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Valuation in a simplified Barndorff-Nielsen Shepard Model

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MEMM for Lévy Process-driven Stochastic Volatility Models

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1 Introduction

Asset processes driven by non-normal Lévy processes have become popular in the last few years. To be mentioned are models, where the asset processes are pure Lévy processes. Such models date back to the work of Mandelbrot (1967). But also more complex models as for example the stochastic volatility model of Barndorff-Nielsen and Shephard (2001) have been developed. This class of models, hereafter termed BN-S models, is constructed via a mean reverting, stationary process of the Ornstein Uhlenbeck type driven by a subordinator, a Lévy process with no Gaussian component and positive increments. In detail, such a model can look the following way:

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= \{\mu + \beta V_{t-}^2\}dt + V_{t-}dB_t + (e^{\rho x} - 1)dY_{\lambda t} \\ dV_t^2 &= -\lambda V_{t-}^2 dt + dY_{\lambda t}, \end{aligned}$$

where the parameters $\mu, \beta, \lambda, \rho$ are real constants with $\lambda > 0$ and $\rho \leq 0$. B is a Brownian motion while the process $Y = (Y_{\eta t})$ is a subordinator. This model allows to deal with the so called leverage type problem, i.e. for equities a fall in the price is associated with an increase in future volatility. The Brownian Motion B and the subordinator Y are independent.

The main reason for the use of Lévy process-driven asset models is the flexibility when fitting a model to observed asset prices. One drawback is the incompleteness of the corresponding financial market, which results in the existence of multiple equivalent martingale measures. A standard approach is to identify an optimal martingale measure on the basis of the utility function of the investor (see e.g. Kallsen (2001) for reference). In this paper, we pick the exponential utility function, which corresponds, as shown by Delbaen et al. (2002), to the minimal entropy martingale measure (MEMM).

The MEMM in case of pure Lévy processes has been analyzed by several authors (e.g. Chan (1999), Miyahara (2001), Fujiwara and Miyahara (2003) with increasing complexity). For more complex asset models as for example the BN-S model, the paper of Nicolato and Venardos (2002), where they analyzed the class of all equivalent martingale measures as well as the subclass of structure preserving martingale measures, is to be mentioned. Their paper adresses the whole range of possible martingales. The purpose of this paper is to investigate one specific measure for the class of complex Lévy models. We are not aware of any other paper addressing and solving this problem.

The main contribution of this paper is the determination of the MEMM in case of Lévy Process-driven stochastic volatility models in form of the solution of a boundary problem. Under appropriate conditions, existence of the solution is ensured. We further present the MEMM for a couple of such models and we show a new approach how to identify the MEMM in case of an additive asset process.

The paper is structured as follows. Section 2 introduces our setup and develops the conditions for the structure of the MEMM in case of any Lévy process-driven asset model. These conditions are very useful, as will be seen in section 3, where we apply them to the additive process case. It will be shown that the identified approach represents an elegant way to determine the MEMM in case of deterministic volatility models. Section 4 handles the case of stochastic volatility models. There, we get our main result of this paper, the determination of the MEMM via a partial differential equation. We conclude this paper in section 5 with a couple of examples. Its main goal is to show the complexity of the MEMM even in very simple stochastic volatility models.

Our approach has been influenced by Rheinländer (2003) as well as Hobson (2002) who analysed the minimal entropy martingale measures for different stochastic volatility models as for example the Stein-Stein and the Heston model. However, their analyses have been performed on models of pure Brownian motion type, i.e. without having any jumps. The appearance of jumps asks for other technics. Becherer's (2001) way to handle interacting systems of semi-linear PDE's helped us to solve these issues.

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2 Preliminaries

We start with some general assumptions, which are valid throughout the paper unless otherwise specified.
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ fulfills the usual conditions and is generated by a Lévy process Y with Y^c, μ_Y and $\nu_Y(dx, dt) = \nu(dx)dt$ being its continuous martingale part, the jump measure and its compensator respectively. For the sake of simplicity, we assume that $\langle Y^c \rangle_t = t$. We refer to Jacod and Shiryaev (1987) with respect to the notation used in this paper.

Remark 2.1 *The advantage of restricting the model to a filtration generated by a Lévy process is the so-called weak property of predictable representation:*

> *Let X be a semimartingale with X^c and μ_X being its continuous martingale part and the jump measure respectively. If the class of all continuous local martingales is equal to $\mathcal{L}(X^c)$ and the class of all discontinous, local martingales is equal to the space $\{W(x) * (\mu_X - \nu_X) \mid W \in \mathcal{G}(\mu_X)\}$[1], then we say that X has* the weak property of predictable representation.

It is a well-known result that if X is a Lévy process and the filtration is generated by X, then the weak representation property is fulfilled (see e.g. He et. al. (1992), theorem 13.49).

[1] $\mathcal{G}(\mu_X)$ is a class of functions ensuring that $W * \mu_X$ is well-defined (see Jacod and Shiryaev (1987), definition II.1.27)

Let $\mathcal{F} = \mathcal{F}_T$, where $T \in (0, \infty)$ is some finite fixed time horizon. We denote by S an $\mathbb{F}$-adapted, locally bounded semimartingale, which has the following canonical decomposition:

$$S_t = S_0 + M_t + A_t$$

with M being a locally bounded local martingale, and A being a predictable process with finite variation. Taking advantage of the weak property of predictable representation, we represent the local martingale M as

$$\begin{aligned} M_t &= M_t^c + M_t^d \\ &= \int_0^t \sigma_s^M dY_s^c + W^M(x) * (\mu_Y - \nu_Y) \end{aligned}$$

with M^c and M^d being the continuous and the discontinuous part of the local martingale M. Besides local boundedness of the asset process, we assume that the asset price process S satisfies the following structure condition:

Assumptions 2.1 (Structure Condition) *There exists a predictable process λ satisfying*

$$A_t = \int_0^t \lambda_s d\langle M \rangle_s,$$

whereas

$$K_T := \int_0^T \lambda_s^2 d\langle M \rangle_s < \infty \ \mathbb{P}\text{-}a.s. \tag{1}$$

Remark 2.2 *Similar to Ansel and Stricker (1992), one can show that the structure condition is a direct consequence of the no arbitrage property under the consideration of a reasonable set of potential strategies, provided that the semimartingale is special. Hence, this condition is very natural in an economics framework.*

Let us now introduce the notion of martingale measures:

Definition 2.1

1. *$\mathcal{V}$ is the linear subspace of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, spanned by the elementary stochastic integrals of the form $f = h(S_{T_2} - S_{T_1})$, whereas $0 \leq T_1 \leq T_2 \leq T$ are stopping times such that the stopped process S^{T_2} is bounded and h is a bounded $\mathcal{F}_{T_1}$-measurable random variable.*

2. *A signed martingale measure is a signed measure $\mathbb{Q} \ll \mathbb{P}$ with $E[\frac{d\mathbb{Q}}{d\mathbb{P}}] = 1$ and $E[\frac{d\mathbb{Q}}{d\mathbb{P}} f] = 0$ for all $f \in \mathcal{V}$.*

We denote by $\mathcal{M}(\mathbb{P})$ those martingale measures with nonnegative density and by $\mathcal{M}^e(\mathbb{P})$ the subset of $\mathcal{M}(\mathbb{P})$ consisting of probability measures which are equivalent to $\mathbb{P}$. Here and in the sequel, we identify measures with their densities. Note that, as S is locally bounded, a probability measure $\mathbb{Q}$ absolutely continuous to $\mathbb{P}$ is in $\mathcal{M}(\mathbb{P})$ if and only if S is a local $\mathbb{Q}$-martingale.
Let us recall the concept of relative entropy which is also known as Kullback-Leibler information.

Definition 2.2

1. *The relative entropy* $I(\mathbb{Q},\mathbb{R})$ *of the probability measure* $\mathbb{Q}$ *with respect to the probability measure* $\mathbb{R}$ *is defined as*

$$I(\mathbb{Q},\mathbb{R}) = \begin{cases} E_{\mathbb{R}}\Big[\frac{d\mathbb{Q}}{d\mathbb{R}}\log\frac{d\mathbb{Q}}{d\mathbb{R}}\Big], & \text{if } \mathbb{Q} \ll \mathbb{R}, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is well known that $I(\mathbb{Q},\mathbb{R}) \geq 0$ *and that* $I(\mathbb{Q},\mathbb{R}) = 0$ *if and only if* $\mathbb{Q} = \mathbb{R}$.

2. *The minimal entropy martingale measure, in the following also abbreviated as MEMM,* $\mathbb{Q}^E$ *is the solution of*

$$\min_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} I(\mathbb{Q},\mathbb{P}).$$

Theorems 1,2 and Remark 1 of Frittelli (2000) as well as the fact that $\mathcal{V} \subset L^\infty(\mathbb{P})$ yield the following.

Theorem 2.1 *If there exists* $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$ *such that* $I(\mathbb{Q},\mathbb{P}) < \infty$*, then the minimal entropy martingale measure exists, is unique and moreover is equivalent to* $\mathbb{P}$.

A criterion for a martingale measure to coincide with the MEMM has been introduced by Grandits and Rheinländer (2002):

Theorem 2.2 *Assume there exists* $\bar{\mathbb{Q}} \in \mathcal{M}^e(\mathbb{P})$ *with* $I(\bar{\mathbb{Q}},\mathbb{P}) < \infty$*. Then,* $\bar{\mathbb{Q}}$ *is the minimal entropy martingale measure if and only if the following hold:*

- $$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} = \exp\Big(c_T + \int_0^T \phi_s dS_s\Big) \tag{2}$$

  for a constant c *and an* S*-integrable predictable process* ϕ.

- $E_{\mathbb{Q}}[\int_0^T \phi_t dS_t] = 0$ *for all equivalent martingale measures with finite relative entropy* $I(\mathbb{Q},\mathbb{P})$.

Remark 2.3 *Due to the above results, the proposed strategy to find the MEMM is the following:*

- *Find some candidate martingale measure which can be represented as in* (2).
- *Check the following elements:*
  1. $\exp\Big(c + \int_0^T \phi_t dS_t\Big)$ *is integrable with*

     $$E\Big[\exp\Big(c_T + \int_0^T \phi_t dS_t\Big)\Big] = 1,$$

     i.e. via $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(c + \int_0^T \phi_t dS_t\Big)$*, a true measure* $\mathbb{Q}$ *is defined.*
  2. $I(\mathbb{Q}^*,\mathbb{P}) < \infty$.
  3. $\int_0^\cdot \phi_t dS_t$ *is a true* $\mathbb{Q}$*-martingale for all* $\mathbb{Q} \in \mathcal{M}^e$ *with finite relative entropy.*

We conclude this section with a condition for the candidate process ϕ_t:

Theorem 2.3 *The optimal strategy ϕ_t must fulfill*

$$
\begin{aligned}
c_T &+ \int_0^T \Big[\frac{1}{2}(\sigma_t^L - \lambda_t\sigma_t^M)^2 + \phi_t\lambda_t(\sigma_t^M)^2 + \phi_t\lambda_t \int_{\mathbb{R}} (W_t^M(x))^2\nu(dx)\Big]dt \\
&= \int_0^T \Big(\sigma_t^L - (\phi_t + \lambda_t)\sigma_t^M\Big)dY_t^c \\
&\quad + \Big(W^L(x) - (\phi+\lambda)W^M(x)\Big) * (\mu_Y - \nu_Y) \\
&\quad + \Big(\log(1 - \lambda W^M(x) + W^L(x)) + \lambda W^M(x) - W^L(x)\Big) * \mu_Y
\end{aligned} \tag{3}
$$

with σ_t^L and $W^L(x)$ are defined in such a way that

$$
\int_0^t \sigma_s^M \sigma_s^L ds + W^M(x)W^L(x) * \mu_Y \tag{4}
$$

is a local martingale.

Proof: Let us consider the density $Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$. We may write the density process $Z = (Z_t)$ as a stochastic exponential of the form

$$
Z_t = \mathcal{E}(-\int \lambda dM + L)_t
$$

with L as well as $[L, M]$ being local $\mathbb{P}$-martingales. This representation is due to the Girsanov theorem, just remember that S is a local $\mathbb{Q}$-martingale if, and only if, $S + \frac{1}{Z_-}[S, Z]$ is a local $\mathbb{P}$-martingale. However, this is here the case since

$$
S + \frac{1}{Z_-}d[S, Z] = \int \lambda d\langle M\rangle + M - \int \lambda d[M] + [L, M].
$$

Let us characterize, due to the weak representation property, the local martingale L the following way:

$$
L_t = \int_0^t \sigma_s^L dY_s^c + W^L(x) * (\mu_Y - \nu_Y).
$$

In combination with the strict orthogonality of L and M, one directly gets condition (4).
Let us now consider $\log Z$ by applying the Itô lemma:

$$
\begin{aligned}
\log Z_t &= \int_0^t \frac{1}{Z_{s-}}dZ_s - \frac{1}{2}\int_0^t \frac{1}{Z_{s-}^2} d\langle Z^c\rangle_s + \sum_{s\le t}(\log Z_s - \log Z_{s-} - \frac{1}{Z_{s-}}\Delta Z_s) \\
&= -\int_0^t \lambda_s dM_s + L_t - \frac{1}{2}\int_0^t \lambda_s^2 d\langle M^c\rangle_s + \int_0^t \lambda_s d\langle M^c, L^c\rangle_s - \frac{1}{2}\langle L^c\rangle_t \\
&\quad + \sum_{s\le t}(\log\frac{Z_s}{Z_{s-}} + \Delta\int_0^s \lambda dM - \Delta L_s) \\
&= \int_0^t (\sigma_s^L - \lambda_s\sigma_s^M)dY_s^c - \frac{1}{2}\int_0^t (\lambda_s\sigma_s^M - \sigma_s^L)^2 ds \\
&\quad + \Big(W^L(x) - \lambda W^M(x)\Big) * (\mu_Y - \nu_Y) \\
&\quad + \Big(\log(1 - \lambda W^M(x) + W^L(x)) + \lambda W^M(x) - W^L(x)\Big) * \mu_Y.
\end{aligned}
$$

On the other hand, due to the theorem 2.2, we may write

$$\begin{aligned}
\log Z_T &= c_T + \int_0^T \phi_t dS_t \\
&= c_T + \int_0^T \phi_t \sigma_t^M dY_t^c + \phi W^M(x) * (\mu_Y - \nu_Y) \\
&\quad + \int_0^T \Big(\phi_t \lambda_t (\sigma_t^M)^2 + \phi_t \lambda_t \int_{\mathbb{R}} (W_t^M(x))^2 \nu(dx)\Big) dt.
\end{aligned}$$

We get equation (3) by combining the two equations introduced above.

q.e.d.

Rheinländer (2003) used a similar approach in the case of continuous processes and got an equation of the form

$$\frac{1}{2} K_T = c_T + \int_0^T (\phi_t + \lambda_t) dS_t + L_T + \frac{1}{2}\langle L\rangle_T.$$

As it is obvious by a direct comparison of the two equations, the discontinuous process case is quite different, such that methods used in the continuous case are not applicable in our situation. However, even though equation (3) looks complexe, interesting results can be directly determined by it as we will see in the following section.

Remark 2.4 *An approach to handle the equation* (3) *is to consider the problem under the assumption that the purely discontinuous local martingales* M^d *and* L^d *belong to the class of processes with finite variation as well as* L *is locally bounded. This helps reducing the complexity of the problem, since then, the processes* M^d, L^d *as well as* $[M, L]$ *belong to the class of processes with locally integrable variation* $\mathcal{A}_{loc}$. *The advantage of this is that one may write*

$$W^M(x) * (\mu_Y - \nu_Y) = W^M(x) * \mu_Y - W^M(x) * \nu_Y$$

as well as the predictable bracket process $\langle M, L\rangle$ *exists, which, since* $[M, L]$ *being a local martingale, must be zero. Therefore, under these simplifications, the conditions* (3) *and* (4) *reduce to*

$$\begin{aligned}
& c_T + \int_0^T \Big[\frac{1}{2}(\sigma_t^L - \lambda_t \sigma_t^M)^2 + \phi_t \lambda_t (\sigma_t^M)^2\Big] dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \Big(W_t^L(x) - (\phi_t + \lambda_t) W_t^M(x) + \phi_t \lambda_t (W_t^M(x))^2\Big) \nu(dx) dt \\
&= \int_0^T \Big(\sigma_t^L - (\phi_t + \lambda_t)\sigma_t^M\Big) dY_t^c \\
&\quad + \Big(\log(1 - \lambda W^M(x) + W^L(x)) - \phi W^M(x)\Big) * \mu_Y
\end{aligned} \tag{5}$$

and

$$\sigma_t^M \sigma_t^L + \int_{\mathbb{R}} W_t^M(x) W_t^L(x) \nu(dx) = 0. \tag{6}$$

3 The Deterministic Volatility Case

The purpose of this section is to present the usefulness of equation (3) with respect to the identification of the MEMM. We will present this at hand of a special case which has attracted some attention in the last few years, namely the determination of the MEMM in case of a pure Lévy process. Several approaches have been presented to identify. However, we believe that the approach presented in this paper is, differently to the others, more powerful and takes better into consideration the wellknown properties of the MEMM.

Let us specify our asset process

$$S_t = S_0 + A_t + \int_0^t \sigma_s^M dY_s^c + W^M(x) * (\mu_Y - \nu_Y)$$

the following way:

Assumptions 3.1

- *There is a strictly positive, adapted càdlàg process V, such that*

$$\begin{aligned} W_t^M(x) &= V_{t-} f_t(x), \\ \sigma_t^M &= V_{t-}\sigma_t, \\ A_t &= \int_0^t V_{s-}\eta_s ds, \end{aligned}$$

  with f_t, σ_t and η_t being deterministic, bounded functions.

- $\sigma_t \geq \sigma_* > 0$ *for all* $t \in [0, T]$.

As discussed in remark 2.4, we will heuristically identify the solution under simplified conditions. The justification of the solution will then be performed on the basis of the model as determined by assumptions 3.1. Let us therefore rewrite equation (5) using the above notation:

$$\begin{aligned} & c_T + \int_0^T \Big(\frac{1}{2}(\sigma_t^L - V_{t-}\lambda_t\sigma_t)^2 + V_{t-}^2\phi_t\lambda_t\sigma_t^2\Big)dt \\ & \quad + \int_0^T \int_{\mathbb{R}} \Big(W_t^L(x) - V_{t-}(\phi_t + \lambda_t)f_t(x) + V_{t-}^2\phi_t\lambda_t f_t^2(x)\Big)\nu(dx)dt \\ & = \int_0^T \Big(\sigma_t^L - V_{t-}(\phi_t + \lambda_t)\sigma_t\Big)dY_t^c \\ & \quad + \Big(\log(1 - V_-\lambda f(x) + W^L(x)) - V_-\phi f(x)\Big) * \mu_Y \end{aligned} \tag{7}$$

Because of

$$\lambda_t = \frac{\eta_t}{V_{t-}(\sigma_t^2 + \int_{\mathbb{R}} f_t^2(x)\nu(dx))},$$

$\widehat{\lambda}_t := V_{t-}\lambda_t$ is a deterministic function. It is obvious that c_T is constant, if $\widehat{\phi}_t := V_{t-}\phi_t$, σ_t^L and $W_t^L(x)$ are deterministic functions as well as both terms with random elements have zero weights, i.e.

$$\sigma_t^L = (\widehat{\phi}_t + \widehat{\lambda}_t)\sigma_t, \tag{8}$$

$$W_t^L(x) = \widehat{\lambda}_t f_t(x) - 1 + \exp(\widehat{\phi}_t f_t(x)). \tag{9}$$

Replacing σ_t^L and $W_t^L(x)$ in equation (6), we immediately get the following condition for $\widehat{\phi}_t$:

$$\begin{aligned} 0 &= \sigma_t^2(\widehat{\phi}_t + \widehat{\lambda}_t) + \widehat{\lambda}_t \int_{\mathbb{R}} f_t^2(x)\nu(dx) + \int_{\mathbb{R}} f_t(x)\Big(\exp\{\widehat{\phi}_t f_t(x)\} - 1\Big)\nu(dx) \\ &= \eta_t + \sigma_t^2 \widehat{\phi}_t + \int_{\mathbb{R}} f_t(x)\Big(\exp\{\widehat{\phi}_t f_t(x)\} - 1\Big)\nu(dx). \end{aligned}$$

Hence, a strategy ϕ as identified by above equation is a potential candidate. In the following, we will present that this candidate not only fulfills all necessary conditions but also is valid without the restrictive conditions of Remark 2.4:

Theorem 3.1 *Suppose that there exists a deterministic function $\widehat{\phi} : [0,T] \to \mathbb{R}$, such that for any $t \in [0,T]$*

$$\int_{\mathbb{R}} |f_t(x)||\exp\{\widehat{\phi}_t f_t(x)\} - 1|\nu(dx) < \infty, \tag{10}$$

$$\eta_t + \sigma_t^2 \widehat{\phi}_t + \int_{\mathbb{R}} f_t(x)\Big(\exp\{\widehat{\phi}_t f_t(x)\} - 1\Big)\nu(dx) = 0, \tag{11}$$

then the MEMM $\mathbb{Q}^*$ exists and is defined as

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \exp\Big\{c_T + \int_0^T \frac{\widehat{\phi}_t}{V_{t-}} dS_t\Big\}.$$

c_T is the normalizing constant.

Proof: Condition (10) is necessary such that the integral in equation (11) is well-defined. Due to assumptions 3.1, the process $\widehat{\phi}_t$ is uniformly bounded:

$$|\widehat{\phi}_t| \leq \Big|\frac{\eta_t}{\sigma_t^2}\Big| \leq \frac{\max_{0\leq s\leq T}|\eta_s|}{\sigma_*^2}.$$

Even if $W^M(x) * \mu_Y$ and $W^L(x) * \mu_Y$ are not in $\mathcal{A}_{loc}$, the following statements are still correct:

- $\Big(W^L(x) - (\phi + \lambda)W^M(x)\Big) * \mu_Y = \Big(\exp\{\widehat{\phi}f(x)\} - 1 - \widehat{\phi}f(x)\Big) * \mu_Y \in \mathcal{A}_{loc}$:

  Local boundedness is obvious, further

  $$\Big|\exp\{\widehat{\phi}f(x)\} - 1 - \widehat{\phi}f(x)\Big| * \nu_Y < \infty$$

  since

  $$(\widehat{\phi}f(x))^2 \geq \Big|\exp\{\widehat{\phi}f(x)\} - 1 - \widehat{\phi}f(x)\Big|$$

  in a neighborhood of the origin $x = 0$.

- $[M, L] \in \mathcal{A}_{loc}$:

  Finite variation is ensured by definition of the quadratic variation; local boundedness is ensured by the structure of L as well as boundedness of $\widehat{\phi}$.

Therefore, the identified solution defined by (8), (9) and (11) is also a solution to theorem 2.3. In the following, we show that $\mathbb{Q}^*$ is an equivalent probability measure, that $I(\mathbb{Q}^*, \mathbb{P}) < \infty$ and that $\int_0^T \frac{\widehat{\phi}_t}{V_t} dS_t$ is a true $\mathbb{Q}$-martingale for all $\mathbb{Q} \in \mathcal{M}^e$ with finite relative entropy:

1. $\mathbb{Q}^*$ is an equivalent probability measure:

   Let us consider the following lemma:

Lemma 3.1 *Let N be a locally bounded local $\mathbb{P}$-martingale. Let $\mathbb{Q}$ be a measure defined by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t = \mathcal{E}(N)_t,$$

where $\Delta N > -1$. Set

$$U_t = \frac{1}{2}\langle N^c\rangle_t + \sum_{s\leq t}\{(1+\Delta N_s)\log(1+\Delta N_s) - \Delta N_s\}$$

and assume $U \in \mathcal{A}_{loc}$ as well as $U_T < \infty$ $\mathbb{P}$-a.s. Then, $\mathbb{Q}$ is an equivalent probability measure.

Proof: $Z_T > 0$ by assumption, hence $\mathbb{P} \approx \mathbb{Q}$. Since $(1+\Delta N)\log(1+\Delta N) \stackrel{\Delta N \to -1}{\longrightarrow} 0$ and N is locally bounded with localizing sequence T_n, U^{T_n} is bounded. Let $\mathbb{Q}^{(n)}$ be the measure with density

$$Z_T^{(n)} := Z_{T\wedge T_n}.$$

By Lepingle and Mémin (1979), Theorem III.1, it is a probability measure. By assumption,

$$\begin{aligned} \mathbb{Q}^{(n)}(T_n < T) &= E[Z_{T_n}^{(n)} 1_{\{T_n<T\}}] \\ &= E[Z_{T_n} 1_{\{T_n<T\}}] \\ &= E_{\mathbb{Q}}(T_n < T) \stackrel{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Then,

$$\begin{aligned} 1 = E[Z_T^{(n)}] &= E[Z_T 1_{\{T_n=T\}}] + E[Z_T^{(n)} 1_{\{T_n<T\}}] \\ &= E[Z_T 1_{\{T_n=T\}}] + \mathbb{Q}^{(n)}(T_n < T). \end{aligned}$$

Hence,

$$1 \geq E[Z_T] \geq E[Z_T 1_{\{T_n=T\}}] \stackrel{n\to\infty}{\longrightarrow} 1.$$

q.e.d.

Let us now consider the local martingale N^* defined as

$$\begin{aligned} N_t^* &:= -\int_0^t \lambda_s dM_s + L_t \\ &= \int_0^t \widehat{\phi}_s \sigma_s dY_s^c + \Big(\exp\{\widehat{\phi} f(x)\} - 1\Big) * (\mu_Y - \nu_Y). \end{aligned} \tag{12}$$

Since $\widehat{\phi}$ and f are bounded, N_t^* is locally bounded and $\Delta N_t^* > -1$. We must further show that

$$U_t = \frac{1}{2}\int_0^t \widehat{\phi}_s^2 \sigma_s^2 ds + \Big(\widehat{\phi} f(x)\exp\{\widehat{\phi} f(x)\} - \exp\{\widehat{\phi} f(x)\} + 1\Big) * \mu_Y \in \mathcal{A}_{loc}.$$

Since U is obviously locally bounded, it is sufficient to prove that

$$\Big(\widehat{\phi} f(x)\exp\{\widehat{\phi} f(x)\} - \exp\{\widehat{\phi} f(x)\} + 1\Big) * \mu_Y$$

has finite variation. For this purpose, let us analyze

$$g(z) = z^2 - z\exp z + \exp z - 1.$$

It can be easily shown that $g(z) \geq 0$ for all $z \leq 1$:

g reaches zero for $z_1 = 0$ and $z_2 = 1$. Further, $g'(z) = 2z - z \exp z$ and we have therefore two local extrema $\dot{z}_1 = 0$ and $\dot{z}_2 = \ln 2$. Analysing the convexity at the extrema, we get $g''(\dot{z}_1) > 0$ and $g''(\dot{z}_2) < 0$. Therefore, $g(z) \geq 0$ for all $z \leq 1$.

In addition, $z \exp z - \exp z + 1 \geq 0$. Hence,

$$\Big|\widehat{\phi} f(x) \exp\{\widehat{\phi} f(x)\} - \exp\{\widehat{\phi} f(x)\} + 1\Big| \leq (\widehat{\phi} f(x))^2$$

in a neighborhood of $f(x) = 0$, and therefore, U has finite variation. Another consequence of the above result is that $U_T < \infty$ $\mathbb{P}$-a.s. The conditions of lemma 3.1 are fulfilled and we have that $\mathbb{Q}$ is an equivalent martingale measure.

2. $I(\mathbb{Q}^*, \mathbb{P}) < \infty$:

The density $Z^* = \frac{d\mathbb{Q}^*}{d\mathbb{P}}$ may be written as $\exp(c + \int_0^T \phi_t dS_t)$. Hence,

$$\begin{aligned} I(\mathbb{Q}^*, \mathbb{P}) &= E_{\mathbb{Q}^*}[c_T + \int_0^T \frac{\widehat{\phi}_t}{V_{t-}} dS_t] \\ &= E_{\mathbb{Q}^*}\Big[c_T + \int_0^T \widehat{\phi}_t \Big(\eta_t dt + \sigma_t dY_t^c\Big) + \widehat{\phi} f(x) * (\mu_Y - \nu_Y)\Big]. \end{aligned}$$

We prove via the following lemma:

Lemma 3.2 *Let us introduce $\nu_Y^{\mathbb{Q}^*} = \exp\{\widehat{\phi} f(x)\} * \nu_Y$. Then, $f(x) * (\mu_Y - \nu_Y^{\mathbb{Q}^*})$ as well as $\int_0^t \sigma_s dY_s^c - \int \widehat{\phi}_s \sigma_s^2 ds$ are true $\mathbb{Q}^*$-martingales.*

Proof: We get the result by a direct application of Girsanov's theorem. Obviously,

$$m_t^d := f(x) * (\mu_Y - \nu_Y),$$
$$m_t^c := \int_0^t \sigma_s dY_s^c$$

are local $\mathbb{P}$-martingales (even true $\mathbb{P}$-martingales). Hence,

$$\begin{aligned} m_t^d - \int_0^t \frac{1}{Z_{s-}} d\langle Z, m^d \rangle_s &= f(x) * (\mu_Y - \nu_Y) \\ &\quad + \int_0^t \int_{\mathbb{R}} \Big(- f_s(x) \exp\{\widehat{\phi}_s f_s(x)\} + f_s(x)\Big) \nu(dx) ds \\ &= f(x) * \mu_Y - \int_0^t \int_{\mathbb{R}} f_s(x) \exp\{\widehat{\phi}_s f_s(x)\} \nu(dx) ds, \\ m_t^c - \int_0^t \frac{1}{Z_{s-}} d\langle Z, m^c \rangle_s &= \int_0^t \sigma_s dY_s^c - \int_0^t \widehat{\phi}_s \sigma_s^2 ds \end{aligned}$$

are both local $\mathbb{Q}^*$-martingales. They are also true martingales since f, σ and $\widehat{\phi}$ are bounded.

q.e.d.

We know from above that $\widehat{\phi} f(x) * (\mu_Y - \nu_Y^{\mathbb{Q}^*})$ is a true $\mathbb{Q}^*$-martingale. Further, since $\widehat{\phi} f(x) * (\nu_Y^{\mathbb{Q}^*} - \nu_Y) < \infty$, we may write

$$\begin{aligned} E_{\mathbb{Q}^*}\Big[\widehat{\phi} f(x) * (\mu_Y - \nu_Y)\Big] &= \widehat{\phi} f(x) * (\nu_Y^{\mathbb{Q}^*} - \nu_Y) \\ &= \int_0^T \Big[\widehat{\phi}_t \int_{\mathbb{R}} f_t(x) (\exp\{\widehat{\phi}_t f_t(x)\} - 1) \nu(dx)\Big] dt \\ &= \int_0^T \widehat{\phi}_t \Big(- \eta_t - \sigma_t^2 \widehat{\phi}_t\Big) dt. \end{aligned}$$

On the other hand,

$$E_{\mathbb{Q}^*}[\int_0^T \widehat{\phi}_t \sigma_t dY_t^c] \quad = \quad \int_0^T \widehat{\phi}_t^2 \sigma_t^2 dt.$$

Therefore,

$$E_{\mathbb{Q}^*}[\int_0^T \frac{\widehat{\phi}_t}{V_t} dS_t] = E_{\mathbb{Q}^*}\Big[\int_0^T \widehat{\phi}_t(\eta_t dt + \sigma_t dY_t^c) + \widehat{\phi} f(x) * (\mu_Y - \nu_Y)\Big] = 0.$$

Hence, we have proven that $I(\mathbb{Q}^*, \mathbb{P})$ is finite.

3. $\int_0^\cdot \frac{\widehat{\phi}_t}{V_{t-}} dS_t$ is a true $\mathbb{Q}$-martingale for all $\mathbb{Q} \in \mathcal{M}^e$ with finite relative entropy:

   Since V_- is locally bounded away from zero, $\frac{\widehat{\phi}_t}{V_{t-}}$ is locally bounded and therefore, $\int_0^\cdot \frac{\widehat{\phi}_t}{V_{t-}} dS_t$ is a local $\mathbb{Q}$-martingale. Let us mention a very useful result by Rheinländer (2003):

   Lemma 3.3 *Let $\mathbb{Q}$ be an equivalent martingale measure with finite relative entropy. Let $\int \psi_t dS_t$ be a local $\mathbb{Q}$-martingale. Then $\int \psi_t dS_t$ is a true $\mathbb{Q}$-martingale if, for some $\beta > 0$ small enough, $\exp\Big\{\beta \int_0^T \psi_t^2 d[S]_t\Big\}$ is $\mathbb{P}$-integrable.*

   This is due to the inequality

$$\beta E_{\mathbb{Q}}\Big[\int_0^T \psi_t^2 d[S]_t\Big] \leq H(\mathbb{Q}, \mathbb{P}) + \frac{1}{e} E\Big[\exp\Big\{\beta \int_0^T \psi_t^2 d[S]_t\Big\} + 1\Big] < \infty$$

   and $\mathbb{Q}$-integrability of the quadratic variation ensures that $\int \psi_t dS_t$ is a true $\mathbb{Q}$-martingale.

   Let us again consider the process $\int \frac{\widehat{\phi}_t}{V_{t-}} dS_t$:

$$\begin{aligned} E_{\mathbb{Q}}\Big[\int_0^T \frac{\widehat{\phi}_t^2}{V_{t-}^2} d[S]_t\Big] &= E_{\mathbb{Q}}\Big[\int_0^T \widehat{\phi}_t^2 \sigma_t^2 dt + \widehat{\phi}^2 (f(x))^2 * \mu_Y\Big] \\ &\leq K^2\Big(H(\mathbb{Q}, \mathbb{P}) + \frac{1}{e} E\Big[\exp\Big\{\int_0^T \sigma_t^2 dt + (f(x))^2 * \mu_Y\Big\} + 1\Big]\Big) \end{aligned}$$

   with $K := \frac{\max_{0 \leq t \leq T} |\eta_t|}{\sigma_*^2}$ such that $|\widehat{\phi}_t| \leq K$. Further, $\exp\{\int_0^T \sigma_t^2 dt\}$ is deterministic and finite and

$$E\Big[\exp\{(f(x))^2 * \mu_Y\}\Big] = \int_0^T \int_{\mathbb{R}} \Big(e^{(f_t(x))^2} - 1\Big) \nu(dx) dt < \infty, \tag{13}$$

   see e.g. He et al. (1992), lemma 14.39.1. Hence, we get

$$E\Big[\exp\Big\{\int_0^T \sigma_t^2 dt + (f(x))^2 * \mu_Y\Big\}\Big] < \infty$$

   and $\int \frac{\widehat{\phi}_t}{V_{t-}} dS_t$ is therefore a true $\mathbb{Q}$-martingale for all $\mathbb{Q} \in \mathcal{M}^e$ with finite relative entropy.

We conclude that

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \exp\Big\{c_T + \int_0^T \frac{\widehat{\phi}_t}{V_t} dS_t\Big\}$$

fulfills all necessary conditions for being the MEMM $\mathbb{Q}^E$.

Q.E.D.

Remark 3.1

1. *It has to be noted that conditions (10) and (11) correspond to well known conditions for the MEMM in case of Lévy processes (e.g. condition (C) in Fujiwara and Miyahara (2003), or condition (4.4) in theorem B in Esche and Schweizer (2003)). Differently to these papers, we assumed local boundedness of the asset process, which, from an economical standpoint, we consider as a natural condition. On the other hand and differently to other papers, we presented the MEMM not only for Lévy processes. The class of additive processes is also covered in the assumptions 3.1.*

2. *The restriction of a nonvanishing continuous martingale part has been used to ensure that $\widehat{\phi}$ is uniformly bounded. However, this condition can be weakened without difficulties. It is sufficient to assume that*
$$\sigma_t^2 + \int_{\mathbb{R}} f_t^2(x)\nu(dx)$$
is bounded away from zero.

4 Models with Stochastic Volatility

Before we start with a detailed analysis, we restrict the Lévy process Y to the case where $\nu(\mathbb{R}) < \infty$ with bounded $supp(\nu)$, i.e. Y^d represents a compound Poisson process with bounded jumps. Let us now consider the class of so-called stochastic volatility models, which consists of asset price models of the following type:

$$\frac{dS_t}{S_{t-}} = \eta(t, V_{t-})dt + \sigma^M(t, V_{t-})dY_t^c + dW^M(t, V_{t-}, x) * (\mu_Y - \nu_Y) \quad (14)$$
$$dV_t = \eta^V(t, V_{t-})dt + \sigma^V(t, V_{t-})dY_t^c + dW^V(t, V_{t-}, x) * \mu_Y \quad (15)$$

with V being defined in a domain $E \subset \mathbb{R}$. We will often abbreviate the processes in such a way that we write η_t, σ_t and $W_t(x)$ instead of $\eta(t, V_{t-})$, $\sigma(t, V_{t-})$ and $W(t, V_{t-}, x)$ respectively.

Remark 4.1 *One could also allow a model where the volatility process V_t is dependent on S_{t-}, provided that the dependency is restricted to the jump component $W^V(t, x)$. It is then simple to show that we are capable to transform our problem into a probability space $(\Omega, \mathbb{P}')$, where the compensator ν_Y' of $\mu_Y' := W_t^V(x) * \mu_Y$ is independent of S_{t-}. The density transformation may then be written as follows:*
$$\frac{d\mathbb{P}}{d\mathbb{P}'} := \mathcal{E}\Big((e^Z - 1) * (\mu_V - \nu_V')\Big)$$
with $Z = \frac{d\nu_V}{d\nu_V'}$, such that V under $(\mathbb{P}, \mathcal{F})$ has the characteristic triplet as introduced in (15). On the other hand, Y^c will not be influenced by this transformation. Therefore, we want to directly work in a model, where V is independent of S_{t-}.

Other constraints are:

Assumptions 4.1

1. $\overline{\mathrm{supp}(\nu)}$ *is connected and contains an open set.*

2. *The coefficients $\eta^V(t, y)$ and $\sigma^V(t, y)$ are Lipschitz-continuous. Further, σ^V must be uniformly bounded in $[0, T] \times E$.*

3. *The coefficients* $\eta(t,y)$, $\sigma^M(t,y)$ *and* $W^M\underline{(t,y,x)}$ *are Lipschitz-continuous. Further,* $W_t^M(x)$ *is continuous and in* $(-1,0]$ *for all* $x \in \overline{\operatorname{supp}(\nu)}$ *as well as* σ_t^M *is uniformly bounded away from zero on* $[0,T]\times E$.

4. W^M *and* W^V *are in* $\mathcal{G}(\mu)$, *whereas* $W \in \mathcal{G}(\mu)$ *means that* W *is predictable,* $\int_{\mathbb{R}} |W_t(x)|\nu(dx) < \infty$ *and* $\sqrt{\sum_{i=1}^{\infty} \tilde{W}_{\tau_i}^2 1_{\tau_i \le t}} \in \mathcal{A}_{loc}$ *with* τ_i *are the jump times and*

$$\tilde{W}_t = \int_{\mathbb{R}} W_t(x)\mu(\{t\},dx) - \int_{\mathbb{R}} W_t(x)\nu(dx).$$

   Further, we assume that we may write

$$W^V(t,V_{t-},x) \quad = \quad w^V(t,V_{t-})\hat{w}^V(t,x)$$

   with w^V *being a Lipschitz-continuous functions in* V_{t-}.

5. $\widehat{\lambda}_t := \frac{\eta_t}{(\sigma_t^M)^2 + \int (W_t^M(x))^2\nu(dx)}$ *is positive and uniformly bounded on* $[0,T]\times E$.

Remark 4.2 *Due to Protter (1990), Theorem V.38, Assumptions 4.1.2 and 4.1.4 ensure that there exists a unique solution*

$$V_t = V_0 + \int_0^t \eta_s^V ds + \int_0^t \sigma_s^V dY_s^c + W^V(x) * \mu_Y,$$

which does not explode in $[0,T]$. *In addition, the same argument and Assumption 4.1.5 ensures on* $[0,T]$ *the existence of a solution*

$$S_t = S_0 + \int_0^t \eta_s ds + \int_0^t \sigma_s^M dY_s^c + W^M(x) * (\mu_Y - \nu_Y).$$

Due to $\nu(\mathbb{R}) < \infty$ and $\operatorname{supp}(\nu)$ is bounded, we turn our attention to equation (5). The jump times may, because of the finite Lévy measure ν, be counted in increasing order $0 =: \tau_0 < \tau_1 < \cdots$, such that we may write

$$\begin{aligned}
&\left[\log\left(1 - \widehat{\lambda} W^M(x) + W^L(x)\right) - \widehat{\phi} W^M(x)\right] * \mu_Y \\
&\quad = \sum_{i=1}^{\infty} \left[\log\left(1 - \widehat{\lambda}_{\tau_i} W_{\tau_i}^M(x_{\tau_i}) + W_{\tau_i}^L(x_{\tau_i})\right) - \widehat{\phi}_{\tau_i} W_{\tau_i}^M(x_{\tau_i})\right] 1_{\tau_i \le T}.
\end{aligned}$$

Let us consider an auxiliary process u which is defined as follows:

$$\begin{aligned}
\Delta u_{t,X_{t-}}(x) &:= u(t, X_{t-} + W_t^X(x)) - u(t, X_{t-}) \\
&:= \log\left(1 - \widehat{\lambda}_t W_t^M(x) + W_t^L(x)\right) - \widehat{\phi}_t W_t^M(x) \qquad (16)
\end{aligned}$$

where X_t is an $\mathcal{F}_t^Y$-adapted stochastic process. The structure of X_t is not obvious, since W^L and $\widehat{\phi}$ are not known. However, we use in the following the

important conjecture $\quad X_t = V_t$.

An indication for the appropriateness of this conjecture is the case when $W_t^V(x) = 0$. Then, u will not be affected and therefore $\Delta u_{t,V_{t-}}(x) = 0$. But this is consistent with the deterministic volatility case in section 3, especially condition (9). Obviously, this is not a stringent evidence

of our conjecture. Its correctness is only shown by the fact that using this conjecture, we will be able to identify the MEMM.
Considering again equation (5), one immediately sees that a jump of Y^d at time T should not have any impact. Therefore, we must have $\Delta u_{T,V_{T-}}(x) = 0$ respectively $u(T, \ .\ )$ has to be constant. Since we are free to choose this constant, we set

$$u(T, \ .\ ) = 0. \tag{17}$$

Applying Itô's lemma and taking in account that there are no jumps in (τ_i, τ_{i+1}), we get

$$\begin{aligned}
&u(\tau_{i+1}, V_{\tau_{i+1}-}) - u(\tau_i, V_{\tau_i -} + W^V_{\tau_i}(x_{\tau_i}))\\
&= \int_{\tau_i}^{\tau_{i+1}} du(t, V_t)\\
&= \int_{\tau_i}^{\tau_{i+1}} \Big[\frac{\partial}{\partial t} u(t, V_{t-}) + \eta^V_t \frac{\partial}{\partial V} u(t, V_{t-}) + \frac{(\sigma^V_t)^2}{2} \frac{\partial^2}{\partial V^2} u(t, V_{t-})\Big] dt\\
&\quad + \int_{\tau_i}^{\tau_{i+1}} \sigma^V_t \frac{\partial}{\partial V} u(t, V_{t-}) dY^c_t\\
&= \int_{\tau_i}^{\tau_{i+1}} \Big[\frac{\partial}{\partial t} u(t, V_{t-}) + (\mathcal{L}_t u)(V_{t-})\Big] dt + \int_{\tau_i}^{\tau_{i+1}} \sigma^V_t \frac{\partial}{\partial V} u(t, V_{t-}) dY^c_t
\end{aligned}$$

with the second-order differential operator

$$(\mathcal{L}_t f)(y) \quad = \quad \eta^V_t \frac{\partial}{\partial y} f(t,y) + \frac{(\sigma^V_t)^2}{2} \frac{\partial^2}{\partial y^2} f(t,y).$$

Obviously, we may therefore rewrite equation (5) as

$$\begin{aligned}
&c_T + u(0, V_0)\\
&= -\sum_{i=0}^{\infty} \int_{\tau_i \wedge T}^{\tau_{i+1}\wedge T} \Big[\frac{1}{2}(\sigma^L_t - \widehat{\lambda}_t \sigma^M_t)^2 + \widehat{\phi}_t \widehat{\lambda}_t (\sigma^M_t)^2 + \frac{\partial}{\partial t} u(t, V_{t-}) + (\mathcal{L}_t u)(V_{t-})\\
&\qquad + \int \Big(W^L_t(x) - (\widehat{\phi}_t + \widehat{\lambda}_t) W^M_t(x) + \widehat{\phi}_t \widehat{\lambda}_t (W^M_t(x))^2\Big) \nu(dx)\Big] dt\\
&\quad + \sum_{i=0}^{\infty} \int_{\tau_i \wedge T}^{\tau_{i+1}\wedge T} \Big[\sigma^L_t - (\widehat{\phi}_t + \widehat{\lambda}_t)\sigma^M_t - \sigma^V_t \frac{\partial}{\partial V} u(t, V_{t-})\Big] dY^c_t.
\end{aligned} \tag{18}$$

A solution to this problem might be

$$\begin{aligned}
&\frac{1}{2}(\sigma^L_t - \widehat{\lambda}_t \sigma^M_t)^2 + \widehat{\phi}_t \widehat{\lambda}_t (\sigma^M_t)^2 + \frac{\partial}{\partial t} u(t, V_{t-}) + (\mathcal{L}_t u)(V_{t-})\\
&+ \int \Big(W^L_t(x) - (\widehat{\phi}_t + \widehat{\lambda}_t) W^M_t(x) + \widehat{\phi}_t \widehat{\lambda}_t \Big(W^M_t(x)\Big)^2\Big) \nu(dx) \quad = \quad 0
\end{aligned} \tag{19}$$

with (17) in combination with

$$\sigma^L_t - (\widehat{\phi}_t + \widehat{\lambda}_t)\sigma^M_t = \sigma^V_t \frac{\partial}{\partial V} u(t, V_{t-}). \tag{20}$$

Let us introduce

$$\begin{aligned}
k^y(t, u_t) \quad := \quad & \frac{1}{2}\Big(\sigma^L_t(y) - \widehat{\lambda}_t(y) \sigma^M_t(y)\Big)^2 + \widehat{\phi}_t(y) \widehat{\lambda}_t(y) (\sigma^M_t(y))^2\\
& + \int \Big(W^L_t(y,x) - \Big(\widehat{\phi}_t(y) + \widehat{\lambda}_t(y)\Big) W^M_t(y,x) + \widehat{\phi}_t(y) \widehat{\lambda}_t(y) (W^M_t(y,x))^2\Big) \nu(dx).
\end{aligned} \tag{21}$$

(19) represents nothing else than a PDE of u of the form

$$\frac{\partial}{\partial t}u(t,y) + \mathcal{L}_t u(y) + k^y(t,u_t) = 0 \quad (22)$$

$$u(T,y) = 0 \quad \text{for all } y \in E, \quad (23)$$

provided that $\widehat{\phi}_t$, σ_t^L and $W_t^L(x)$ are functions of u_t.
By equation (20) in combination with condition (6), we get

$$\widehat{\phi}_t = -\frac{\sigma_t^V}{\sigma_t^M}\frac{\partial}{\partial V}u(t,V_{t-}) - \frac{\int W_t^M(x)W_t^L(x)\nu(dx)}{(\sigma_t^M)^2} - \widehat{\lambda}_t, \quad (24)$$

which, replaced in equation (16), leads to

$$\exp\Big\{\Delta u_{t,V_{t-}}(x) + \Big[-\frac{\sigma_t^V}{\sigma_t^M}\frac{\partial}{\partial V}u(t,V_{t-}) - \frac{\int W_t^M(z)W_t^L(z)\nu(dz)}{(\sigma_t^M)^2} - \widehat{\lambda}_t\Big]W_t^M(x)\Big\}$$
$$= 1 - \widehat{\lambda}_t W_t^M(x) + W_t^L(x). \quad (25)$$

With the help of the following lemma, we can show that under suitable conditions, $\widehat{\phi}_t$, σ_t^L and $W_t^L(x)$ may be written as functions of u_t.

Lemma 4.1 *Let us assume that u is a continuous bounded function in $\mathcal{C}_b([0,T]\times E)$ with $u_{t,V_{t-}} \in \mathcal{C}^{0,1}([0,T]\times E)$. Hence, equation (25) is uniquely solved by a function $W^L(t,V_{t-},x) \in \mathcal{C}(\overline{\text{supp}(\nu)})$.*

Proof: We are aware that fixed point theorems are fairly standard. However, since we did not find the appropriate reference, we present the proof for our specific case for the sake of completeness.
We denote with $K := \overline{\text{supp}(\nu)}$ a compact set. Let us assume $t < T$. The existence of a function $f_t(x) := W_t^L(x)$ fulfilling equation (25) can be shown by Schauder's Fixed Point Theorem. Let us consider the continuous operator $S^t : \mathcal{C}(K) \to \mathcal{C}(K)$ with the Banach spaces $(\mathcal{C}(K), \|\cdot\|_\infty$, defined as

$$(S^t f)(x) := \exp\Big\{\Delta u_{t,V_{t-}}(x) + \Big[-\frac{\sigma_t^V}{\sigma_t^M}\frac{\partial}{\partial V}u(t,V_{t-}) - \widehat{\lambda}_t - \frac{\int W_t^M(z)f(z)\nu(dz)}{(\sigma_t^M)^2}\Big]W_t^M(x)\Big\}$$
$$-1 + \widehat{\lambda}_t W_t^M(x).$$

Since $W_t^M(x)W_t^M(y) \geq 0$ for any $x, y \in K$, we have for all $f \in \mathcal{C}(K)$

$$\underline{f}_t(x) \leq S^t f(x) \leq \overline{f}_t(x) \quad (26)$$

with

$$\underline{f}_t(x) := -1 + \widehat{\lambda}_t W_t^M(x)$$

$$\overline{f}_t(x) := e^{\Delta u_{t,V_{t-}}(x) + \Big[-\frac{\sigma_t^V}{\sigma_t^M}\frac{\partial}{\partial V}u(t,V_{t-}) - \widehat{\lambda}_t - \frac{\int W_t^M(z)\left(-1+\widehat{\lambda}_t W_t^M(z)\right)\nu(dz)}{(\sigma_t^M)^2}\Big]W_t^M(x)} - 1 + \widehat{\lambda}_t W_t^M(x).$$

We therefore define the closed, bounded, nonempty convex set

$$\mathcal{M} = \{f \in \mathcal{C}(K) \mid \underline{f}_t(x) \leq f(x) \leq \overline{f}_t(x)\},$$

for which we have $S^t(\mathcal{M}) \subset \mathcal{M}$. We are left with showing that the operator S^t is compact. For this purpose, let us consider a bounded set $U \in \mathcal{C}_b^1(K)$. We have to show that the set

$$V := \{S^t f \mid f \in U\} \subset \mathcal{C}(K)$$

is relatively compact, which is, due to the theorem of Arzela-Ascoli, equivalent to prove

- $\{g(x)\ ;\ g \in V\}$ is relatively compact in $\mathbb{R}$ for all $x \in K$.

  Due to (26) and the conditions on $\Delta u_{t,V_{t-}}$ and W_t^M, it is obvious that

$$-\infty < \inf_{z \in K} \underline{f}_t(z) \leq g(x) \leq \sup_{z \in K} \overline{f}_t(z) < \infty \tag{27}$$

  for all $x \in K$.

- For all $x \in K$ and every $\epsilon > 0$, there exists an $\delta(\epsilon, x) > 0$, independent of f, such that

$$\sup_{g \in V} |g(x) - g(y)| < \epsilon \ \text{ whenever } y \in K \text{ and } |x - y| < \delta(\epsilon, x).$$

  Let us write $|g(x) - g(y)|$:

$$\begin{aligned}
&|g(x) - g(y)| \\
&= \Big| \exp^{\Delta u_{t,V_{t-}}(x) + \left[-\frac{\sigma_t^V}{\sigma_t^M}\frac{\partial}{\partial V}u(t,V_{t-}) - \widehat{\lambda}_t - \frac{\int W_t^M(z)f(z)\nu(dz)}{(\sigma_t^M)^2}\right]W_t^M(x)} \\
&\quad - \exp^{\Delta u_{t,V_{t-}}(y) + \left[-\frac{\sigma_t^V}{\sigma_t^M}\frac{\partial}{\partial V}u(t,V_{t-}) - \widehat{\lambda}_t - \frac{\int W_t^M(z)f(z)\nu(dz)}{(\sigma_t^M)^2}\right]W_t^M(y)} \\
&\quad + \widehat{\lambda}_t\Big(W_t^M(x) - W_t^M(y)\Big)\Big|.
\end{aligned}$$

  We know that

$$\sup_{f \in U} \|f\|_\infty \int_a^b W_t^M(z)\nu(dz) \leq \int_{\mathbb{R}} W_t^M(z)f(z)\nu(dz) \leq -\sup_{f \in U}\|f\|_\infty \int_{\mathbb{R}} W_t^M(z)\nu(dz).$$

  On the other hand, by assumption, $\Delta u_{t,V_{t-}}$ is continuous in x, u is continuously differentiable in V and $W_t^M(.)$ is continuous. Hence, for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in K$ with $|x - y| < \delta$,

$$\sup_{g \in V} |g(x) - g(y)| < \epsilon.$$

The uniqueness of the solution can be shown as follows:

> Let us assume that there are two solutions f_t and g_t. Hence, we directly see that the following relation must be fulfilled:
>
> $$\frac{f_t(x)}{g_t(x)} = \exp\Big\{\frac{W_t^M(x)}{(\sigma_t^M)^2}\Big[-\int W_t^M(z)[f_t(z) - g_t(z)]\nu(dz)\Big]\Big\}.$$
>
> Let us now assume that $f_t(x) > g_t(x)$. However, we get a contradiction because the LHS of above equation is > 1 and the RHS is < 1. Therefore, f_t and g_t must be equal.

The case T is similar. We have only to take in account that $u(T,.) = 0$ and therefore $\frac{\partial}{\partial V}u(T, V_{T-}) = 0$.

q.e.d.

Remark 4.3 *In case of $\sigma_t^V = 0$, u only has to be a continuous bounded function without being differentiable in the second variable.*

Having introduced the PDE of the form (22) - (23), we will in the following present that there exists a solution $\hat{u}$ to this PDE, which defines a candidate measure. We will then finish this section by proving the conditions in remark 2.3 ensuring that this measure is the true MEMM. Let us therefore start with proving the existence of a solution for the above PDE. The following theorem is a slightly different version of a theorem presented by Becherer (2001):

Theorem 4.1 *Let D be a domain in $\mathbb{R}^d$, i.e. an open connected subset. For $(t,z) \in [0,T] \times D$, consider the following diffusion process*

$$Z_t^{t,z} = z \ \in \ D, \quad dZ_s = b(s, Z_s^{t,z})ds + \sum_{j=1}^{m} \Sigma_j(s, Z_s^{t,z})dW_s^j \tag{28}$$

for continuous functions $b : [0,T] \times D \to \mathbb{R}^d$ and $\Sigma_j : [0,T] \times D \to \mathbb{R}^d$, $j = 1, ..., m$, with an m-dimensional Brownian motion $W = (W^1, ..., W^m)^{tr}$. We write b and each Σ_j as a $(1 \times d)$ columns vector and define the $(d \times m)$ matrix-valued function Σ by $\Sigma^{ij} := (\Sigma_j)^i$. Let us define

$$(\mathcal{A}_t f)(x) = \sum_{i=1}^{d} b^i(t,x) \frac{\partial}{\partial x^i} f(t,x) + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t,x) \frac{\partial}{\partial x^i \partial x^j} f(t,x)$$

with

$$a^{ij}(t,x) := (\Sigma(t,x)\Sigma^{tr}(t,x))^{ij}.$$

We assume that

- ***a-1*** *The coefficients b and Σ_j, $j = 1, ..., m$, are on $[0,T] \times D$ locally Lipschitz-continuous in x, uniformly in t, i.e.*

  for each compact subset K of D, there is a constant c_K such that

  $|G(t,x) - G(t,y)| \leq c_K |x - y|$ *for all* $t \in [0,T], x, y \in K$ *and* $G \in \{b, \Sigma_j, j = 1, ..., m\}$.

- ***a-2*** *For all $(t,x) \in [0,T) \times D$, the solution $Z^{t,z}$ neither explodes nor leaves D before T, i.e. $\mathbb{P}\Big[\sup_{s \in [t,T]} \big|Z_s^{t,z}\big| < \infty\Big] = 1$ and $\mathbb{P}\Big[Z_s^{t,z} \in D$ for all $s \in [t,T]\Big] = 1$.*

- ***a-3*** *Let $g : [0,T] \times \mathcal{C}_b(D) \to \mathcal{C}_b(D)$ be a Lipschitz-continuous function, uniformly in t, i.e. there exists a constant $L < \infty$ such that*

  $$\|g(t, v_1) - g(t, v_2)\|_\infty \leq L \|v_1 - v_2\|_\infty \quad \forall t \in [0,T], \ v_1, v_2 \in \mathcal{C}_b(D).$$

- ***a-4*** *There exists a sequence $(D_n)_{n \in \mathbb{N}}$ of bounded domains with closure $\overline{D_n} \subset D$ such that $\cup_{n=1}^{\infty} D_n = D$, each D_n has a $\mathcal{C}^2$-boundary , and for each n,*

  - ***a-4.1*** *the coefficients b and a^{ij} are uniformly Lipschitz-continuous on $[0,T] \times \overline{D_n}$,*
  - ***a-4.2*** $\det a(t,x) \neq 0$ *for all $(t,x) \in [0,T] \times \overline{D_n}$,*
  - ***a-4.3*** *$(t,z,u) \to g^z(t,u)$ is uniformly Hölder-continuous on $[0,T] \times \overline{D_n} \times \mathcal{C}_b(D)$.*

Hence, there is a unique classical solution $u \in \mathcal{C}_b([0,T] \times D)$ with $u \in \mathcal{C}^{1,2}([0,T) \times D)$ to the problem

$$\frac{\partial}{\partial t} u(t,z) + (\mathcal{A}_t u)(z) + g^z(t, u_t) = 0, \tag{29}$$

$$u(T,z) = 0 \text{ for all } z \in E. \tag{30}$$

In addition, the solution is unique.

Proof: By Theorem V.38 in Protter (1990), the Lipschitz condition ***a-1*** implies that (28) has a unique strong solution on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ up to a possibly finite random explosion time. We impose by ***a-2***, that the solution $Z^{t,z}$ neither explodes to infinity nor leaves D before T.
Let us introduce the operator $F : \mathcal{C}_b([0,T] \times D) \to \mathcal{C}_b([0,T] \times D)$ with

$$(Fu)^z(t) := (Fu)(t,z) = E\Big[\int_t^T g^{Z_s^{t,z}}(s, u_s)ds\Big].$$

We now prove that F is a contraction on the space $\mathcal{C}_b([0,T] \times D)$ with respect to the norm

$$\|u\|_\beta := \sup_{(t,z)\in[0,T]\times D} e^{-\beta(T-t)}|u(t,z)|$$

for $\beta < \infty$ large enough. It is obvious that this norm is equivalent to the supremums-norm.
Due to ***a-3***, we obtain for $v, w \in \mathcal{C}_b([0,T] \times D)$ and $\beta > 0$

$$\begin{aligned}
&e^{-\beta(T-t)}|(Fv)(t,z) - (Fw)(t,z)| \\
&= \frac{1}{e^{\beta(T-t)}} \left| E\Big[\int_t^T \Big(g^{Z_s^{t,z}}(s, v(s)) - g^{Z_s^{t,z}}(s, w(s))\Big)ds\Big]\right| \\
&\leq \frac{1}{e^{\beta(T-t)}} \int_t^T E\left[\left|g^{Z_s^{t,z}}(s, v(s)) - g^{Z_s^{t,z}}(s, w(s))\right| e^{-\beta(T-s)}e^{\beta(T-s)}\right] ds \\
&\leq \frac{1}{e^{\beta(T-t)}} L\|v-w\|_\beta \int_t^T e^{\beta(T-s)}ds \\
&\leq \frac{L}{\beta}\|v-w\|_\beta
\end{aligned}$$

for all $t \in [0,T]$ and $z \in D$. Thus,

$$\|Fv - Fw\|_\beta \leq \|v-w\|_\beta$$

and F is therefore a contraction with respect to the norm $\|\cdot\|_\beta$ for $\beta > L$. Due to the Banach Fixed Point theorem, F has then a unique fixed point $\hat{u}$.
Let $w \in \mathcal{C}_b([0,T] \times D)$ with $w \in \mathcal{C}^{1,2}([0,T) \times D)$ and $u := Fw$. By the definition of F, we already have $u \in \mathcal{C}_b([0,T] \times D)$. Let us now consider the following partial differential equation:

$$\frac{\partial}{\partial t}u(t,z) + (\mathcal{A}_t u)(z) + g^z(t, w_t) = 0, \quad \text{on } [0,T) \times D \tag{31}$$

with boundary condition

$$u(T,z) = 0 \;\forall\; z \in D.$$

It is evident from the definition of u and F that u satisfies the terminal conditions. To prove the claim, it suffices to show for any $\varepsilon > 0$ that u is in $\mathcal{C}^{1,2}([0,T-\varepsilon) \times D)$ and satisfies (31) on $[0, T-\varepsilon) \times D$ instead of $[0,T) \times D$. Let us fix an arbitraty $\varepsilon \in (0,T)$ and set $T' := T - \varepsilon$. For any $(t,z) \in [0,T'] \times D$, using the above representation of $u(t,z)$, we may write

$$\begin{aligned}
u(t,z) &= (Fw)(t,z) \\
&= E\Big[\int_t^T g^{Z_s^{t,z}}(s, w_s)ds\Big] \\
&= E\Big[\,E\Big[\int_t^T g^{Z_s^{t,z}}(s, w_s)ds\Big|\mathcal{F}_{T'}\Big]\Big] \\
&= E\Big[u(T', Z_{T'}^{t,z}) + \int_t^{T'} g^{Z_s^{t,z}}(s, w_s)ds\Big]
\end{aligned}$$

using the strong Markov property of $Z^{t,z}$ for the last equation.
Therefore, via Theorem 1 in Heath and Schweizer (2000), we know that u is in $\mathcal{C}^{1,2}([0,T') \times E)$ and satisfies the PDE (31) on the set $(t,z) \in [0,T') \times E$, provided we can verify the assumptions (A1), (A2), and (A3') for their general Feynman-Kac type result. (A1) and (A2) are exactly ***a-1*** and ***a-2***. Conditions (A3') and (A3a') are exactly ***a-4*** and ***a-4.1***. Condition (A3c') is automatically fulfilled. The continuity of Σ in combination with assumption ***a-4.2*** implies the condition (A3b') and $u \in \mathcal{C}_b([0,T] \times D)$ implies (A3e').
In order to verify (A3d'), first note that w is continuously differentiable on $[0,T) \times D$ and therefore Lipschitz-continuous on bounded closed subsets $[0,T'] \times \overline{D_n}$ and via ***a-4.3***, this implies that the composition

$$\tilde{g} : (t,z) \to g^z(t, w_t) \text{ is uniformly Hölder-continuous on } [0,T'] \times \overline{D_n}.$$

So, (A3d') in Heath and Schweizer (2000) holds as well and we can apply their theorem 1 and obtain that u is in $\mathcal{C}^{1,2}([0,T') \times D)$ and satisfies (31) on the set $(t,z) \in [0,T') \times D$. Since $\varepsilon > 0$ was arbitrarily chosen, this implies that $u = Fw$ is in $\mathcal{C}^{1,2}([0,T) \times D)$ and satisfies the PDE (31) on $[0,T) \times D$. In particular, we have shown that

$$F \;\text{ maps }\; \mathcal{C}_b^{1,2}([0,T) \times D) \;\text{ into }\; \mathcal{C}_b^{1,2}([0,T) \times D)$$

and conclude from the contraction argument that there exists a fixed point $\hat{u}$, which is in $\mathcal{C}_b^{1,2}([0,T) \times D)$ and satisfies the PDE (29) with boundary conditions (30).

q.e.d.

In case of $a(t,x) = 0$ for all $(t,x) \in [0,T] \times D$, we get the following result:

Corollary 4.1 *Let D be a domain in $\mathbb{R}^d$. For $(t,z) \in [0,T] \times D$, consider the function*

$$Z_s^{t,z} = z + \int_0^s b(r, Z_r^{t,z})dr, \qquad \textit{with} \;\; b \neq 0$$

which is assumed to stay in D. In addition, we assume the following:

Let $g : [0,T] \times \mathcal{C}_b(D) \to \mathcal{C}_b(D)$ be a Lipschitz-continuous function, uniformly in t.

Hence, there exists a unique classical solution $u \in \mathcal{C}_b([0,T] \times D)$ with $u \in \mathcal{C}^{1,1}([0,T) \times D)$ to the problem

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,z) + b(t,z) \frac{\partial}{\partial z} u(t,z) + g^z(t, u_t) = 0 \\ &u(T,z) = 0 \;\; \forall \;\; z \in D. \end{aligned}$$

In addition, the solution is unique.

Proof: The proof is along the argumentation above. The operator $F : \mathcal{C}_b([0,T] \times D) \to \mathcal{C}_b([0,T] \times D)$ with

$$(Fv)^z(t) = \int_t^T g^{Z_s^{t,z}}(s, v_s) ds$$

obviously is a contraction in the corresponding normed space. Further, let $w \in \mathcal{C}_b([0,T] \times D)$ with $w \in \mathcal{C}^{1,1}([0,T) \times D)$, then $u = Fw$ is a solution to the differential equation

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,z) + b(t,z) \frac{\partial}{\partial z} u(t,z) + g^z(t, w_t) = 0 \\ &u(T,z) = 0 \;\; \forall \;\; z \in D, \end{aligned}$$

since

$$\begin{aligned} u(T,Z_T^{t,z}) &= u(t,z)+\int_t^T dv(s,Z_s^{t,z}) \\ &= u(t,z)+\int_t^T \Big[\frac{\partial}{\partial s}u(s,Z_s^{t,z})+b(s,Z_s^{t,z})\frac{\partial}{\partial V}u(s,Z_s^{t,z})\Big]ds \\ &= u(t,z)-\int_t^T g^{Z_s^{t,z}}(s,w(s))ds = 0. \end{aligned}$$

Hence, there is a fixed point $\hat{u}$ which uniquely solves the problem.

q.e.d.

Let us now apply these results to the case where $g^y(t,u_t)$ has the form (21). In this case, $g^y(t,u_t)$ does not have to be a Lipschitz-continuous function and is generally not well-defined on the whole set $[0,T]\times\mathcal{C}_b(E)$. With the introduction of an auxiliary function $\tilde{g}$, we then can show the following result:

Theorem 4.2 *We consider all assumptions 4.1. Let us assume that either*

$$a) \qquad \sigma^V = 0 \text{ and } \eta^V \neq 0$$

or

$$b) \qquad \sigma^V(t,y)\neq 0 \text{ for all } (t,y)\in[0,T]\times E.$$

Then, there is a classical solution $\hat{u}\in\mathcal{C}_b([0,T]\times E)$ *with - for a) -* $\hat{u}\in\mathcal{C}^{1,1}([0,T)\times E)$ *and - for b) -* $\hat{u}\in\mathcal{C}^{1,2}([0,T)\times E)$ *to the PDE*

$$\frac{\partial}{\partial t}u(t,y)+(\mathcal{L}_t)u(y)+g^y(t,u_t)=0 \tag{32}$$

with boundary condition

$$u(T,y)=0. \tag{33}$$

$\hat{u}$ *satisfies the Feynman-Kac representation*

$$\hat{u}(t,y)=E\Big[\int_t^T g^{\hat{V}_s^{t,y}}(s,\hat{u}_s)ds\Big] \tag{34}$$

with

$$d\hat{V}_s^{t,y}=\eta^V(s,\hat{V}_s^{t,y})ds+\sigma^V(s,\hat{V}_s^{t,y})dY_s^c$$

and $\hat{V}_t^{t,y}=y$.

Proof: We only consider the case $\eta^V(t,y)\neq 0$, the other case is proven similarly. Let us consider an auxiliary function

$$\tilde{g}_\varepsilon^y(t,v):=g^y(t,\kappa^\varepsilon(t,v)),$$

defined on $[0,T]\times E$, with, for any $\varepsilon>0$ small enough, the function κ^ε, Lipschitz-continuous for corresponding norms, transforms the continuous, uniformly bounded $v\in\mathcal{C}_b(E)$ into a differentiable function $\tilde{v}_t^\varepsilon\in\mathcal{C}_b^1(E)$, whereas $\|\cdot\|_\infty$ and $|||\cdot|||:=\|\cdot\|_\infty+\|\frac{\partial\cdot}{\partial y}\|_\infty$ are the norms of the corresponding spaces $\mathcal{C}(E)$ and $\mathcal{C}^1(E)$. In detail, the transformation looks the following:

- $\check{v}_t(y) := \min\Big(\max(-(T-t)(C-\varepsilon), v(y)), (T-t)(C-\varepsilon)\Big)$ with a suitably chosen constant C.

- $\check{v}_t$ is transformed into a function $\tilde{v}_t^\varepsilon \in \mathcal{C}^1(E)$, uniformly bounded with $|\tilde{v}_t^\varepsilon(y)| \leq (T-t)C$. In addition, $\tilde{v}_t^\varepsilon$ is constructed in such a way that it converges in the supremumsnorm of $\mathcal{C}_b(E)$ for $\varepsilon \to 0$ to $\check{v}_t$.

Hence, due to the boundary conditions of assumptions 4.1.2 - 4.1.4 as well as (27), for any $v \in \mathcal{C}_b^1(E)$, $\tilde{g}(t,v)$ is uniformly bounded. Further, for any fixed $(t_0, y_0) \in [0,T] \times E$, it can be easily shown that $g^{y_0}(t_0, v)$ is Lipschitz-continuous, uniformly in (t_0, y_0), on the set

$$\Big\{v \in \mathcal{C}_b^1(E); \|v\|_\infty \leq CT\Big\}:$$

Lipschitz-continuity is shown if one can show the Lipschitz-continuity of W^L with respect to v. However, since v is bounded, the Lipschitz-continuity is obvious with respect to the term Δv. The Lipschitz-continuity with respect to $\frac{\partial}{\partial y}v(y)$ can be directly seen by the following simplified example:

$$f(x) + \alpha = \exp\{\gamma - \beta g(x)\}$$

with α, γ being any constants and $\beta > 0$. x takes the role of $\frac{\partial}{\partial y}v(y)$. One directly sees that

$$\left|g'(x)\right| = \Bigg|\frac{1}{\frac{\partial x}{\partial g(x)}\Big|_{g^{-1}(x)}}\Bigg| = \left|\frac{x+\alpha}{\beta(x+\alpha)+1}\right| < \left|\frac{1}{\gamma\beta}\right|.$$

As a direct consequence and due to the assumptions 4.1.2-4.1.3, $\tilde{g} : [0,T] \times \mathcal{C}_b(E) \to \mathcal{C}_b(E)$ is a Lipschitz-continuous function, uniformly in t as well as $(t,y,u) \to \tilde{g}^y(t,u)$ is uniformly Hölder-continuous on $[0,T] \times \overline{E_n} \times \mathcal{C}_b(E)$ with E_n being any bounded domain with closure $\overline{E_n} \subset E$. Since all necessary conditions are fulfilled, Theorem 4.2 can be applied to the PDE (32) with $\tilde{g}$ instead of g, which gives us a unique bounded solution $\hat{u}^\varepsilon \in \mathcal{C}_b([0,T] \times E)$ with $\hat{u}^\varepsilon \in \mathcal{C}^{1,2}([0,T) \times E)$. We will show below that

$$|\hat{u}^\varepsilon(t,y)| \leq (T-t)C \tag{35}$$

for $(t,y) \in [0,T] \times E$. Admitting this result for a moment, let us consider the compact interval $[0,T'] \times \overline{E_n} \subset [0,T) \times E$ with $T' < T$. Since $\hat{u}^\varepsilon \in \mathcal{C}^{1,2}([0,T) \times E)$, $\hat{u}^\varepsilon$ is in the Banach space $\mathcal{C}_b^{1,2}([0,T'] \times \overline{E_n})$. Hence, we set $\varepsilon \to 0$ and we get that $\hat{u}_t^\varepsilon$ converges against a function $\hat{u}_t \in \mathcal{C}_b^{1,2}([0,T'] \times \overline{E_n})$, which in addition is uniformly bounded with $|\hat{u}_t(z)| \leq (T-t)C$. Since T' and $\overline{E_n}$ can be chosen arbitrarily, we get $\hat{u}_t \in \mathcal{C}_b([0,T] \times E)$ with $\hat{u}_t \in \mathcal{C}^{1,2}([0,T) \times E)$, which solves the PDE (32) with function g and satisfies (34).
To finish the proof, it remains to establish (35). In the following, any $\varepsilon > 0$ may be chosen, hence we leave beside the index ε. Let us fix arbitrary an $t \in [0,T]$ and $y \in E$ and define the time τ_y:

$$\tau_y := \inf\{s \in [t,T] \mid \hat{u}(s, \hat{V}_s^{t,y}) < (T-s)C\} \wedge T.$$

Obviously, $\hat{u}(s, \hat{V}_s^{t,y}) \geq (T-s)C$ for all $s \in [t, \tau_y)$ and $\hat{u}(\tau_y, \hat{V}_{\tau_y}^{t,y}) \leq (T-\tau_y)C$. Since $\hat{u}(s,y) \geq (T-s)C$ for all $s \in [t,\tau_y)$, we get $\Delta\tilde{u}_{s,\hat{V}_s^{t,y}} \leq 0$ as well as $\frac{\partial}{\partial y}\tilde{u}(s, \hat{V}_s^{t,y}) = 0$. We therefore get that

$$-1 + \widehat{\lambda}_t W_t^M(x) \leq W_t^L(x) \leq e^{\left[-\widehat{\lambda}_t - \frac{\int W_t^M(z)(-1+\widehat{\lambda}_t W_t^M(z))\nu(dz)}{(\sigma_t^M)^2}\right] W_t^M(x)} - 1 + \widehat{\lambda}_t W_t^M(x).$$

Therefore, $g^y(t,u_t)$ is uniformly bounded, i.e. there exists a constant C_1 such that $g^y(t,u_t) < C_1$. We may write

$$\begin{aligned}\hat{u}(t,y) &= E\Big[\int_t^T \tilde{g}^{\hat{V}_s^{t,y}}(s,\hat{u}_s)ds\Big] \\ &= E\Big[\int_t^{\tau_y} \tilde{g}^{\hat{V}_s^{t,y}}(s,\hat{u}_s)ds + \int_{\tau_y}^T \tilde{g}^{\hat{V}_s^{t,y}}(s,\hat{u}_s)ds\Big] \\ &= E\Big[\int_t^{\tau_y} \tilde{g}^{\hat{V}_s^{t,y}}(s,\hat{u}_s)ds\Big] + E\Big[\hat{u}(\tau_y,\hat{V}_{\tau_y}^{t,y})\Big] \\ &\leq C_1 E[\tau_y - t] + CE[T-\tau_y]\end{aligned}$$

due to the strong Markov property of $\hat{u}$.
The lower bound will be shown similarly. Let us define the time

$$\delta_y := \inf\{s\in[t,T] \mid \hat{u}(s,\hat{V}_s^{t,y}) > -(T-s)C\} \wedge T.$$

Hence, $\hat{u}(s,\hat{V}_s^{t,y}) \leq -(T-s)C$ for all $s\in[t,\delta_y)$ and $\hat{u}(\delta_y,\hat{V}_{\tau_y}^{t,y}) \geq -(T-\delta_y)C$. Symmetrically, we get $\Delta\tilde{u}_{s,\hat{V}_s^{t,y}} \geq 0$ as well as $\frac{\partial}{\partial y}\tilde{u}(s,\hat{V}_s^{t,y}) = 0$. Let us now show that there is a constant $C_2 > 0$ such that $\tilde{g}^y(s,\hat{u}_s) \geq -C_2$. Due to (27), W_s^L, $\widehat{\phi}_s$ and $\widehat{\sigma}_t^L$ are bounded from below. In addition, due to assumption 4.1.3,

$$\widehat{\phi}_s W_s^M(x)\Big(-1+\widehat{\lambda}_s W_s^M(x)\Big)$$

is also bounded from below. Therefore, one directly sees that $\tilde{g}^y$ as a whole is uniformly bounded from below, i.e. a constant C_2 exists. The same analysis as above provides

$$\hat{u}(t,y) \geq -C_2E[\delta_y - t] - CE[T-\delta_y].$$

Let us now choose $C = C_1 \vee C_2$ and we directly get (35).

q.e.d.

Remark 4.4 *The case ($\sigma^V = 0$, $\eta^V = 0$) is degenerate. Even though the above theorems are not applicable, one directly sees by the same arguments that one gets a classical solution $\hat{u}$ to the problem*

$$\frac{\partial}{\partial t}u(t,y) + g^y(t,u) = 0, \quad u(T,y) = 0$$

with, for any fixed $y\in E$, $\hat{u}(.,y)\in\mathcal{C}_b^1([0,T])$.

We conclude this section with the definition and justification of the MEMM in case of stochastic volatility asset models:

Theorem 4.3 *We assume all assumptions 4.1. In addition, we assume that there exists an $\alpha > 0$ such that*

$$E\Big[\exp\Big\{\alpha\int_0^T(\sigma_t^M)^2dt\Big\}\Big] < \infty \tag{36}$$

The solution $\hat{u}$ to the PDE (32) with boundary condition (33) defines the Minimal Entropy Martingale measure via $W^L(t,V_{t-},x)$ in lemma 4.1 and may be written as

$$Z_T = \frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}\Big(-\Big[\int(\widehat{\lambda}_t\sigma_t^M - \sigma_t^L)dY_t^c + (\widehat{\lambda}W^M(x) - W^L(x)) * (\mu_Y - \nu_Y)\Big]\Big)_T$$

with

$$\sigma_t^L = -\frac{\int W_t^M(x)W_t^L(x)\nu(dx)}{\sigma_t^M}.$$

Proof: To ensure that $\mathbb{Q}^*$ defines the MEMM, we know from remark 2.1, that one has to ensure the following three conditions:

1. $\mathbb{Q}^*$ is an equivalent probability measure.
2. $I(\mathbb{Q}^*, \mathbb{P}) < \infty$.
3. $\int_0^\cdot \frac{\widehat{\phi}_t}{V_t} dS_t$ is a true $\mathbb{Q}$-martingale for all $\mathbb{Q} \in \mathcal{M}^e$ with finite relative entropy.

We know that the density Z_T may be written as

$$Z_T = \exp\Big\{c_T + \int_0^T \frac{\widehat{\phi}_t}{S_{t-}} dS_t\Big\}$$

with $\widehat{\phi}_t$ being the strategy defined as in (24). Obviously, the strategy process is uniformly bounded. The above conditions can be proven in a similar manner as in Theorem 3.1.

With respect to condition 1, it has to be noted that Z_t never reaches zero, since

$$-\widehat{\lambda}_t W_t^M(x) + W_t^L(x) > -1$$

due to constraint (26). Further, since U of Lemma 3.1 fulfills $U \in \mathcal{A}_{loc}$ as well as $U_T < \infty$ $\mathbb{P}$-a.s., $\mathbb{Q}^*$ is an equivalent martingale measure.

Introducing

$$\nu_Y^{\mathbb{Q}^*} = (\widehat{\lambda}_s W_s^M(x) - W_s^L(x) - 1) * \nu_Y,$$

we get that $W^M(x) * (\mu_Y - \nu_Y^{\mathbb{Q}^*})$ as well as $\int_0^t \sigma_s^M dY_s^c - \int_0^t (\widehat{\lambda}_s \sigma_s^M - \sigma_s^L)\sigma_s^M ds$ are local $\mathbb{Q}^*$-martingales. Let us apply Lemma 3.3 to ensure that both terms are true $\mathbb{Q}$-martingales, i.e. we have to show that there is an $\beta > 0$ such that

$$E\Big[\exp\Big\{\beta (W^M(x))^2 * \mu_Y\Big\}\Big] < \infty, \tag{37}$$

$$E\Big[\exp\Big\{\beta \int_0^T (\sigma_t^M)^2 dt\Big\}\Big] < \infty. \tag{38}$$

However, we know from equation (13) that

$$\begin{aligned} E\Big[\exp\Big\{\beta \int_0^T (\sigma_t^M)^2 dt\Big\}\Big] &= \int_0^T \int_{\mathbb{R}} \Big(e^{(W_t^M(x))^2} - 1\Big)\nu(dx)dt \\ &\leq \int_0^T \int_{\mathbb{R}} (e-1)\nu(dx)dt < \infty \end{aligned}$$

since $W_t^M(x) \in (-1, 0]$. Inequality (38) corresponds to assumption (36) for $\beta \leq \alpha$. Further, since

$$\frac{dS_t}{S_{t-}} = \sigma_t^M dY_t^c - (\widehat{\lambda}_t \sigma_t^M - \sigma_t^L)\sigma_t^M dt + dW^M(x) * (\mu_Y - \nu_Y^{\mathbb{Q}^*})$$

as well as $\widehat{\phi}_t$ is uniformly bounded, we get $E\Big[\int \frac{\widehat{\phi}_t}{S_{t-}} dS_t\Big] = 0$, i.e. condition 2 is ensured.

Condition 3 can be ensured in a similar way. It is obvious that $\int \frac{\widehat{\phi}_t}{S_{t-}} dS_t$ is a local $\mathbb{Q}$-martingale. It will be a true $\mathbb{Q}$-martingale if we can show that, for some $\beta > 0$,

$$E\Big[\exp\Big\{\beta \int_0^T \frac{\widehat{\phi}_t^2}{S_{t-}^2} dS_t\Big\}\Big]$$

is finite. Let us take $\beta = \frac{\alpha}{2}$ and remind (37) and (38), the Schwarz inequality ensures that

$$E\Big[\exp\Big\{\beta\int_0^T\Big(\frac{\widehat{\phi}_t}{S_{t-}}\Big)^2 dS_t\Big\}\Big] = E\Big[\exp\Big\{\beta\Big(\int_0^T(\sigma_t^M)^2dt + (W^M(x))^2 * \mu_Y\Big)\Big\}\Big] < \infty.$$

Since all three conditions are fulfilled, the assumed solution (19) and (20) to problem (18) is correct.

q.e.d.

Remark 4.5 *The result of this section may be generalized to the case where M^d and V^d are finite variation processes, i.e. the case where we might have $\nu(\mathbb{R}) = \infty$, but*

$$\int |W_t^M(x)|\nu(dx) < \infty \quad \text{and} \quad \int |W_t^V(x)|\nu(dx) < \infty. \tag{39}$$

We then know that L^d has also finite variation since

$$W_t^L(x) = \exp\Big\{\Delta u_{t,V_{t-}}(x) + \Big[\frac{\sigma_t^V}{\sigma_t^M}\frac{\partial}{\partial V}u(t,V_{t-}) - \widehat{\lambda}_t - \frac{\int S_t^M(z)W_t^L(z)\nu(dz)}{(\sigma_t^M)^2}\Big]W_t^M(x)\Big\} - 1 + \widehat{\lambda}_t W_t^M(x)$$

and

$$\begin{aligned}\int \Delta u_{t,V_{t-}}(x)\nu(dz) &= \int \Big[u(t,V_{t-}+W_t^V(x)) - u(t,V_{t-})\Big]\nu(dx)\\ &= \int\int_{V_{t-}}^{V_{t-}+W_t^V(x)} \frac{\partial}{\partial y}u(t,y)dy\nu(dx)\\ &\leq u'_* \int W_t^V(x)\nu(dx) < \infty\end{aligned}$$

with

$$u'_* := \max_{x\in \text{supp}(\nu)} \frac{\partial}{\partial y}u(t,V_{t-}+W_t^V(x)).$$

Therefore, equation (5) is still applicable.
In the general case of $\nu(\mathbb{R}) = \infty$, the jumping times of Y are countable. We enumerate them as $t_1, t_2, \ldots$. Let us write

$$Y_n^d(t) := \sum_{j=1}^n (Y_{t_j}^d - Y_{t_j-}^d)1_{(0,t]}(t_j) \quad \text{for} \quad 0 \leq t < \infty.$$

We know that Y_n^d converges uniformly on any bounded inveral of t to Y_t^d as $n \to \infty$. Let us perform on $Y_n = Y_t^c + Y_n^d(t)$ the same argument as above. Via a limes argument, one can easily show that the resulting strategy still is bounded and therefore all conditions of remark 2.3 are fulfilled, i.e. ensuring the MEMM.

5 Examples

5.1 The Orthogonal Volatility Process

Let us consider the asset process

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= \eta(t,V_{t-})dt + \sigma^M(t,V_{t-})dY_t^c + dW^M(t,V_{t-},x) * (\mu_Y - \nu_Y)\\ dV_t &= \eta^V(t,V_{t-})dt + \sigma^V(t,V_{t-})dY_t^c + dW^V(t,V_{t-},x) * \mu_Y,\end{aligned}$$

fulfilling the assumptions 4.1, (36) as well as the additional condition

$$\sigma_t^M \sigma_t^V + \int W_t^M(x) W_t^V(x) \nu(dx) = 0.$$

This condition ensures that the volatility process is strongly orthogonal to the asset process. In this general case, this model restriction does not easen the problem. But let us consider the special case ($W_t^M(x) = 0,\ \sigma_t^V = 0$). Then, we get the following result:

Corollary 5.1 *The optimal strategy is*

$$\widehat{\phi}_t = -\widehat{\lambda}_t, \tag{40}$$

and the density process of the MEMM is defined via

$$\begin{aligned} W^L(t, V_{t-}, x) &= \exp\{u(t, V_{t-} + W_t^V(x)) - u(t, V_{t-})\} - 1 \\ \sigma^L(t, V_{t-}) &= 0, \end{aligned}$$

whereas u is the classical solution of the PDE

$$\begin{aligned} \frac{\partial}{\partial t} u(t,y) + \eta_t^V \frac{\partial}{\partial y} u(t,y) + \Big[-\frac{1}{2} \widehat{\lambda}_t^2 (\sigma_t^M)^2 + \int W^L(t, V_{t-}, x) \nu(dx) \Big] &= 0, \\ u(T, y) &= 0. \end{aligned}$$

Proof: (40), which is a direct consequence of equation (24), leads via (25) and (6) directly to the above structures of W^L and σ^L. The rest is a direct consequence of theorem 4.3.

q.e.d.

Remark 5.1

1. *The form (40) of the optimal strategy in this specific case has already been identified by Grandits and Rheinländer (2002). However, while the density of the MEMM at a fixed time T can be described very nicely, the corresponding density process turns out to be quite complex and not intuitiv at all. A solution may be found only with numerical methods.*

2. *Becherer used a model of this type in his PhD thesis. In his case, the process V jumps within m different states:*

$$\begin{aligned} \frac{dS_t}{S_t} &= \eta(t, V_{t-}) dt + \sigma^M(t, V_{t-}) dY_t^c \\ dV_t &= \sum_{j,k=1}^{m} 1_k(V_{t-}) dN_t^{kj}, \end{aligned}$$

   where η and σ^M are functions of class $\mathcal{C}^1$ with respect to $t \in [0, T]$, 1_k denotes the indicator function on $\{k\}$ and $N = (N^{kj})$ is a multivariate adapted point process. This model represents the degenerate case ($\sigma^V = 0,\ \eta^V = 0$). In this case, we have the following differential equation to solve:

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,k) - \frac{1}{2} (\lambda_t \sigma_t)^2 + \sum_{j=1}^{m} 1_{j \neq k} \mu^{kj} (e^{u(t,j) - u(t,k)} - 1) = 0. \\ &u(T, k) = 0, \end{aligned}$$

   where μ^{kj} represents the jump density of a jump from state k to state j.

5.2 The Barndorff-Nielsen Shephard Model

In Barndorff-Nielsen and Shephard (2001), a price process of the stock $S = (S_t)_{t\in[0,T]}$ is defined by the exponential $\exp(X_t)$ with $X = (X_t)_t$ satisfying

$$\begin{aligned} dX_t &= (\mu + \beta\sigma_{t-}^2)dt + \sigma_{t-}dY_t^c + d\rho x * \tilde{\mu}_Y, \\ d\sigma_t^2 &= -\lambda\sigma_{t-}^2 dt + dx * \tilde{\mu}_Y, \end{aligned}$$

where the parameters $\mu, \beta, \rho, \lambda$ are real constants with $\lambda > 0$ and $\rho \leq 0$, $\tilde{\mu}_Y$ has compensator $\tilde{\nu}_Y := \lambda\nu_Y$. In addition, Y^d is assumed to be a subordinator, i.e. with positive increments only. V_0 is assumed to be strictly positive.
It can be easily shown that the process S_t may then be written as

$$\frac{dS_t}{S_{t-}} = \Big(\mu + \lambda\int(e^{\rho x} - 1)\nu(dx) + \sigma_{t-}^2(\beta + \frac{1}{2})\Big)dt + \sigma_{t-}dY_t^c + d(e^{\rho x} - 1) * (\tilde{\mu}_Y - \tilde{\nu}_Y).$$

We apply the results of section 4 and get the following result:

Corollary 5.2 *We assume $\nu(\mathbb{R}) < \infty$ with bounded $\mathrm{supp}(\nu)$. Let us assume $\mu + \lambda\int(e^{\rho x} - 1)\nu(dx) > 0$ and $\beta + \frac{1}{2} > 0$. Hence, the MEMM in case of the Barndorff-Nielsen Shephard Model is determined as follows:*

Let us define

$$\begin{aligned} g^y(t, u_t) &= \frac{1}{2}(\sigma_t^L - \widehat{\lambda}_t\sqrt{y})^2 + \widehat{\phi}_t\widehat{\lambda}_t y \\ &\quad + \lambda\int\Big[W_t^L(y,x) - (\widehat{\phi}_t + \widehat{\lambda}_t)(e^{\rho x} - 1) + \widehat{\phi}_t\widehat{\lambda}_t(e^{\rho x} - 1)^2\Big]\nu(dx) \end{aligned}$$

with $W^L(t,y,x)$ to solve

$$\begin{aligned} &\exp\Big\{\Delta u_{t,V_{t-}}(x) + \Big[-\frac{\lambda\int(e^{\rho z} - 1)W^L(t,y,z)\nu(dz)}{y} - \widehat{\lambda}_t\Big](e^{\rho x} - 1)\Big\} \\ &= 1 - \widehat{\lambda}_t(e^{\rho x} - 1) + W^L(t,y,x), \end{aligned}$$

and

$$\begin{aligned} \widehat{\phi}_t &= -\frac{\lambda\int(e^{\rho x} - 1)W^L(t,y,x)\nu(dx)}{y} - \widehat{\lambda}_t, \\ \sigma_t^L &= -\frac{\int\lambda(e^{\rho x} - 1)W^L(t,y,x)\nu(dx)}{\sqrt{y}}. \end{aligned}$$

Hence, the classical solution $\hat{u}$ of the PDE

$$\begin{aligned} \frac{\partial}{\partial t}u(t,y) - \lambda y\frac{\partial}{\partial y}u(t,y) + g^y(t,u_t) &= 0, \\ u(T,y) &= 0 \;\; \forall \;\; y \in \mathbb{R}_+ \end{aligned}$$

defines the MEMM via W^L and σ^L:

$$Z_t = \frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}\Big(\int(-\widehat{\lambda}_s\sigma_s + \sigma_s^L)dY_s^c + (-\widehat{\lambda}(e^{\rho x} - 1) + W^L(x)) * (\tilde{\mu}_Y - \tilde{\nu}_Y)\Big)_t.$$

Proof: Let us denote $V_t := \sigma_t^2$ and

$$\begin{aligned}
\eta^V(t,y) &:= y, \\
W^V(t,y,x) &:= x \\
\eta(t,y) &:= \mu + \lambda \int (e^{\rho x} - 1)\nu(dx) + \sigma_{t-}^2(\beta + \frac{1}{2}), \\
\sigma^M(t,y) &:= \sqrt{y}, \\
W^M(t,y,x) &:= e^{\rho x} - 1.
\end{aligned}$$

To apply theorem 4.3, we have to check whether all assumptions 4.1 as well as (36) are fulfilled:

1. Since Y^d is a subordinator, $\overline{\text{supp}(\nu)} = \mathbb{R}_+$.
2. $\eta^V(t,y)$ obviously is Lipschitz-continuous. σ^V is set equal to zero.
3. It is clear that V_t is positive and on $[0,T]$ uniformly bounded away from zero. Hence, there is an $\epsilon > 0$ such that $V_t \in (\epsilon, \infty) := E$. It is obvious that η_t, σ_t^M and $W_t^M(x)$ are Lipschitz-continuous in $(t,y) \in [0,T] \times E$. The continuity of $W_t^M(x)$ is as obvious as $W_t^M(\text{supp}(\nu)) = (-1,0]$. Since $V_t \in (\epsilon,\infty)$, $\sigma_t^M \in (\sqrt{\epsilon},\infty)$ and therefore uniformly bounded from zero.
4. It can be easily shown that W^M and W^V are in $\mathcal{G}(\mu)$. Further, $w^V(t, V_{t-}) = 1$ and is therefore Lipschitz-continuous.
5. $\widehat{\lambda}_t = \frac{\mu+\lambda\int(e^{\rho x}-1)\nu(dx)+V_{t-}(\beta+\frac{1}{2})}{V_{t-}+\lambda\int(e^{\rho x}-1)^2\nu(dx)} > 0$ and is uniformly bounded by
$$\max(\frac{\mu + \lambda \int(e^{\rho x} - 1)\nu(dx)}{\lambda \int (e^{\rho x} - 1)^2 \nu(dx)}, \beta + \frac{1}{2}).$$
6. $E\Big[\exp\Big\{\int_0^T (\sigma_t^M)^2 dt\Big\}\Big] < \infty$ due to a result of Benth et al. (2002), lemma 3.1.

Corollary 5.2 therefore follows directly from theorem 4.3.

q.e.d.

Remark 5.2 *The restrictions $\mu + \lambda \int(e^{\rho x} - 1)\nu(dx) > 0$ and $\beta + \frac{1}{2} > 0$ are natural. They express that independent of the value σ_t, the drift is always positive. Positive risk premiums are to be expected in a financial market when the investors are risk-averse.*

5.3 The Single Jump Asset Model

As a third example, let us consider a very simplified asset model of the following type:

$$\frac{dS_t}{S_{t-}} = (1 + 1_{\tau<t})^2 dt + (1 + 1_{\tau<t}) dY_t^c + \alpha d(1_{\tau \leq t} - \mu \int_0^t (1 - 1_{\tau<t} dz) \tag{41}$$

with $\mu := \nu(\mathbb{R})$ and $\alpha \in (-1, 0]$. We consider this model to show the complexity of the MEMM in case of even such a simple stochastic volatility model. This model is also covered by the class of stochastic volatility models described by (14) and (15) and restricted by assumptions 4.1. It might be written as

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= \eta(t,V_{t-})dt + \sigma^M(t,V_{t-})dY_t^c + d\alpha W^M(t,V_{t-},x) * (\mu_Y - \nu_Y)\\ dV_t &= dW^V(t,V_{t-},x) * \mu_Y,\end{aligned}$$

with $V_0 = 0$,

$$\begin{aligned}W^V(t,V_{t-},x) &:= \begin{cases} 1 & V_{t-}=0\\ 0 & V_{t-}=1\end{cases}\\ \eta(t,V_{t-}) &:= (1+V_{t-})^2\\ \sigma^M(t,V_{t-}) &:= 1+V_{t-}, \quad \text{and } W^M(t,V_{t-},x) = \begin{cases} \alpha & V_{t-}=0\\ 0 & V_{t-}=1\end{cases}.\end{aligned}$$

It can be easily shown that all assumptions 4.1 are fulfilled. As in the Becherer model case, this model is in the degenerate model class since $(\eta^V = 0, \sigma^V = 0)$. Similarly as in the other examples, we get the following characterisation for the identification of the MEMM:

Corollary 5.3 *Let us define*

$$\begin{aligned}g^0(t,u_t) &= \frac{\alpha^2\mu^2}{2}(w_t^l)^2 + \Big[\mu + \alpha^2\mu - \frac{\alpha^3\mu^2}{1+\alpha^2\mu}\Big]w_t^l - \frac{1}{2}\frac{1}{(1+\alpha^2\mu)^2}(1+2\alpha^2\mu)\\ g^1(t,u_t) &= -2,\end{aligned}$$

with

$$\begin{cases} \exp\Big\{2(t-T) - u(t,0) + \alpha[-\alpha w_t^l\mu - 1]\Big\} = 1 - \frac{\alpha}{1+\alpha^2\mu} + w_t^l & \text{if } V_{t-}=0\\ w_t^l = 0 & \text{if } V_{t-}=1\end{cases}.$$

Hence, the classical solution $\hat{u}(.,0)$ *of this ordinary differential equation*

$$\frac{\partial}{\partial t}u(t,0) + g^0(t,u_t) \text{ with } u(T,1) = 0$$

defines the MEMM. $\widehat{\phi}_t$ *and* σ_t^L *are of the form*

$$\begin{aligned}\widehat{\phi}_t &= \begin{cases} -\alpha\mu w_t^l - \frac{1}{1+\alpha^2\mu} & \text{if } V_{t-}=0\\ -1 & \text{if } V_{t-}=1\end{cases},\\ \sigma_t^L &= \begin{cases} -\alpha\mu w_t^l & \text{if } V_{t-}=0\\ 0 & \text{if } V_{t-}=1\end{cases}.\end{aligned}$$

Proof: It is obvious that $g^1(t,u_t) = -2$ and therefore, $u(t,1) = -2(T-t)$ and we may write

$$\Delta u_{t,V_{t-}}(x) = \begin{cases} 2(t-T) - u(t,0) & \text{if } V_{t-}=0\\ 0 & \text{if } V_{t-}=1\end{cases}.$$

$\widehat{\phi}_t$ and σ_t^L are due to equation (20).

q.e.d.

For this specific case, the following theorem presents the MEMM in a slightly different form. This result helps to visualize the complexity of the solution even in case of a such a simple process as the single-jump asset process:

Theorem 5.1 *The MEMM may be written as*

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \exp\Big\{c_T + \int_0^T \frac{\widehat{\phi}_t}{S_{t-}} dS_t\Big\}$$

with

$$\widehat{\phi}_t = \begin{cases} -\frac{1}{\alpha u_t} + \frac{1}{\alpha} - 1 + \alpha\mu & \text{for} \quad t \in [0, \tau \wedge T] \\ -1 & \text{for} \quad t \in (\tau \wedge T, T] \end{cases}$$

and u_t fulfills the initial value problem

$$u_t' = \Big(f_1 + f_2 u_t^2\Big)(u_t - 1) \tag{42}$$

$$\frac{1 - u_T}{\alpha^2 \mu u_T} = \exp\Big(-\frac{1}{u_T} + 1 - \alpha + \alpha^2\mu\Big), \tag{43}$$

with

$$f_1 = \frac{1}{2\alpha^2}$$

$$f_2 = \frac{3}{2} + \alpha\mu - \frac{\alpha^2\mu^2}{2} - \mu - \frac{1}{2\alpha^2}.$$

Proof: Using equations (42) and (42), we write w_t^l and σ_t^L as functions of $\widehat{\phi}_t$. It is easy show that equation (5) may then be written the following way:

$$c_T + \int_0^{\tau\wedge T} \widehat{\phi}_t\Big(1 - \alpha\mu - \frac{1}{\alpha}\Big)dt + \frac{1}{2}\int_0^{\tau\wedge T} \widehat{\phi}_t^2 dt - \frac{1}{\alpha}\tau \wedge T - 2(T - \tau \wedge T)$$

$$= \Big(\log\Big(1 - \frac{1 + \widehat{\phi}_\tau}{\alpha\mu}\Big) - \alpha\widehat{\phi}_\tau\Big)1_{\tau \leq T}. \tag{44}$$

Let us now consider a function k_t, which is defined by the equality

$$-\frac{1 + k_t}{\alpha\mu} = \exp\Big(-(-\alpha k_t + c_{T-t}^{[1]} - c_{T-t}^{[0]})\Big) - 1,$$

where

$$c_{T-t}^{[n]} = -\log E[\exp \int_0^{\max(T-t,0)} \widehat{\phi}_t dS_t]$$

with $n = 0$ and $n = 1$ meaning that the jump did not yet appear and did appear, respectively. Obviously, k_t is a differentiable function with

$$-\frac{1 + k_T}{\alpha\mu} = \exp(\alpha k_T) - 1. \tag{45}$$

However, $\widehat{\phi}_t = k_t$ for $t \in [0, \tau \wedge T]$, as can be seen from the following:

Let us consider the density process prior and after the jump:

$$Z_{\tau-} = E[\exp(c_T + \int_0^T \phi dS_t)|\mathcal{F}_{\tau-}]$$

$$= \exp c_T \exp \int_0^{\tau-} \phi_t dS_t E[\exp \int_\tau^T \phi_t dS_t|\mathcal{F}_{\tau-}]$$

$$
\begin{aligned}
&= \exp c_T \exp \int_0^\tau \phi_t dS_t \exp -c^{[0]}_{T-\tau}. \\
Z_{\tau_1} &= E[\exp(c_T + \int_0^T \phi dS_t)|\mathcal{F}_\tau] \\
&= \exp c_T \exp\Big\{ \int_0^{\tau-} \phi_t dS_t + \widehat{\phi}_\tau \alpha \Big\} E[\exp \int_\tau^T \phi_t dS_t|\mathcal{F}_\tau] \\
&= \exp c_T \exp\Big\{ \int_0^{\tau-} \phi_t dS_t + \widehat{\phi}_\tau \alpha \Big\} \exp -c^{[1]}_{T-\tau}
\end{aligned}
$$

Hence, the relative jump of the density process Z at time τ is

$$
\frac{\Delta Z_\tau}{Z_{\tau-}} = \exp\Big\{ -(-\alpha\widehat{\phi}_\tau + c^{[1]}_{T-\tau} - c^{[0]}_{T-\tau})\Big\} - 1.
$$

On the other hand,

$$
\begin{aligned}
\frac{\Delta Z_\tau}{Z_\tau} &= -\lambda_\tau \Delta M_\tau + \Delta L_\tau = -\alpha\widehat{\lambda}_\tau - \frac{\widehat{\phi}_\tau + \widehat{\lambda}_\tau}{\alpha\mu} \\
&= -\frac{\widehat{\lambda}_\tau + 1}{\alpha\mu}.
\end{aligned}
$$

Since τ is not predictable, we must have

$$
-\frac{\widehat{\lambda}_t + 1}{\alpha\mu} = \exp\Big\{ -(-\alpha\widehat{\phi}_t + c^{[1]}_{T-t} - c^{[0]}_{T-t})\Big\} - 1
$$

for all $t \in [0, \tau \wedge T]$.

Hence, we may write equation (44) as follows:

$$
\begin{aligned}
&c_T + \int_0^{\tau\wedge T} k_t\Big(1 - \alpha\mu - \frac{1}{\alpha}\Big)dt + \frac{1}{2}\int_0^{\tau\wedge T} k_t^2 dt - \frac{1}{\alpha}\tau\wedge T - 2(T - \tau\wedge T) \\
&= \Big[\log\Big(1 - \frac{1+k_\tau}{\alpha\mu}\Big) - \alpha k_\tau\Big] 1_{\tau\le T}.
\end{aligned}
$$

The equation is correct for any value of τ, therefore one may differentiate it at any point $t = \tau$. Since the equation is constant for $t > T$, we restrict the analysis to $t \le T$, for which we get the following differential equation:

$$
k_t\Big(1 - \alpha\mu - \frac{1}{\alpha}\Big) - \frac{1}{\alpha} + \frac{1}{2}k_t^2 + 2 = -\frac{1}{1 - \frac{1+k_t}{\alpha\mu}} \frac{k_t'}{\alpha\mu} - \alpha k_t'. \tag{46}
$$

In addition, with equation (45), we have a boundary condition for the above differential equation, hence a classical initial value problem. The initial value problem (42), (43) can be achieved by standard transformations, replacing k_t by $u_t = -\frac{1}{\alpha k_t - 1 + \alpha - \alpha^2\mu}$.
Boundedness of the strategy $\widehat{\phi}$ has been already ensured by corollary 5.3 and therefore, the identified measure $\mathbb{Q}^*$ is truely the MEMM.

q.e.d.

Remark 5.3

1. *Even in this simple case, we do not have an explicit representation for u_t. However, the solution u_t is defined by the relationship*

$$\int_{u_t}^{u_T} \frac{dy}{(y-1)(f_1+f_2y^2)} = T-t.$$

Let us rewrite

$$\int_{u_t}^{u_T} \frac{dy}{(y-1)(f_1+f_2y^2)} = \int_{u_t-1}^{u_T-1} \frac{dz}{zQ(z)}$$

with $Q(z) = f_2z^2 + 2f_2z + f_1 + f_2$. This integral may be explicitly solved:

$$\int \frac{dz}{zQ(z)} = \frac{1}{2(f_1+f_2)} \ln \frac{z^2}{Q(z)} - \frac{1}{f_1+f_2} \begin{cases} \sqrt{\frac{f_2}{f_1}} atan \sqrt{\frac{f_2}{f_1}}(z+1) & \text{for } f_2 > 0, \\ \sqrt{\frac{-f_2}{f_1}} atanh - \sqrt{\frac{-f_2}{f_1}}(z+1) & \text{for } f_2 < 0 \text{ and } z+1 < \sqrt{\frac{-f_1}{f_2}}, \\ -\sqrt{\frac{-f_2}{f_1}} \ln \frac{f_2(z+1)-\sqrt{-f_1f_2}}{f_2(z+1)+\sqrt{-f_1f_2}} & \text{for } f_2 < 0 \text{ and } z+1 > \sqrt{\frac{-f_1}{f_2}} \end{cases} \tag{47}$$

However, for $f_2 < 0$, it can be easily shown that $z+1 < \sqrt{-\frac{f_1}{f_2}}$, since $u_T < u_* := \sqrt{-\frac{f_1}{f_2}}$. Therefore, we will never have the third case in the one-jump asset model.

2. *The compensator of $1_{\tau<t}$ under the minimal entropy martingale measure is*

$$\mu\Big(1 - \widehat{\lambda}_t(1-1_{\tau<t}) - (1+1_{\tau<t})\frac{\widehat{\lambda}_t + \widehat{\phi}_t}{\alpha\mu}\Big).$$

Since $\widehat{\phi}_t$ is not constant for $t \in [0,\tau]$, the price process under $\mathbb{Q}^*$ may not be written by a model of type (41). Similarly, in case of a BN-S model, the asset process under the MEMM $\mathbb{Q}^*$ will in general not belong anymore to the class of BN-S models. Hence, the MEMM does not belong to the class of structure preserving martingale measures.

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