-Risk Portfolio Selection and the Asset Allocation Puzzle\*

**Enrico De Giorgi**<sup>†</sup>  
University of Lugano

**Thorsten Hens**<sup>‡</sup>  
University of Zurich

**János Mayer**<sup>§</sup>  
University of Zurich

First Draft: 10th June 2005

This Version: 22nd May 2006

## Abstract

In this paper we suggest a behavioral foundation for the reward-risk approach to portfolio selection based on prospect theory. We identify sufficient conditions for mutual fund separation in reward-risk models in general and for prospect theory in particular. It is shown that a prospect theory investor satisfies mutual fund separation if the reference point is the risk-free rate. For other reference points mutual fund separation fails for the prospect theory reward-risk model and, as we will show, prospect theory can explain the asset allocation puzzle.

**JEL Classification:** G11, D81.

---

\*Financial support from the BSI Gamma Foundation, the Foundation for Research and Development at the University of Lugano, and the National Center of Competence in Research “Financial Valuation and Risk Management” (NCCR-FINRISK) is gratefully acknowledged. An earlier version of the paper was presented at the Universities of Zurich and Lugano, at the Zurich Wealth Forum, at the NHH Workshop in Bergen, and at the BSI Gamma Foundation conference on Behavioural Finance held in Frankfurt. We thank the participants for their helpful comments and suggestions.

<sup>†</sup>Institute of Finance, University of Lugano, via Buffi 13, CH-6900 Lugano, Switzerland, Tel. +41 +91 9124677, Fax. +41 +91 9124647 and NCCR-FINRISK, email: enrico.degiorgi@lu.unisi.ch.

<sup>‡</sup>Institute for Empirical Research in Economics, University of Zurich, Plattenstrasse 32, CH-8032 Zürich, Switzerland, Tel. +41 +44 6343706, Fax. +41 +44 6374970, email: thens@iew.unizh.ch.

<sup>§</sup>Institute for Operations Research, University of Zurich, Moussonstrasse 15, CH-8044 Zürich, Tel. +41 +44 6343774, Fax. +41 +44 6344920, email: mayer@ior.unizh.ch.

# A Behavioral Foundation of Reward-Risk Portfolio Selection and the Asset Allocation Puzzle

## **Abstract**

In this paper we suggest a behavioral foundation for the reward-risk approach to portfolio selection based on prospect theory. We identify sufficient conditions for mutual fund separation in reward-risk models in general and for prospect theory in particular. It is shown that a prospect theory investor satisfies mutual fund separation if the reference point is the risk-free rate. For other reference points mutual fund separation fails for the prospect theory reward-risk model and, as we will show, prospect theory can explain the asset allocation puzzle.

**JEL Classification:** G11, D81.

## Introduction

The modern portfolio theory of Markowitz (1952) is a rich source of intuition and also the basis for many practical decisions. Markowitz's seminal idea was to evaluate portfolios by using two opposing criteria: reward and risk. From a reward-risk perspective investors may differ with respect to the degree they are willing to trade off reward against risk but all investors will choose from the set of efficient portfolios: those portfolios that for any given level of risk achieve maximal reward. Moreover, under certain conditions, the reward-risk model of portfolio selection leads to mutual fund separation, i.e. all investors hold a combination of the same portfolio of risky assets combined with the risk-free asset. Mutual fund separation greatly simplifies the advice one should give to a heterogenous set of investors since the proportion of risky assets in the optimal portfolio is then independent of the investor's risk aversion. Moreover, mutual fund separation implies a simple asset pricing structure in which a single risk factor based on the optimal portfolio of risky assets explains the rewards investors get in a financial market equilibrium.

While the asset pricing implication of the general reward-risk model has good empirical support (see De Giorgi and Post 2004), practitioners' advice does not seem to follow the mutual fund separation property. This so-called asset allocation puzzle was first observed by Canner, Mankiw, and Weil (1997) who found that in practitioners' advice the more risk-averse the client, the smaller the stocks-to-bonds ratio. Finally, the reward-risk perspective is sometimes criticized as "ad hoc", meaning that it should instead be derived from the

microeconomic principle of utility maximization.

Markowitz (1952) suggested the expected return as the reward and the variance of returns as the risk measure. The mean-variance reward-risk model can be given a microeconomic foundation since mean-variance preferences arise from risk-averse expected utilities under restrictive assumptions on assets' return distributions (Bigelow 1993, Meyer 1987), or assuming quadratic utility indexes. In this case, as Tobin (1958) has shown, mutual fund separation is obtained if all investors can trade the risk-free asset without running into borrowing constraints. The asset pricing implication of the reward risk model is then the Capital Asset Pricing Model, CAPM, which goes back to Sharpe (1964), Lintner (1965), and Mossin (1966).

The foundation of reward-risk on quadratic utilities is implausible since this would imply increasing relative risk aversion and monotonicity of preferences would be lost for high payoff. Moreover, there exists a wealth of empirical evidence to suggest that the return of many assets exhibits skewness and fat tails and hence variance fails to fully describe the return distribution. Being aware of this critique, Markowitz (1959) realized that variance might not be an appropriate risk measure because with asymmetric returns it would still penalize positive and negative deviations from the mean in the same way. As a result, risk was then associated only with the returns below the mean. To achieve such a reward-risk model Markowitz suggested semi-variance as the risk measure. Others followed by suggesting general lower partial moments (Bawa and Lindenberg 1977, Harlow and Rao 1989), value-at-risk (Jorion 1997), coherent risk measures (Artzner, Delbaen, Eber, and Heath 1999),

convex risk measures (Föllmer and Schied 2002, Frittelli and Rosazza Gianin 2004), spectral risk measures (Acerbi 2002) or general deviation measures (Rockafeller, Urysev, and Zabarankin 2006). While some of these extensions of Markowitz (1952) have been justified from the reward-risk perspective they have however no microeconomic foundation based on expected utility, because they violate second-order stochastic dominance. For this point see De Giorgi (2005), who also suggests a class of risk measures that are consistent with second-order stochastic dominance. Moreover, for most of these cases it is not known whether the central property of the reward-risk model, the mutual fund separation, still holds. Generalizations of mutual-fund separation have mainly been derived for the microeconomic model of portfolio selection, generalizing the case of quadratic utility to expected utility with hyperbolic absolute risk aversion. The seminal paper in this literature is Cass and Stiglitz (1970). For a recent monograph including this approach to mutual fund separation see Magill and Quinzii (1996, Section 16).

The purpose of this paper is twofold. First we prove mutual fund separation for the general reward-risk model and secondly we suggest giving the reward-risk model a *behavioral* microeconomic foundation based on prospect theory (see Kahneman and Tversky 1979).

In the general reward-risk model we consider, investors' utilities are increasing functions of a reward measure and decreasing functions of a risk measure. We show that mutual fund separation holds if reward and risk measures can be transformed by means of strictly increasing maps into positive homogeneous, translation invariant or translation equivariant functionals. This result generalizes the observation made by Grootveld and Hallerbach

(1999) and Brogan and Stidham (2005), that mutual fund separation holds for the mean-lower partial moment (LPM) portfolio model of Harlow and Rao (1989) if the target value corresponds to the risk-free asset, or to the expected portfolio return. Indeed, for those target values, LPM can be transformed into a positive homogeneous, translation invariant risk measure. Moreover, it also follows that mutual fund separation holds if investors possess mean-risk preferences, where risk is evaluated according to coherent risk measures (Artzner, Delbaen, Eber, and Heath 1999), or general deviation measures (Rockafeller, Urysev, and Zabarankin 2006). Consequently, the asset allocation puzzle cannot be rationalized by assuming that investors assess portfolio's risk with the latter measures.

The behavioral reward-risk model that we propose is based on the trade-off between gains and losses that is incorporated in prospect theory. For given asset payoffs, we defined reward as the value function applied over gains, while risk is minus the value function applied over losses and normalized by the index of loss aversion. These reward and risk measures obviously depend on the parametrization of prospect theory by means of the utility index, as well as on the choice of the reference point which defines what is perceived as gain and what is perceived as loss. We show that positive homogeneous utility indexes and positive homogeneous and translation equivariant reference points imply the separation property for optimal portfolio allocations. An example of reward and risk measures leading to this result are those defined by means of the piecewise power utility index suggested by Kahneman and Tversky (1979) and having the risk-free gross return as reference point. An additional benefit of the reward-risk perspective on prospect theory is that it solves the infinite leverage problem. De Giorgi,

Hens, and Levy (2003) show that an investor with prospect theory preferences as specified by Kahneman and Tversky (1979) will either not invest at all in risky assets or will infinitely leverage an optimal portfolio of risky assets. Maximizing reward under a risk constraint sets upper bounds on the leverage the investor takes since the risk increases with leveraging.

In general, optimal solutions to the prospect theory reward-risk model do not satisfy the mutual fund separation property. The proportion of stocks to bonds varies according to investors' risk preferences and there is no clear relationship between investors' risk tolerance and the corresponding bond/stock ratio. We perform an empirical analysis on US data. We show that prospect theory reward-risk investors will increase their bond/stock ratio when they become more averse to losses, if they possess a piecewise linear utility index and deterministic reference points that are greater than the risk free gross return. By these investors, the risk free gross return, being below the reference point, is perceived as a loss. Therefore as they become more loss averse, they shift some of their wealth from cash to bonds, which provide a stronger upside potential. Our empirical analysis also shows that more loss averse prospect theory reward-risk investors have a bond/stock ratio that is much higher than that of mean-variance investors. This result is also consistent with the observation of Canner, Mankiw, and Weil (1997) concerning the recommendations of popular advisors.

In the literature several approaches have been suggested in order to rationalize the asset allocation puzzle. Canner, Mankiw, and Weil (1997) relax the key assumptions leading to the mutual fund separation theorem, which are: (i) There exists a riskless asset, (ii) investors possess mean-variance objective functions, (iii) investors uses historical distributions, (iv)

assets can be freely traded (i.e. there are no short-sale or borrowing constraints), (v) investors operate over a one-period planning horizon. However, they found that deviating from these assumptions fails to explain satisfactorily the recommended portfolio allocations, does not provide satisfactory explanations of the recommended portfolio allocations, in particular the relationship between bond/stock ratio and risk aversion. Indeed, the authors conclude that it is hard to explain recommended portfolio allocations with a rational model. Some authors are skeptical about the conclusion of Canner, Mankiw, and Weil (1997) concerning the inconsistency of financial advisors with respect to the modern portfolio theory. Elton and Gruber (2000), for example, claim that the bond/stock ratio test is not sufficient to assert that financial advisors do not follow the modern portfolio theory. In particular they point out that the violation of two-fund separation may result from short-sale constraints that the advisors are obliged to satisfy. Shalit and Yitzhaki (2003) test the efficiency of financial advisors' portfolio allocations with respect to the second-order stochastic dominance (SSD) and show that these allocations are not inefficient, i.e. not dominated with respect to SSD by any alternative allocation. Therefore, even if the recommended allocations are not mean-variance efficient, providing advice to a wide range of investors, financial advisors are rational in recommending portfolios that are not dominated by SSD. Other authors explain the asset allocation puzzle by the fact that a static portfolio model is not able to capture important aspects of the portfolio decision process. Brennan and Xia (2000, 2002) and Campbell and Viceira (2001, 2002), for example, explain the asset allocation puzzle using a dynamic model of portfolio selection, exploiting the effect of inflation on long-term portfolio decisions.

Bajeux-Besnainou, Jordan, and Portait (2001) explain the puzzle by assuming that investors' horizon may exceed the maturity of cash assets and investors rebalance the portfolio as time passes. Consequently, it is not reasonable to assume that a risk-less asset exists. Mougeot (2003) introduces a dynamic model with estimation risk, i.e. uncertainty about market's excess return, and show that estimation risk and interest rate risk can rationalize the asset allocation puzzle. Gomes and Michaelides (2004) provide a human capital explanation of the puzzle: investors face a stochastic uninsurable labor income and more risk averse investors hold a smaller percentage of their assets in stocks. Siebenmorgen and Weber (2003), being skeptical about the fact that financial advisors are able to manage complex dynamic models, take a static approach and provide a behavioral model for asset allocation, where investors are assumed to evaluate the risky component of their portfolio without taking correlation into account (pure risk), while a naive diversification criterion is used. Indeed, Siebenmorgen and Weber (2003) provides experimental results (from a questionnaire submitted to German advisors) showing that financial advisors are bad at dealing with correlations, while they understand very well the benefits of diversification.

Our behavioral explanation of the asset allocation puzzle works in the presence of a risk-free asset and a one-period planning horizon. Moreover, we do not refer to background risks like human capital. Instead we address the second key assumption pointed out by Canner, Mankiw, and Weil (1997): investors may apply a reward-risk perspective but this does not mean that they are mean-variance optimizers. They may instead be prospect theory investors with a reference point different from the risk free rate.

The remainder of the paper is organized as follows. In Section I we introduce the general reward-risk model for portfolio selection and we derive sufficient conditions for reward and risk measures in order to obtain the separation property for optimal allocations. In Section II we define a new reward-model which is linked to prospect theory. Section III presents an empirical analysis of the asset allocation puzzle on US data. Section IV concludes our proposal.

## I General reward-risk model and mutual fund separation

We consider a two-period finance economy. Let  $\Omega = \{1, \dots, S\}$ ,  $S < \infty$ , denote the states of nature in the second period.  $\mathcal{F} = 2^\Omega$  is the power algebra on  $\Omega$ , i.e. the set of all possible events arising from  $\Omega$ . Uncertainty is modelled by the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where probability measure  $\mathbb{P}$  on  $\Omega$  satisfies  $p_s = \mathbb{P}[\{s\}] > 0$  for all  $s = 1, \dots, S$ , i.e. every state of the world has strictly positive probability to occur. The space of real-valued measurable functions on  $(\Omega, \mathcal{F}, \mathbb{P})$  is denoted by  $\mathcal{G}$ .

The investment universe consists of  $K + 1$  assets with payoffs  $A_k$  and price  $q_k$ ,  $k = 0, \dots, K$ .  $R_k = \frac{A_k}{q_k}$  is the gross return of asset  $k$  and  $\mathbf{R} = (R_0, \dots, R_K)'$  is the vector of gross returns. We denote by  $\mathbf{R}_1 = (R_1, \dots, R_K)'$  the vector of risky gross returns. Asset 0 is the risk-free asset with payoff  $A_0 = (1, \dots, 1)'$  and price  $q_0 = \frac{1}{1+r}$ , where  $r = R_0 - 1$  is the

risk-free rate of return. The marketed subspace  $\mathcal{X}$  is the span of  $(A_k)_{k=0,1,\dots,K}$ . Indeed, we assume here that trading is done without constraints.

Let  $\mu : \mathcal{G} \rightarrow \mathbb{R}$  and  $\rho : \mathcal{G} \rightarrow \mathbb{R}$  be real-valued functions on  $\mathcal{X}$ . We call  $\mu$  the *reward measure* and  $\rho$  the *risk measure*. We state the following assumption:

**Assumption 1 (( $\mu, \rho$ )-preferences).** *The investor's utility function can be represented as*

$$U(X) = V(\mu(X), \rho(X)), \quad \text{for all } X \in \mathcal{X}$$

where  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuously differentiable and satisfies

$$(i) \quad \frac{\partial V}{\partial \mu}(\mu, \rho) > 0,$$

$$(ii) \quad \frac{\partial V}{\partial \rho}(\mu, \rho) < 0,$$

(iii)  $V(\mu, \rho)$  is quasi-concave.

The investor maximizes his utility under his budget constraint, i.e. the portfolio choice problem is

$$(1) \quad \max_{X \in \mathcal{X}} U(X), \quad q(X) \leq w_0,$$

where  $w_0$  is the investor's initial wealth and  $q : \mathcal{X} \rightarrow \mathbb{R}$  is the linear pricing functional on  $\mathcal{X}$ , i.e. for  $X \in \mathcal{X}$ ,  $X = \sum_{k=0}^K \theta_k A_k$  for  $\boldsymbol{\theta} \in \mathbb{R}^{K+1}$ , we have  $q(X) = \sum_{k=0}^K \theta_k q_k$ . An alternative representation of the portfolio optimizing problem (1) is to maximize the reward given a certain level of risk. The precise relation of these two perspectives is given by the following

observations. Given Assumption 1, for any solution  $X^*$  to (1) there exists a parameter  $\bar{\rho}$  such that  $X^*$  solves

$$(2) \quad \max_{X \in \mathcal{X}} \mu[X] \quad \text{s.t.} \quad \rho[X] \leq \bar{\rho}, \quad q(X) \leq w_0.$$

This can be easily seen by putting  $\bar{\rho} = \rho[X^*]$ , where  $X^*$  is a solution to the optimization problem (1). On the other hand, we need stronger conditions on  $\mu$ ,  $\rho$ , and  $U$  such that any optimal solution to (2) also solves (1). We will discuss some advantages of representation (2) in Section II, where we introduce a  $(\mu, \rho)$ -representation of prospect theory.

We assume that the budget restriction in (2) holds with equality, i.e.  $q(X) = w_0$  for any optimal allocation. This assumption is satisfied if the reward measure is increasing in the risk-free asset.

Let  $\lambda_k = \frac{\theta_k q_k}{w_0}$  be the proportion of initial wealth invested in asset  $k$  for  $k = 0, \dots, K$  and let  $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_K)'$ . Then, the  $(\mu, \rho)$ -optimization problem (2) can equivalently be written as follows:

$$(3) \quad \max_{\boldsymbol{\lambda} \in \mathbb{R}^{K+1}} \mu[\boldsymbol{\lambda}' \mathbf{R} w_0] \quad \text{s.t.} \quad \rho[\boldsymbol{\lambda}' \mathbf{R} w_0] \leq \bar{\rho}, \quad \boldsymbol{\lambda}' \mathbf{e} = 1,$$

where  $\mathbf{e} = (1, \dots, 1)' \in \mathbb{R}^{K+1}$  and  $\boldsymbol{\lambda}' \mathbf{R} w_0$  is the portfolio's payoff for the investment  $\boldsymbol{\lambda}$  and initial wealth  $w_0$ .

The following definition introduces some basic properties of reward and risk measures.

**Definition 1.** Let  $\zeta : \mathcal{G} \rightarrow \mathbb{R}$  be a real-valued function on  $\mathcal{G}$ . We say that  $\zeta$  is:

(i) translation invariant<sup>1</sup> if

$$\zeta(X + a) = \zeta(X)$$

for all  $a \in \mathbb{R}$  and  $X \in \mathcal{G}$ ,

(ii) translation equivariant if

$$\zeta(X + a) = \zeta(X) + a$$

for all  $a \in \mathbb{R}$  and  $X \in \mathcal{G}$ , or

$$\zeta(X + a) = \zeta(X) - a$$

for all  $a \in \mathbb{R}$  and  $X \in \mathcal{G}$ . In the first case, we say that  $\zeta$  is positive translation equivariant, while in the second case it is negative translation equivariant.

If investors possess mean-variance preferences, i.e.  $\mu(X) = \mathbb{E}[X]$  and  $\rho(X) = \mathbb{E}[(X - \mu(X))^2]$  in Assumption 1, any solution  $X^*$  to the portfolio selection problem (2) satisfies the mutual fund separation property (Tobin 1958), i.e.

$$X^* = \lambda_0 w_0 A_0 + (1 - \lambda_0) w_0 X^M,$$

where  $X^M \in \mathcal{X}$ . Written in terms of portfolio shares  $\boldsymbol{\lambda}^*$  (i.e.  $X^* = \mathbf{R}'\boldsymbol{\lambda}^* w_0$ ), the mutual fund separation property implies:

$$\boldsymbol{\lambda}^* = \lambda_0 \mathbf{e}_0 + (1 - \lambda_0) \boldsymbol{\lambda}^M,$$

where  $\boldsymbol{\lambda}^M$  satisfies  $X^M = \mathbf{R}'\boldsymbol{\lambda}^M$ ,  $\mathbf{e}'\boldsymbol{\lambda}^M = 1$ , and  $\mathbf{e}_0 = (1, 0, \dots, 0)' \in \mathbb{R}^{K+1}$ . Consequently, mean-variance investors optimally allocate their wealth between the risk free asset and a single mutual fund  $\boldsymbol{\lambda}^M$ , which is also called the tangency portfolio (for a detailed description, see De Giorgi 2002). The proportion of wealth invested in the tangency portfolio depends

on the investor's risk aversion, which is measured by means of his target risk  $\bar{\rho}$  (see equation (2)). By contrast, the tangency portfolio does not depend on investors' preferences, but only on investors' beliefs concerning expected returns, variances, and correlations between asset returns. Therefore, optimal allocations of investors with homogeneous beliefs only differ by the proportion of wealth invested in the tangency portfolio, i.e. the quantity

$$\frac{1}{1 - \lambda_0^*} \lambda_1^*$$

is identical for all mean-variance investors with same beliefs.

The mean-variance portfolio choice model is a special case of a general model with reward-risk preferences. However, as we will show in this section, mutual fund separation does not require mean-variance preferences but also holds for the general reward-risk model under few assumptions on reward and risk measures. We now investigate these assumptions and we give sufficient conditions on reward and risk measures in order to derive the separation property of optimal portfolio choices.

**Proposition 1.** *Let  $\mu : \mathcal{G} \rightarrow \mathbb{R}$  and  $\rho : \mathcal{G} \rightarrow \mathbb{R}$  be a reward and risk measure on  $\mathcal{G}$ , respectively. Suppose that strictly increasing transformations  $T_\mu$  and  $T_\rho$  exist, such that  $\tilde{\mu} = T_\mu \circ \mu$  and  $\tilde{\rho} = T_\rho \circ \rho$  are positive homogeneous and satisfy one the following two properties:*

- (i) *translation invariance,*
- (ii) *translation equivariance.*

Then, there exists a portfolio allocation  $\boldsymbol{\lambda}_p$ , such that for any  $(\mu, \rho)$ -optimal portfolio allocation  $\hat{\boldsymbol{\lambda}}$ , there exists  $\lambda_0$  with

$$\hat{\boldsymbol{\lambda}} = \lambda_0 \mathbf{e}_0 + (1 - \lambda_0) \hat{\boldsymbol{\lambda}}_p,$$

where  $\mathbf{e}_0 = (1, \mathbf{0})' \in \mathbb{R}^{K+1}$ .

The proof of proposition 2 can be found in the Appendix.

For many applications the reward and the risk measures are homogeneous so that it makes sense to emphasize this case. Mutual fund separation can then easily be derived from proposition 2.

### Corollary 1.

Let  $\mu : \mathcal{G} \rightarrow \mathbb{R}$  and  $\rho : \mathcal{G} \rightarrow \mathbb{R}$  be reward and risk measures on  $\mathcal{G}$ , respectively. Suppose that one of the following two properties holds for  $\mu$  and  $\rho$ :

- (i) translation invariance and (positive) homogeneity of degree  $\gamma > 0$ ,
- (ii) Translation equivariance and (positive) homogeneity of degree 1.<sup>2</sup>

Then two-fund separation holds.

*Proof.* Let  $\gamma_\mu, \gamma_\rho$  be the degree of homogeneity of  $\mu$  and  $\rho$ , respectively. Take the strictly increasing transformations  $T_\mu : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^{\frac{1}{\gamma_\mu}}$  and  $T_\rho : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^{\frac{1}{\gamma_\rho}}$ . Then the transformed measures  $\tilde{\mu} = T_\mu \circ \mu$  and  $\tilde{\rho} = T_\rho \circ \rho$  are positive homogeneous and translation invariant or equivariant. Thus, by Proposition 2, two-fund separation holds for  $(\mu, \rho)$ .  $\square$

Note that the expected value (the mean) is a linear measure, thus is positive homogeneous and translation equivariant. Variance is positive homogeneous of degree 2 and translation invariant, since in fact  $\text{Var}(\lambda X) = \lambda^2 \text{Var}(X)$  for all  $\lambda \in \mathbb{R}$  and  $\text{Var}(X + a) = \text{Var}(X)$  for all  $a \in \mathbb{R}$ . Consequently, according to the corollary to Proposition 2 mutual fund separation holds for mean-variance optimal allocations, as discussed at the beginning of this section. The same argument applies to the lower partial moment with the risk-free return as target value. In fact, the lower partial moment with target  $\tau$  is defined by:

$$(4) \quad \text{LPM}(X; \alpha, \tau) = \sum_{s=1}^S p_s [\max(0, \tau(X) - X(s))]^\alpha$$

for some  $\alpha \geq 1$ . In general  $\tau$  might depend on portfolio  $X$ , thus it is a function on the space of asset payoffs. If the target value is the risk-free return, then  $\tau(X) = q(X)R_0$  where  $q$  is the pricing functional.<sup>3</sup> Consequently,

$$\begin{aligned} \text{LPM}(X + a; \alpha, \tau) &= \sum_{s=1}^S p_s [\max(0, q(X + a)R_0 - X(s) - a)]^\alpha \\ &= \sum_{s=1}^S p_s [\max(0, q(X)R_0 + a - X(s) - a)]^\alpha \\ &= \text{LPM}(X; \alpha, \tau) \end{aligned}$$

for all  $a \in \mathbb{R}$ . Moreover,

$$\begin{aligned} \text{LPM}(\kappa X; \alpha, \tau) &= \sum_{s=1}^S p_s [\max(0, q(\kappa X)R_0 - \kappa X(s))]^\alpha \\ &= \kappa^\alpha \sum_{s=1}^S p_s [\max(0, q(X)R_0 - X(s))]^\alpha \\ &= \kappa^\alpha \text{LPM}(X; \alpha, \tau) \end{aligned}$$

for all  $\kappa > 0$ . Thus, the lower partial moment with  $\tau = w_0 R_0$  is translation invariant and positive homogeneous of degree  $\alpha$ . The same conclusion applies to any target value  $\tau(X)$  which is positive translation equivariant and positive homogeneous of degree 1 as, for example,  $\tau(X) = \mathbb{E}[X]$  or  $\tau(X) = -VaR_\gamma(X)$ , i.e. the  $\gamma$ -quantile of  $X$ .  $VaR_\gamma(X)$  corresponds to the value-at-risk of  $X$  with confidence level  $\gamma$ , which is defined as minus the  $\gamma$ -quantile. These results on two funds separation for LPM generalizes the observation made by Grootveld and Hallerbach (1999) and Brogan and Stidham (2005). Note that for  $\tau(X) = -VaR_\gamma(X)$  and  $\alpha = 1$  we have

$$\begin{aligned}
LPM(X; 1, -VaR_\gamma(X)) &= \sum_{s=1}^S p_s [\max(0, -VaR_\gamma(X) - X(s))] \\
&= -\mathbb{E}[X 1_{\{X \leq -VaR_\gamma(X)\}}] - VaR_\gamma(X) \mathbb{P}[X \leq -VaR_\gamma(X)] \\
&= \gamma (ES_\gamma(X) - VaR_\gamma(X))
\end{aligned}$$

where  $ES_\gamma(X) = -\frac{1}{\gamma} (\mathbb{E}[X 1_{\{X \leq -VaR_\gamma(X)\}}] + VaR_\gamma(X) (\mathbb{P}[X \leq -VaR_\gamma(X)] - \gamma))$  is the expected shortfall with confidence level  $\gamma$  (Acerbi and Tasche 2002). The relationship between  $LPM$ ,  $VaR$  and  $ES$  has already been derived in Jarrow and Zhao (2006), who consider the special case  $\mathbb{P}[X \leq -VaR_\gamma(X)] = \gamma$ . Expected shortfall is an example of a coherent measure of risk, as Artzner, Delbaen, Eber, and Heath (1999) have pointed out. By definition, coherent risk measures are positive homogeneous and translation equivariant. Similarly, deviation measures are positive homogeneous and translation invariant. Therefore, on the basis of Proposition 2, mutual fund separation also holds if investors have mean-risk preferences where risk is measured by means of any coherent risk measure or deviation measure.

## II The prospect theory reward-risk model

The prospect theory of Kahneman and Tversky (1979) deviates from the expected utility hypothesis in four important aspects:

- Investors evaluate assets according to gains and losses and not according to final wealth.
- Investors are loss averse, i.e. they dislike losses by a factor of approximately 2-2.5 (as found in laboratory experiments) as compared to their liking of gains.
- Investors' von Neumann-Morgenstern utility functions are s-shaped with turning point at the origin.
- Investors' probability assessments are biased in the way that extremely small probabilities (extremely high probabilities) are over-/undervalued.

The cumulative prospect theory (CPT) (Tversky and Kahneman 1992) suggests using distortions of the cumulative distribution function instead of distortions of probabilities. This important adjustment of the original theory solves the inconsistency of prospect theory with first-order stochastic dominance. Prospect theory suggests to modelling investors' preferences as follows:

**Assumption 2 (Prospect Theory Preferences).** *Investors evaluate portfolio payoffs according to the value function:*

$$(5) \quad V(X) = \sum_{s=1}^S u(X(s) - RP(X)) \pi_s$$

for all  $X \in \mathcal{X}$  where

- $u$  is a two-time differentiable function on  $\mathbb{R} \setminus \{0\}$ , strictly increasing on  $\mathbb{R}$ , strictly concave on  $(0, \infty)$  and strictly convex on  $(-\infty, 0)$ , with  $u(0) = 0$ ,
- $\pi_s = w(p_s)$  and  $w$  is a differentiable, non-decreasing function from  $[0, 1]$  onto  $[0, 1]$  with  $w(p) = p$  for  $p = 0$  and  $p = 1$  and with  $w(p) > p$  ( $w(p) < p$ ) for  $p$  small (large),
- $RP : \mathcal{X} \rightarrow \mathbb{R}$  is a reference point. If  $RP(X) = RP$  for all  $X$ , then the reference point is fixed.

We rewrite value function (5) by separating portfolio outcomes which are above the reference point and portfolio outcomes which are below the reference point, in the spirit of prospect theory. We obtain:

$$(6) \quad V(X) = \sum_{s=1}^S u(\max(0, X(s) - RP(X))) \pi_s + \sum_{s=1}^S u(\min(0, X(s) - RP(X))) \pi_s$$

The trade-off between gains and losses is determined by investors' loss aversion. Following Benartzi and Thaler (1995), Köbberling and Wakker (2005), and Schmidt and Zank (2002), we define the index of loss aversion by:

$$\beta = \frac{\lim_{x \nearrow 0} u'(x)}{\lim_{x \searrow 0} u'(x)} > 0.$$

and we rewrite equation (6) as follows:

$$(7) \quad \mathcal{V}(X) = \sum_{s=1}^S u(\max(0, X(s) - RP(X))) \pi_s - \beta \sum_{s=1}^S \frac{-1}{\beta} u(\min(0, X(s) - RP(X))) \pi_s.$$

We now define reward and risk measures by

$$(8) \quad PT^+(X) = \sum_{s=1}^S u(\max(0, X(s) - RP(X))) \pi_s,$$

$$(9) \quad PT^-(X) = -\frac{1}{\beta} \sum_{s=1}^S u(\min(0, X(s) - RP(X))) \pi_s$$

respectively.  $PT^+$  measures positive deviations of assets' returns with respect to the reference point. By contrast, the reward measure ignores assets' payoffs which are below the reference point, i.e. losses. Rather than affecting portfolio reward, we believe losses have an impact on portfolio risk. In fact, losses are determined by means of risk measure  $PT^-$ . Finally, the index of loss aversion  $\beta$  measures the investor's tradeoff between gains and losses.

The reward measure  $PT^+$  is monotone, but in general not positive homogeneous, translation equivariant, or invariant. Risk measure  $PT^-$  is anti-monotone, but in general not positive homogeneous, translation equivariant, or invariant.<sup>4</sup> Moreover, risk measure  $PT^-$  is generally not convex. Convexity is a common assumption for risk measures since it guarantees that investors benefit from diversification, i.e. assign lower risk to the diversified portfolio than to the single positions. This reflects investors' preferences for diversified portfolios, as described by a concave utility index. However, prospect theory suggests that investors show a different risk attitude when faced with losses, and undiversified portfolios might be preferred to diversified portfolios since they offer a greater probability to reduce losses.

Obviously,

$$V(X) = PT^+(X) - \beta PT^-(X)$$

and therefore prospect theory investors possess  $(PT^+, PT^-)$ -preferences (see Assumption 1).

Portfolio choice problem (2) becomes:

$$(10) \quad \max_{\boldsymbol{\lambda} \in \mathbb{R}^{K+1}} PT^+ [\boldsymbol{\lambda}' \mathbf{R} w_0] \quad \text{s.t.} \quad PT^- [\boldsymbol{\lambda}' \mathbf{R} w_0] \leq pt^-, \quad \boldsymbol{\lambda}' \mathbf{e} = 1.$$

Parameter  $pt^-$  represents an investor's risk aversion and is closely related to his index of loss aversion  $\beta$ . In the sequel, we will consider  $pt^-$  as the parameter reflecting investors' loss aversion. As discussed in Section I, under some conditions on  $PT^-$  and  $PT^+$ , optimization problem (10) is equivalent to the maximization of the investor's value function (5) under his budget constraint, provided that solutions to this latter problem exist. However, if we take the utility index

$$(11) \quad u(x) = \begin{cases} x^{\alpha^+} & x \geq 0 \\ -\beta (-x)^{\alpha^-} & x < 0. \end{cases},$$

suggested by Kahneman and Tversky (1979), investors would optimally leverage infinitely any risky portfolio with strictly positive value, under mild conditions on parameters  $\alpha^+$ ,  $\alpha^-$  and  $\beta$ . In fact, for  $RP(X) = w_0 q(X) = w_0 R_0$  (risk-free return),  $\alpha^- = \alpha^+ = \alpha$ , and assuming  $\lambda_0 < 1$ , we have

$$V(w_0 \boldsymbol{\lambda}' \mathbf{R}) = w_0^\alpha (1 - \lambda_0)^\alpha \sum_{s=0}^S \pi_s u \left( \sum_{k=1}^K \tilde{\lambda}_k (R_k(s) - R_0) \right) = w_0^\alpha (1 - \lambda_0)^\alpha V(\tilde{\boldsymbol{\lambda}}' \mathbf{R}_1).$$

where  $\tilde{\lambda}_k = \frac{\lambda_k}{1 - \lambda_0}$  for  $k = 1, \dots, K$  and  $\tilde{\lambda}_0 = 0$ , i.e.  $\tilde{\boldsymbol{\lambda}}$  is a portfolio of risky assets only. Thus, if  $V(\tilde{\boldsymbol{\lambda}}' \mathbf{R}_1) > 0$ , then the investor optimally leverages infinitely the portfolio  $\tilde{\boldsymbol{\lambda}}$  and no solution to the maximization problem exists. The existence of a risky portfolio with positive

value can be easily checked.<sup>5</sup> For example, for a yearly horizon, and assuming  $\alpha = 0.88$  and  $\beta = 2.25$ , the prospect theory value of the market portfolio of Fama and French (1993) is strictly positive, if we take each observation as a scenario and we assume that all scenarios have the same probability. The infinite leverage problem with power utility for prospect theory was first described by De Giorgi, Hens, and Levy (2004), who studied the existence of CAPM equilibria with cumulative prospect theory preferences and suggested an alternative parametrization of the prospect theory by means of a negative exponential utility index.

The reward-risk representation of prospect theory solves the infinite leverage problem by assuming that prospect theory investors also limit their risk exposure, as measured by  $PT^-$ . Indeed, prospect theory models investors' preferences in terms of a trade-off between gains and losses. As long as investors are compensated by gain opportunities, they accept to face potential losses. The extreme case is that of an infinite leverage, where investors' optimal portfolios provide unbounded losses and unbounded gains, while losses are compensated by gains. In other words, an infinite leverage is an optimal portfolio choice with unbounded risk. By contrast, the reward-risk representation of prospect theory focuses on both the reward and the risk dimension, such that investors maximize the portfolio's reward, while they also bound the portfolio's risk. Consequently, the reward-risk perspective incorporates both investors' preferences toward risk (which are described by prospect theory) and their risk ability. This is the main difference compared to the value function maximization.

The conditions on reward and risk measures given in Proposition 2 in order to obtain mutual fund separation can be reformulated for the utility index and the reference point

defining prospect theory measures  $PT^+$  and  $PT^-$ . The following corollary sets sufficient conditions on the utility index  $u$  and the reference point  $RP$  such  $(PT^+, PT^-)$ -optimal allocations satisfy mutual fund separation.

**Corollary 2.** *Let  $PT^+$  and  $PT^-$  be defined as in equations (8) and (9), respectively. Suppose that  $u$  is positive homogeneous of degree  $\gamma_+$  on  $\mathbb{R}^+$ , and positive homogeneous of degree  $\gamma_-$  on  $\mathbb{R}^-$ . Moreover, assume that  $RP$  is positive translation equivariant and positive homogeneous, then mutual fund separation holds for all optimal solutions to the portfolio choice problem (10).*

*Proof.* Let  $\kappa > 0$  then

$$\begin{aligned}
PT^+(\gamma X) &= \sum_{s=1}^S u(\max(0, \kappa X(s) - RP(\kappa X))) \pi_s = \sum_{s=1}^S u(\max(0, \kappa X(s) - \kappa RP(X))) \pi_s \\
&= \sum_{s=1}^S u(\kappa \max(0, X(s) - RP(X))) \pi_s = \kappa^{\gamma_+} \sum_{s=1}^S u(\max(0, X(s) - RP(X))) \pi_s \\
&= \kappa^{\gamma_+} PT^+(X).
\end{aligned}$$

Let  $a \in \mathbb{R}$ , then

$$\begin{aligned}
PT^+(X + a) &= \sum_{s=1}^S u(\max(0, X(s) + a - RP(X + a))) \pi_s \\
&= \sum_{s=1}^S u(\max(0, X(s) + a - RP(X) - a)) \pi_s = \sum_{s=1}^S u(\max(0, X(s) - RP(X))) \pi_s \\
&= PT^+(X).
\end{aligned}$$

Therefore  $PT^+$  is translation invariant and positive homogeneous of degree  $\gamma_+$ . Similarly, one can show that  $PT^-$  is translation invariant and positive homogeneous of degree  $\gamma_-$ . Con-

sequently, according to Corollary 1, mutual fund separation holds for any optimal portfolio choice. □

The assumption that the reference point is positive translation equivariant implies that adding a sure payoff to the current portfolio will affect the reference point by the same amount. In other words, investors' reward and risk measures only capture gains and losses deriving from the risky part of their portfolios, while certain payoffs are already incorporated in the reference point, because investors can achieve them in all states of nature.

Corollary 2 immediately implies the following result for Kahneman and Tversky (1979) specification of the prospect theory value function.

**Corollary 3.** *Let  $u$  be defined as in equation (11) and suppose that  $RP$  is positive translation equivariant and positive homogeneous. Then mutual fund separation holds for all optimal solutions to the portfolio choice problem (10).*

Note that if the utility index  $u$  is defined as in equation (11) and there is no probability weighting ( $w(p) = p$  for  $p \in [0, 1]$ , see Assumption 2), then

$$PT^-(X) = LPM(X; \alpha-, RP).$$

Moreover, for  $\alpha = 1$  and any target value  $RP$ , the ratio

$$\frac{PT^+(X)}{PT^-(X)}$$

corresponds to the omega ratio introduced by Shadwick and Keating (2002) for the valuation of hedge fund portfolios.

### III Empirical analysis

#### A Data description

We analyze investors' optimal choices between three assets: cash, bonds, and stocks. The stock market portfolio is proxied by the CRSP all-share index, a value-weighted average of common stocks listed on NYSE, AMEX, and NADAQ. This index closely matches the common stock employed by Canner, Mankiw, and Weil (1997). The stock portfolio data are obtained from Kenneth French's online data library. The bond index is defined as the intermediate government bond index maintained by Ibbotson. Finally, as in Canner, Mankiw, and Weil (1997), we suppose that cash offers a riskless return equal to the mean return of the Treasury Bill. We use annual real returns for the period from January 1927 to December 2002. In Table 1 we summarize the main statistics of our data.

[Table 1 about here.]

#### B Prospect theory and mutual fund separation

We compare the optimal portfolio choices of investors with mean-variance preferences and with prospect theory ( $PT^+$ ,  $PT^-$ )-preferences. In order to be in accordance with real-world recommended allocations, we assume that investors face short-sale and borrowing constraints. This assumption does not matter for our analysis, since the asset allocation puzzle arises when short-sale and borrowing constraints are not binding. In fact, as dis-

cussed by Canner, Mankiw, and Weil (1997), when the borrowing constraint is binding, the bond/stock ratio increases with increasing risk aversion also with mean-variance preferences. This is consistent with the recommendation of financial advisors. Figure 1 reports the bond/stock ratio of mean-variance optimal allocations with borrowing and short sale constraints as a function of investors' target standard deviation, which describes investors' risk aversion. The bond/stock ratio is constant, equal to 1.5158, as long as the borrowing constraint is not binding. When the borrowing constraint becomes active (this is the case for a target standard deviation greater than 9.88%), the mean-variance optimal strategy of borrowing money and leveraging the risky portfolio with a bond/stock ratio of 1.5158 is not feasible anymore. Thus, when facing borrowing constraints, investors who otherwise want to borrow cash, optimally choose a portfolio of only stocks and bonds. Consequently, if such an investor becomes less risk averse, then he will increase the proportion of stocks and diminish that of bonds, while still preferring not to hold any cash.

Nevertheless, the asset allocation puzzle is the observation that the bond/stock ratio of popular financial advisors decreases with decreasing risk aversion also when short-sale and borrowing constraints are not binding. By contrast, in the latter case, mean-variance investors will hold the same portfolio of risky assets, thus have constant bond/stock ratio, as shown in Figure 1.

[Figure 1 about here.]

We now compute the bond/stock ratios for the prospect theory reward-risk investors. In

this framework, the portfolio selection problem presents several challenges. First, from the theoretical point of view, risk and reward measures  $PT^-$  and  $PT^+$  depend on the utility index  $u$  and the reference point  $RP$  (see Section II). Both reflect investors' preferences and affect optimal portfolio choices. In this empirical analysis we adopt the parametrization of prospect theory suggested by Kahneman and Tversky (see equation 11) and we limit ourselves to the piecewise linear case  $\alpha^+ = \alpha^- = 1$ , following previous applications of prospect theory to portfolio selection and asset pricing (see Benartzi and Thaler 1995, Barberis, Huang, and Santos 2001, Grüne and Semmler 2005). The reference point is assumed to be deterministic and we take values ranging from 1.00635 (which corresponds to the risk-free gross return) to 1.015. Finally, working with sample data, we consider each sample point as one equally likely state. Therefore, the probability weighting function characterizing prospect theory preferences (see Assumption 2) does not have any impact on the portfolio selection problem and we simply assume that no probability weighting takes place, i.e.  $\pi_s = p_s$  for all  $s = 1, \dots, S$ .

The second challenge refers to the numerical computation of  $(PT^-, PT^+)$ -optimal allocations. In general, the problem is not convex and several local maxima might exist. We adopt the algorithm described in De Giorgi, Hens, and Mayer (2006).

Figure 2 displays the bond/stock ratios for prospect theory investors as function of their target risk  $pt^-$ , which describes the degree of loss aversion. The reference point corresponds to the risk-free asset. As expected from Corollary 3, the bond/stock ratio is constant when the borrowing constraint is not binding. This reflects the mutual fund separation property of

prospect theory optimal allocations, when the utility index follows the Kahneman and Tversky specification and the reference point is positive homogeneous and positive translation invariant. The main observation from Figure 2 is that the bond/stock ratio for unconstrained optimal allocations (2.342) is much higher than that of unconstrained mean-variance optimal allocations (1.516). Accordingly, the proportion of bonds increased from 60% to 70%. This difference reflects the trade-off between gains and losses that is incorporated in prospect theory. This observation is consistent with the recommendations of popular advisors, who also suggest a much higher bond/stock ratio for conservative and moderate investors compared to the bond/stock ratio of mean-variance optimal choices.

[Figure 2 about here.]

We increase the reference point from 1.00635 to 1.01 (net returns below 1% are considered losses). Figure 3 shows the corresponding bond/stock ratio. The main difference with Figure 2 is that now the bond/stock ratio is overall not constant, thus also when the borrowing constraint is not binding. This result shows that when the reference point differs from the risk-free gross return, then mutual fund separation does not hold anymore. Note that the risk-free gross return is the unique fixed referent point that is positive homogeneous and positive translation equivariant. Figure 3 also shows that the bond/stock ratio increases with increasing loss aversion. The risk-free return, being below the reference point, is now perceived as a loss, thus contributing to portfolio's risk but not to portfolio's reward. Consequently, when investors become more loss averse (and thus allow a smaller target value  $pt^-$ )

they shift some of their wealth from cash to bonds. Indeed, bonds provide a greater upside potential (compared to cash) but a limited downside risk (compared to stocks). Note that in our data set, the average net return over gains<sup>6</sup> for bonds (stocks) is 3.29% (13.06%), while the average net return over losses is -2.09% (-5.47%).

The effect of increasing bond/stock ratio with increasing loss aversion becomes more pronounced if we further increase the reference point to 1.015, as shown in Figure 4.

[Figure 3 about here.]

[Figure 4 about here.]

We also analyzed the bond/stock ratio of optimal portfolio choices for prospect theory reward-risk investors with a deterministic reference point below the risk-free return. This is the case if, for example, investors evaluate gains and losses with respect to their current wealth. Having the reference point below the risk-free return, investors assign to the risk-free asset zero risk and strictly positive reward. Consequently, more loss averse investors can now achieve a positive reward without taking any risk and, consequently, do not need to create upside potential by investing in bonds. Contrary to the case with a reference point above the risk-free return, we now find that the bond/stock ratio increases with decreasing risk aversion, as long as short-sale constraints are not binding. As investors become less loss averse, they reduce the proportion of cash, while they also increase the proportion of bonds compared to stocks in order to control for portfolio's downside risk.

The results for the case where the reference point is below the risk-free gross return are not consistent with the portfolio choices recommended by financial advisors. Supported by our previous results for a reference point above the risk-free gross return, we conclude that financial advisors seem to behave as their reference point were greater than the risk-free return.

## IV Conclusion

We presented a reward-risk model for portfolio selection that is founded in the prospect theory of Kahneman and Tversky (1979). The reward measure captures portfolio payoffs above the reference point, while the risk measure captures portfolio payoffs below the reference point. Indeed, in prospect theory, the reference point describes investors' perception of gains and losses. We first analyzed general reward-risk models and we showed that only a few properties for reward and risk measures implied the mutual fund separation theorem for optimal allocations. If reward and risk measures can be transformed by means of increasing transformations into positive homogeneous and translation invariant or equivariant functionals, then reward-risk optimal allocations satisfy the separation property.

We applied these results to the prospect theory reward-risk model and we showed that the mutual fund separation property holds if investors possess piecewise power utility indexes and positive homogeneous and positive translation equivariant reference points. In the latter case, the bond/stock ratio remains constant with changing loss aversion, as long as borrowing

or short-sale constraint are not binding. However, in the general case the bond/stock ratio changes with changing loss aversion.

Finally, an empirical analysis with US data showed that investors with prospect theory reward-risk preferences having piecewise linear utility functions and a fixed reference point that is higher than the risk free gross return, will increase the bond/stock ratio in their optimal allocation if they become more loss averse. This result is consistent with the recommendations of financial advisors studied by Canner, Mankiw, and Weil (1997). Thus, the prospect theory reward-risk model rationalizes the asset allocation puzzle.

## References

- ACERBI, C. (2002): “Spectral Measure of Risk: A Coherent Representation of Subjective Risk Aversion,” *Journal of Banking and Finance*, 26(7), 1505–1518.
- ACERBI, C., AND D. TASCHE (2002): “Expected Shortfall: A Natural Coherent Alternative to Value at Risk,” *Economic Notes by Banca Monte dei Paschi di Siena SpA*, 31(2), 379–388.
- ARTZNER, P., F. DELBAEN, J.-M. EBER, AND D. HEATH (1999): “Coherent Measure of Risk,” *Mathematical Finance*, 9(3), 203–228.
- BAJEUX-BESNAINOU, I., J. JORDAN, AND R. PORTAIT (2001): “An Asset Allocation Puzzle: Comment,” *The American Economic Review*, 91(4), 1170–1179.

- BARBERIS, N., M. HUANG, AND T. SANTOS (2001): “Prospect Theory and Asset Prices,” *The Quarterly Journal of Economics*, 116(1), 1–53.
- BAWA, V., AND E. LINDENBERG (1977): “Capital Market Equilibrium in a Mean-Lower Partial Moment Framework,” *Journal of Financial Economics*, 5, 189–200.
- BENARTZI, S., AND R. THALER (1995): “Myopic Loss Aversion and the Equity Premium Puzzle,” *The Quarterly Journal of Economics*, 110(1), 75–92.
- BIGELOW, J. (1993): “Consistency of Mean-Variance Analysis and Expected Utility Analysis,” *Economics Letters*, 43, 187–192.
- BRENNAN, M., AND Y. XIA (2000): “Stochastic Interest Rates and the Bond-Stock Mix,” *European Finance Review*, 4(2), 197–210.
- (2002): “Dynamic Asset Allocation under Inflation,” *Journal of Finance*, 57(3), 1201–1238.
- BROGAN, A., AND S. STIDHAM (2005): “Non-Separation in the Mean-LPM Portfolio Optimization Problem,” Working paper, University of North Carolina.
- CAMPBELL, J., AND L. VICEIRA (2001): “Who Should Buy Long-Term Bonds?,” *The American Economic Review*, 91(1), 99–127.
- (2002): *Strategic Asset Allocation*. Oxford University Press, Oxford.

- CANNER, N., G. MANKIW, AND D. WEIL (1997): “An Asset Allocation Puzzle,” *The American Economic Review*, 87(1), 181–191.
- CASS, D., AND J. STIGLITZ (1970): “The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds,” *Journal of Economic Theory*, 2(2), 122–160.
- DE GIORGI, E. (2002): “A Note on Portfolio Selection Under Various Risk Measures,” NCCR-FINRISK Working paper, no. 9, Available at SSRN: <http://ssrn.com/abstract=762104>.
- (2005): “Reward-Risk Portfolio Selection and Stochastic Dominance,” *Journal of Banking and Finance*, 29(4), 895–926.
- DE GIORGI, E., T. HENS, AND H. LEVY (2003): “Existence of CAPM Equilibria with Prospect Theory Preferences,” Working Paper, University of Zurich.
- (2004): “Existence of CAPM Equilibria with Prospect Theory Preferences,” NCCR-FINRISK Working Paper, no. 85, Available at SSRN: <http://ssrn.com/abstract=420184>.
- DE GIORGI, E., T. HENS, AND J. MAYER (2006): “Computational Aspects of Prospect Theory and Asset Pricing Applications,” NCCR-FINRISK Working paper, no. 274, Available at SSRN: <http://ssrn.com/abstract=878086>.

- DE GIORGI, E., AND T. POST (2004): “Second Order Stochastic Dominance, Reward-Risk Portfolio Selection and the CAPM,” Working paper, University of Zurich and University of Lugano, Available at SSRN: <http://ssrn.com/abstract=643481>.
- ELTON, E., AND M. GRUBER (2000): “The Rationality of Asset Allocation Recommendations,” *Journal of Financial and Quantitative Analysis*, 35(1), 27–41.
- FAMA, E., AND K. FRENCH (1993): “Common risk-factors in the returns on stocks and bonds,” *Journal of Financial Economics*, 33(1), 3–56.
- FÖLLMER, H., AND A. SCHIED (2002): “Convex Measures of Risk and Trading Constraints,” *Finance and Stochastics*, 6(4), 429–447.
- FRITTELLI, M., AND E. ROSAZZA GIANIN (2004): “Dynamic Convex Risk Measures,” in *Risk Measures for the 21st Century*, ed. by G. Szegö. J. Wiley.
- GOMES, F., AND A. MICHAELIDES (2004): “A Human Capital Explanation for an Asset Allocation Puzzle,” Working paper, London Business School and CEPR.
- GROOTVELD, H., AND W. HALLERBACH (1999): “Variance vs Downside Risk: Is There Really That Much Difference?,” *European Journal of Operational Research*, 114(2), 304–319.
- GRÜNE, L., AND W. SEMMLER (2005): “Asset Prices and Loss Aversion,” mimeo Bielefeld University, Germany.

- HARLOW, W., AND R. RAO (1989): “Asset Pricing in a Generalized Mean-Lower Partial Moment Framework: Theory and Evidence,” *Journal of Financial and Quantitative Analysis*, 25, 285–311.
- JARROW, R., AND F. ZHAO (2006): “Downside Losses and Portfolio Management,” *Management Science*, forthcoming.
- JORION, P. (1997): *Value at Risk*. Irwin, Burr Ridge, IL.
- KAHNEMAN, D., AND A. TVERSKY (1979): “Prospect Theory: An Analysis of Decisions Under Risk,” *Econometrica*, 47(2), 263–291.
- KÖBBERLING, V., AND P. WAKKER (2005): “An Index of Loss Aversion,” *Journal of Economic Theory*, 122(1), 119–131.
- LINTNER, J. (1965): “The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets,” *Review of Economics and Statistics*, 47(1), 13–37.
- MAGILL, M., AND M. QUINZII (1996): *Theory of Incomplete Markets, Vol.1*. MIT-Press, Cambridge MA.
- MARKOWITZ, H. (1952): “Portfolio Selection,” *Journal of Finance*, 7(1), 77–91.
- (1959): *Portfolio Selection*. John Wiley & Sons, New York.

- MEYER, J. (1987): “Two-Moment Decision Models and Expected Utility Maximization,” *The American Economic Review*, 77(3), 421–430.
- MOSSIN, J. (1966): “Equilibrium in a Capital Asset Market,” *Econometrica*, 34(4), 768–783.
- MOUGEOT, N. (2003): “Interest Rate, Learning and the Asset Allocation Puzzle,” EFA 2003 Annual Conference Paper, No. 89, Available at SSRN: <http://ssrn.com/abstract=419740>.
- ROCKAFELLER, R., S. URYSEV, AND M. ZABARANKIN (2006): “Generalized Deviations in Risk Analysis,” *Finance and Stochastic*, 10(1), 51–74.
- SCHMIDT, U., AND H. ZANK (2002): “Risk Aversion in Cumulative Prospect Theory,” Discussion Paper 0207, School of Economic Studies, University of Manchester, UK.
- SHADWICK, W., AND C. KEATING (2002): “A Universal Performance Measure,” *The Journal of Performance Measurement*, 6(3), 59–84.
- SHALIT, H., AND S. YITZHAKI (2003): “An Asset Allocation Puzzle: Comment,” *The American Economic Review*, 93(3), 1002–1008.
- SHARPE, W. (1964): “Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk,” *Journal of Finance*, 19(3), 425–442.
- SIEBENMORGEN, N., AND M. WEBER (2003): “A Behavioural Model for Asset Allocation,” *Financial Markets and Portfolio Management*, 17(1), 15–42.

TOBIN, J. (1958): “Liquidity Preference as Behaviour Towards Risk,” *Review Economic Studies*, 25(2), 65–86.

TVERSKY, A., AND D. KAHNEMAN (1992): “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty*, 5, 297–323.

## V Appendix

We prove the following theorem:

**Proposition 2.** *Let  $\mu : \mathcal{G} \rightarrow \mathbb{R}$  and  $\rho : \mathcal{G} \rightarrow \mathbb{R}$  be a reward and risk measure on  $\mathcal{G}$ , respectively. Suppose that strictly increasing transformations  $T_\mu$  and  $T_\rho$  exist, such that  $\tilde{\mu} = T_\mu \circ \mu$  and  $\tilde{\rho} = T_\rho \circ \rho$  are positive homogeneous and satisfy one the following two properties:*

- (i) *translation invariance,*
- (ii) *translation equivariance.*

*Then, there exists a portfolio allocation  $\boldsymbol{\lambda}_p$ , such that for any  $(\mu, \rho)$ -optimal portfolio allocation  $\hat{\boldsymbol{\lambda}}$ , there exists  $\lambda_0$  with*

$$\hat{\boldsymbol{\lambda}} = \lambda_0 \mathbf{e}_0 + (1 - \lambda_0) \hat{\boldsymbol{\lambda}}_p,$$

where  $\mathbf{e}_0 = (1, \mathbf{0})' \in \mathbb{R}^{K+1}$ .

*Proof.* The  $(\tilde{\mu}, \tilde{\rho})$  portfolio optimization problem is

$$(12) \quad \max_{\boldsymbol{\lambda} \in \mathbb{R}^{K+1}} \tilde{\mu}[\boldsymbol{\lambda}' \mathbf{R} w_0] \quad \text{s.t.} \quad \tilde{\rho}[\boldsymbol{\lambda}' \mathbf{R} w_0] \leq \hat{\rho}, \quad \boldsymbol{\lambda}' \mathbf{e} = 1.$$

We first show that the  $(\mu, \rho)$  portfolio optimization problem (3) with target risk  $\bar{\rho}$  is equivalent to the  $(\tilde{\mu}, \tilde{\rho})$  optimization problem (12) with target risk  $\hat{\rho} = T_\rho(\bar{\rho})$ .

Let  $\hat{\boldsymbol{\lambda}}$  be a solution to the  $(\mu, \rho)$  optimization (3) and suppose that  $\tilde{\boldsymbol{\lambda}}$  exists such that

$$\tilde{\rho}[\tilde{\boldsymbol{\lambda}}' \mathbf{R} w_0] \leq T_\rho(\bar{\rho}) = \hat{\rho}$$

and

$$\tilde{\mu} \left[ \tilde{\boldsymbol{\lambda}}' \mathbf{R} w_0 \right] > \tilde{\mu} \left[ \hat{\boldsymbol{\lambda}}' \mathbf{R} w_0 \right].$$

Then, since  $T_\mu$  and  $T_\sigma$  are strictly increasing,

$$\rho \left[ \tilde{\boldsymbol{\lambda}}' \mathbf{R} w_0 \right] \leq \bar{\rho}$$

and

$$\mu \left[ \tilde{\boldsymbol{\lambda}}' \mathbf{R} w_0 \right] > \mu \left[ \hat{\boldsymbol{\lambda}}' \mathbf{R} w_0 \right].$$

This contradicts the optimality of  $\hat{\boldsymbol{\lambda}}$ . The same argument applies to  $T_\mu^{-1}$  and  $T_\sigma^{-1}$ , since the inverses are also strictly increasing. Thus the equivalence of (3) and (12) follows.

Now, we show mutual separation for the  $(\tilde{\mu}, \tilde{\rho})$  portfolio choice. Suppose first that  $\tilde{\mu}$  and  $\tilde{\rho}$  are positive homogeneous and *positive translation equivariant*. Let us take a portfolio  $\hat{\boldsymbol{\lambda}}_p$  that solves the optimization problem (12) for target risk  $\hat{\rho}_p$ . The portfolio gross return is denoted by  $R_p = w_0 \boldsymbol{\lambda}'_p \mathbf{R}$ . Let us now take another target risk  $\hat{\rho}$  and find  $\lambda_0$  such that  $\hat{\rho} = \lambda_0 w_0 R_0 + (1 - \lambda_0) \hat{\rho}_p$ , i.e.  $\lambda_0 = \frac{\rho - \hat{\rho}_p}{w_0 R_0 - \hat{\rho}_p}$ . Without loss of generality we assume  $1 - \lambda_0 > 0$ .<sup>7</sup>

Consider the linear combination

$$\boldsymbol{\lambda} = \lambda_0 \mathbf{e}_0 + (1 - \lambda_0) \hat{\boldsymbol{\lambda}}_p.$$

Then

$$\tilde{\rho} [\boldsymbol{\lambda}' \mathbf{R} w_0] = \lambda_0 w_0 R_0 + (1 - \lambda_0) \tilde{\rho} [R_p] \leq \lambda_0 w_0 R_0 + (1 - \lambda_0) \hat{\rho}_p = \hat{\rho}$$

and

$$\tilde{\mu} [\boldsymbol{\lambda}' \mathbf{R} w_0] = \lambda_0 w_0 R_0 + (1 - \lambda_0) \tilde{\rho}[R_p]$$

is minimal. In fact, if a portfolio  $\hat{\boldsymbol{\lambda}}$  exists that further minimizes the reward  $\tilde{\mu}$  for the target risk  $\hat{\rho}$ , then one can find a linear combination of this portfolio and the risk-free asset that further minimizes the reward  $\tilde{\mu}$  for the target reward  $\hat{\rho}_p$ , a contradiction to the definition of  $\hat{\boldsymbol{\lambda}}_p$ . Indeed, let  $\hat{\rho}$  be the risk of the portfolio  $\hat{\boldsymbol{\lambda}}$  and let  $\tilde{\lambda}_0 = -\frac{\lambda_0}{1-\lambda_0}$ . Then

$$\tilde{\boldsymbol{\lambda}}_p = \tilde{\lambda}_0 \mathbf{e}_0 + (1 - \tilde{\lambda}_0) \hat{\boldsymbol{\lambda}}.$$

satisfies

$$\tilde{\rho} [\tilde{\boldsymbol{\lambda}}_p' \mathbf{R} w_0] \geq \hat{\rho}_p$$

and

$$\begin{aligned} \tilde{\mu} [\tilde{\boldsymbol{\lambda}}_p' \mathbf{R} w_0] &= \tilde{\lambda}_0 w_0 R_0 + (1 - \tilde{\lambda}_0) w_0 \tilde{\mu} [\hat{\boldsymbol{\lambda}}' \mathbf{R}] \\ &> \tilde{\lambda}_0 w_0 R_0 + (1 - \tilde{\lambda}_0) w_0 \tilde{\mu} [\boldsymbol{\lambda}' \mathbf{R}] \\ &= \tilde{\lambda}_0 w_0 R_0 + (1 - \tilde{\lambda}_0) \{ \lambda_0 w_0 R_0 + (1 - \lambda_0) \tilde{\mu}[R_p] \} = \tilde{\mu} [R_p]. \end{aligned}$$

Note that  $1 - \tilde{\lambda}_0 = \frac{1}{1-\lambda_0} > 0$ . Finally, given an optimal solution  $\hat{\boldsymbol{\lambda}}_p$  for a given target risk  $\hat{\rho}_p$ , then any optimal solution for any target risk  $\tilde{\rho}$  corresponds to a linear combination of  $\hat{\boldsymbol{\lambda}}_p$  and the risk-free asset. This proves mutual fund separation for  $(\tilde{\mu}, \tilde{\rho})$ . Because of equivalence of (3) and (12) from the first step of this proof, the statement of the Proposition follows directly. The same argument applies to the case of translation invariant and negative equivariant reward or risk measures.  $\square$

## Notes

<sup>1</sup>Artzner, Delbaen, Eber, and Heath (1999) use the terminology “translation invariant” for the property that we call “negative translation equivariant” (see point (ix) of this definition). We prefer to associate the word “invariant” to the property of being invariant with respect to deterministic translation, while we associate the word “equivariant” to the property of varying in the same way (in absolute terms) as the underlying payoff.

<sup>2</sup>Suppose that  $\zeta$  is translation equivariance and homogeneous of degree  $\gamma > 0$ . Then

$$\zeta(\kappa(X + a)) = |\kappa|^\gamma \zeta(X + a) = |\kappa|^\gamma (\zeta(X) + a) = |\kappa|^\gamma \zeta(X) + |\kappa|^\gamma a.$$

On the other hand, we also have:

$$\zeta(\kappa(X + a)) = \zeta(\kappa X + \kappa a) = \zeta(\kappa X) + \kappa a = |\kappa|^\gamma \zeta(X) + \kappa a.$$

Since this is true for all  $\kappa \in \mathbb{R}$  (or  $\kappa > 0$ ) and  $a \in \mathbb{R}$ , we must have  $\gamma = 1$ .

<sup>3</sup>The risk-free return depends on portfolio initial value, which is the price in period one. Therefore, the risk-free rate of return is in general a moving target. But, if we assume that investors invests the total wealth  $w_0$ , then any optimal portfolio  $X^*$  satisfies  $q(X^*) = w_0$  and thus  $\tau = w_0 R_0$  which is fix and independent from the portfolio choice.

<sup>4</sup>Let  $\zeta : \mathcal{G} \rightarrow \mathbb{R}$  be a real-valued function on  $\mathcal{G}$ . We say that  $\zeta$  is:

(i) *monotone* if

$$\zeta(X) \leq \zeta(Y)$$

for all  $X, Y \in \mathcal{G}$  with  $X \leq Y$ ,

(ii) *anti-monotone* if

$$\zeta(X) \geq \zeta(Y)$$

for all  $X, Y \in \mathcal{G}$  with  $X \leq Y$ .

<sup>5</sup>The infinite leverage problem is relevant if there exists a risky portfolio  $\tilde{\boldsymbol{\lambda}}$  such that  $V(\tilde{\boldsymbol{\lambda}}' \mathbf{R}) > 0$ . Let  $Z(s) = \sum_{k=1}^K \tilde{\lambda}_k (R(s) - R_0)$  for  $s = 1, \dots, S$ ;  $Z(s)$  corresponds to the excess return of portfolio  $\tilde{\boldsymbol{\lambda}}$  in state  $s$ . Then  $V(\tilde{\boldsymbol{\lambda}}' \mathbf{R}) > 0$  if and only if

$$\frac{\sum_{Z(s) > 0} \pi_s Z(s)^\alpha}{\sum_{Z(s) < 0} \pi_s |Z(s)|^\alpha} > \beta.$$

For  $\alpha \approx 1$  and  $\beta > 1$ , this condition is usually stronger than

$$\frac{\sum_{Z(s) > 0} p_s Z(s)}{\sum_{Z(s) < 0} p_s |Z(s)|} > 1,$$

which is equivalent to  $\sum_{s=1}^S p_s Z(s) > 0$ , i.e. the expected excess return is positive. We computed these ratios for the market portfolio of Fama and French (1993) assuming that each observation represents a scenario and all scenarios have the same probability. For a yearly horizon, this We obtain 1.07, 1.40 and 2.60 for daily, monthly, and yearly excess returns, respectively. The ratios for  $\alpha = 0.88$  are slightly different, i.e. 1.08, 1.41, and 2.53, respectively. Thus, for yearly returns, the market portfolio of Fama and French (1993) represents an infinite leverage opportunity for prospect theory investors with power utility index with  $\alpha = 0.88$  and  $\beta = 2.25$ .

<sup>6</sup>Gains and losses are defined with respect to the reference point. Here we take  $RP = 1.01$ .

<sup>7</sup>The assumption  $1 - \lambda_0 > 0$  holds in general because investors' target risk  $\hat{\rho}_p$  and  $\hat{\rho}$  are greater than the risk free return. We also assume that a risky portfolio exists, that provides greater reward than the riskfree asset.

(a)			
Asset	Cash	Bond	Stock
Mean	0.635	2.204	8.587
Std deviation	4.225	6.909	21.046
Skewness	-1.086	0.195	-0.193
Kurtosis	4.880	0.585	-0.361

(b)			
	Cash	Bond	Stock
Cash	1	0.734	0.129
Bond	0.734	1	0.148

Table 1: Summary statistics of yearly real returns (in %) for Treasury bills, bond, and stock index. There are 76 observations ranging from 1927 to 2002. The bond index is from Ibbotson Associates, the stock index is from Kenneth French data library (<http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/>). Panel (b) reports the correlation between any pair of assets.

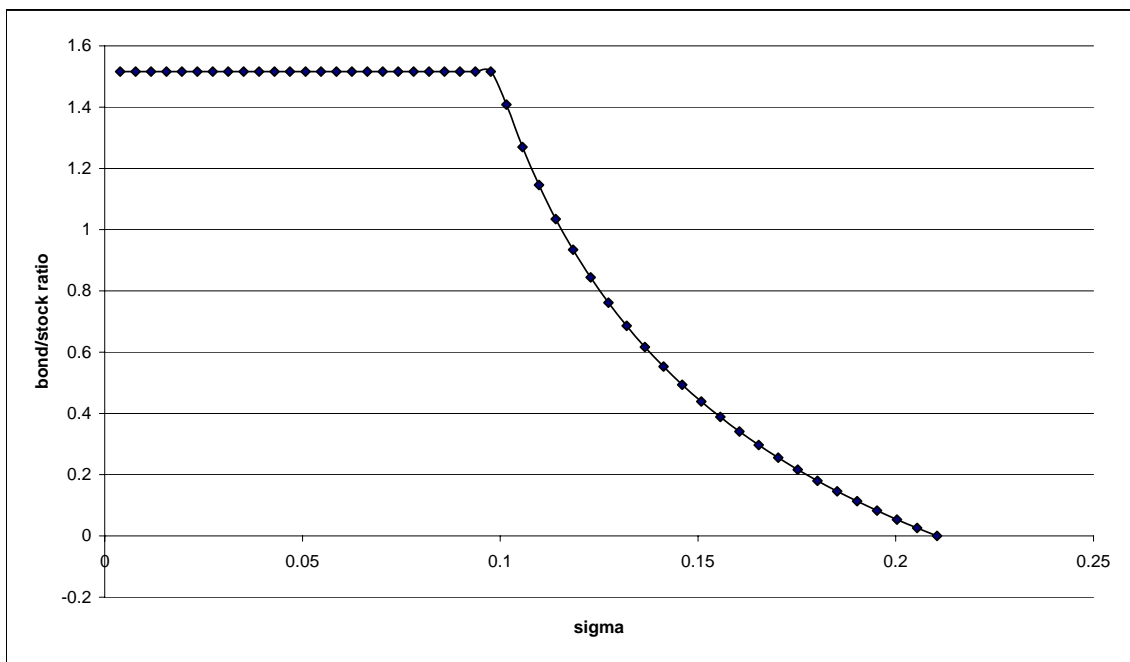


Figure 1: Bond/stock ratio as by investors' risk tolerance expressed in term of the target standard deviation, for the mean-variance optimal portfolio allocation with borrowing and short-sale constraints. With  $\sigma > 9.88\%$ , the borrowing constraint becomes binding and investors only hold stocks and bonds.

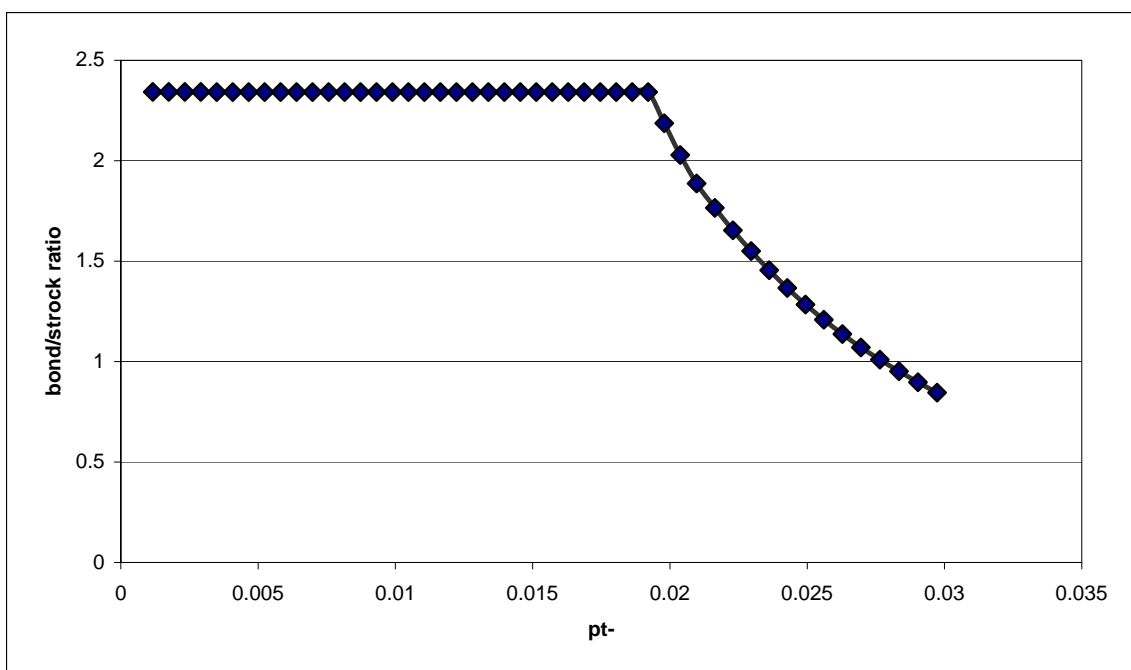


Figure 2: Bond/stock ratio as by investors' risk tolerance  $pt^-$  for the piecewise linear utility index. The reference point corresponds to the risk-free gross return 1.00635.

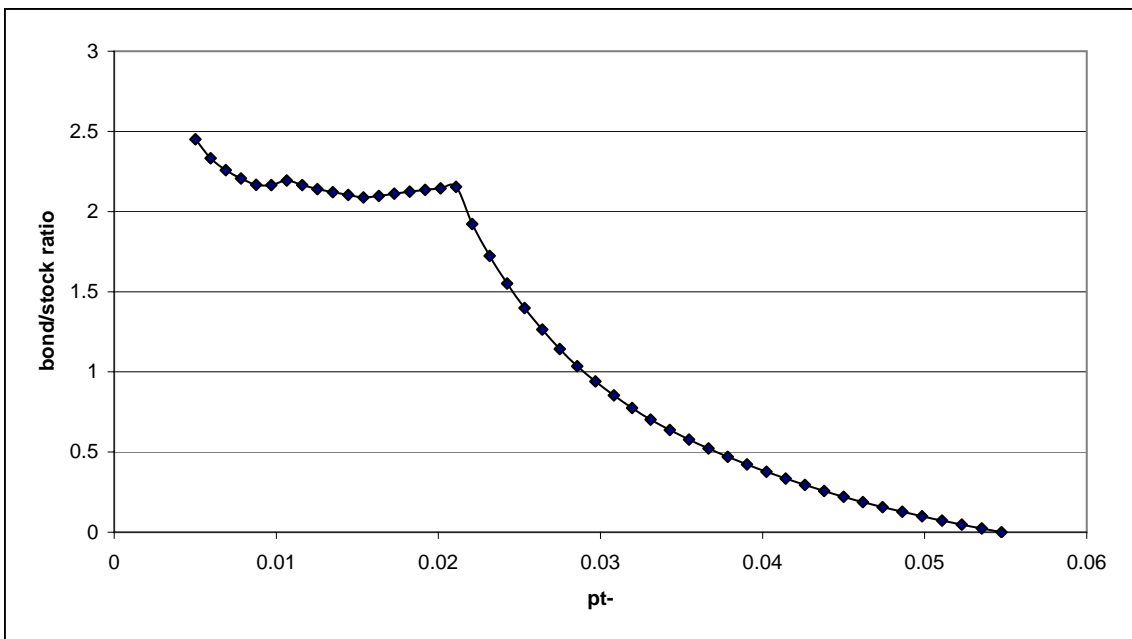


Figure 3: Bond/stock ratio as by investors' risk tolerance  $pt^-$  for the piecewise linear utility index. The reference point is  $RP=1.01$

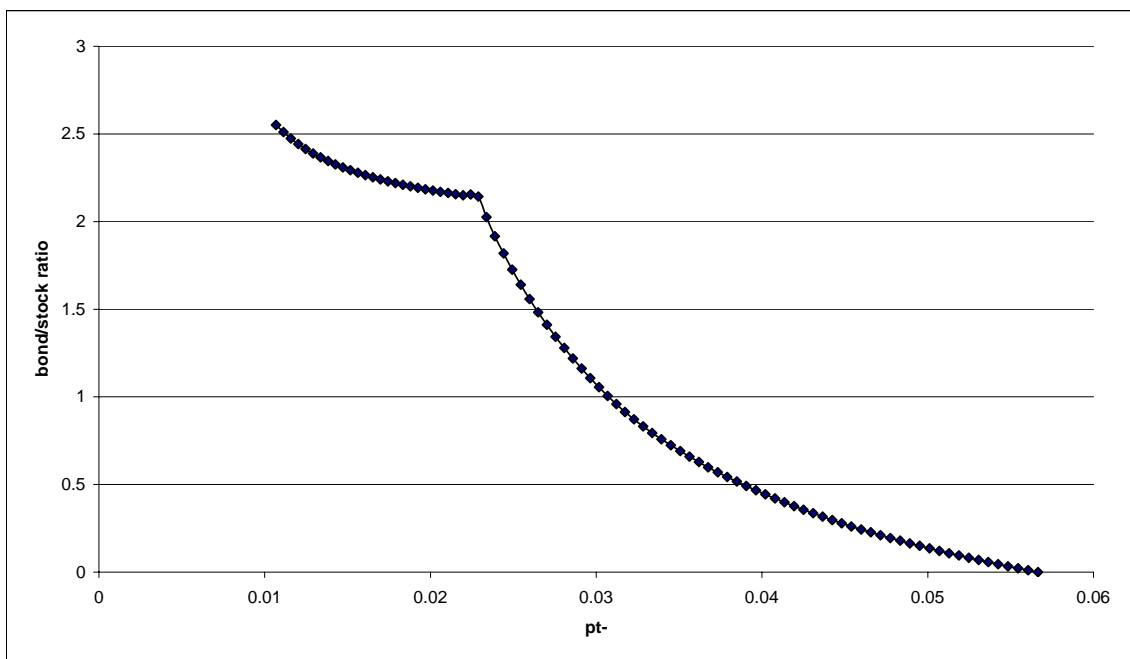


Figure 4: Bond/stock ratio as by investors' risk tolerance  $pt^-$  for the piecewise linear utility index. The reference point is  $RP=1.015$ .