SFB 649 Discussion Paper 2006-061

A Control Approach to Robust Utility Maximization with Logarithmic Utility and Time-Consistent Penalties

Daniel Hernández–Hernández*
Alexander Schied**

* Centro de Investigación en Matemáticas, Guanajuato, Mexico
** Institute of Mathematics, Berlin University of Technology, Germany

This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

http://sfb649.wiwi.hu-berlin.de
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin
Spandauer Straße 1, D-10178 Berlin

SFB 649 ECONOMIC RISK BERLIN

A control approach to robust utility maximization with logarithmic utility and time-consistent penalties

Daniel Hernández–Hernández and Alexander Schied*

This version: July 5, 2006

Abstract: We propose a stochastic control approach to the dynamic maximization of robust utility functionals that are defined in terms of logarithmic utility and a dynamically consistent convex risk measure. The underlying market is modeled by a diffusion process whose coefficients are driven by an external stochastic factor process. In particular, the market model is incomplete. Our main results give conditions on the minimal penalty function of the robust utility functional under which the value function of our problem can be identified with the unique classical solution of a quasilinear PDE within a class of functions satisfying certain growth conditions. The fact that we obtain classical solutions rather than viscosity solutions is important for the use of numerical algorithms, whose applicability is demonstrated in examples.

1 Introduction

One of the fundamental problems in mathematical finance and mathematical economics is the construction of investment strategies that maximize the utility functional of a risk-averse investor. In the majority of the corresponding literature, the optimality criterion is based on a classical expected utility functional of von Neumann-Morgenstern form, which requires the choice of a single probabilistic model $\mathbb{P}$. In reality, however, the choice of $\mathbb{P}$ is often subject to model uncertainty. Schmeidler [24] and Gilboa and Schmeidler [10] therefore proposed the use of *robust utility functionals* of the form

$$X \longmapsto \inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,], \tag{1}$$

where $\mathcal{Q}$ is a set of prior probability measures. In analogy to the move from coherent to convex risk measures, Maccheroni et al. [16] recently suggested to model investor preferences by robust utility functionals of the form

$$X \longmapsto \inf_{Q} \big(E_Q[\,U(X)\,] + \gamma(Q)\big), \tag{2}$$

where γ is a penalty function defined on the set of all possible probabilistic models.

*Supported by Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

Optimal investment problems for robust utility functionals (1) were considered, among others, by Talay and Zheng [25], Quenez [20], Schied [21], Burgert and Rüschendorf [3], Schied and Wu [23], Föllmer and Gundel [8], and the authors [12]. For the generalized utility functionals of type (2), the most popular choice for the penalty function has so far been the entropic penalty function $\gamma(Q) = kH(Q|\mathbb{P})$ for a constant $k > 0$ and a reference probability measure $\mathbb{P}$; see, e.g., Hansen and Sargent [11] and Bordigoni et al. [2] for studies of the optimal consumption problem. The duality theory for the optimal investment problem with a general penalty function γ was developed by Schied [22]. Robust utility maximization is also closely related to other optimization problems involving convex and coherent risk measures, and these problems also received a lot of attention recently; see, for instance, Barrieu and El Karoui [1], Jouini et al. [13], or Klöppel and Schweizer [14, 15].

In this paper, we propose a stochastic control approach to the dynamic maximization of robust utility functionals of the form (2). The penalty function γ will be defined in a Brownian setting and, apart from certain basic requirements such as time consistency, has a rather general form. In particular, we will go beyond the very particular situation of entropic penalties and include the 'coherent' setting (1) as a special case. Our setting will involve logarithmic utility $U(x) = \log x$ and an incomplete financial market model, whose volatility, interest rate process, and trend are driven by an external stochastic factor process.

Our goal consists in characterizing the value function and the optimal investment strategy via the solution of a quasilinear Hamilton-Jacobi-Bellman PDE. As a byproduct, we also obtain a formula for the least-favorable martingale measure in the sense of Föllmer and Gundel [8]. In contrast to earlier approaches such as [25], we avoid the use of viscosity solutions and concentrate our effort on obtaining strong regularity results, which allow us to identify the value function as a unique classical solution of the PDE in question. Regularity of solutions is important because it justifies the use of standard numerical methods for solving the PDE, and we will use such methods in illustrating some interesting qualitative properties of the optimal strategy.

Our method consists in combining the duality results from [22] with a PDE approach to the dual problem of determining optimal martingale measures. This technique has already been applied successfully by Castañeda-Leyva and Hernández-Hernández [4, 5] to the maximization of von Neumann-Morgenstern expected utility and by the authors [12] in the maximization of 'coherent' robust utility functionals of the form (1). It turns out, however, that the introduction of the penalty function γ yields new types of problems, in particular if certain measures Q with $\gamma(Q) < \infty$ have to be described by unbounded control processes (this is the case, e.g., for entropic penalties). To deal with this case, we have to introduce new arguments both on the probabilistic and on the analytic side of the problem.

This paper is organized as follows. In Section 2 we describe the set-up of the problem and state the theorems containing our main findings. These theorems will be proved in the subsequent sections. Section 3 analyzes how certain classes of probability measures $Q \ll \mathbb{P}$ can be described by a suitable sets of control processes. The dual problem for our robust utility maximization problem is formulated in Section 4. In Section 5 we derive the Hamilton-Jacobi-Bellman PDE for the value function via the dual problem and we prove a verification result. This verification result will suffice to prove our results in the special case where the effective domain of γ is a compact set of probability measures that are all equivalent to the reference measure $\mathbb{P}$. In Section 6 we consider the case in which $\gamma(Q)$ can be finite for measures Q that are not equivalent but only absolutely continuous with respect to $\mathbb{P}$. Since the market model may admit arbitrage opportunities under such a measure Q, it is clear that the corresponding problem must become more involved, and it turns out that complications also appear on the analytical side of the problem.

2 Statement of main results

We consider a financial market model with a locally riskless money market account

$$dS_t^0 = S_t^0 r(Y_t)\,dt$$

and a risky asset defined under a reference measure $\mathbb{P}$ through the SDE

$$dS_t = S_t b(Y_t)\,dt + S_t \sigma(Y_t)\,dW_t^1.$$

Here W^1 is a standard $\mathbb{P}$-Brownian motion and Y denotes an external economic factor process modeled by the SDE

$$dY_t = g(Y_t)\,dt + \rho\,dW_t^1 + \overline{\rho}\,dW_t^2, \tag{3}$$

where $\rho \in [-1,1]$ is some correlation factor, $\overline{\rho} := \sqrt{1-\rho^2}$, and W^2 is a standard $\mathbb{P}$-Brownian motion, which is independent of W^1 under $\mathbb{P}$. We suppose that the economic factor cannot be traded directly so that the market model will typically be incomplete.

We assume that $g(\cdot)$ is in $C^2(\mathbb{R})$ with derivative $g' \in C_b^1(\mathbb{R})$, and $r(\cdot)$, $b(\cdot)$, and $\sigma(\cdot)$ belong to $C_b^2(\mathbb{R})$, where $C_b^k(\mathbb{R})$ denotes the class of bounded functions with bounded derivatives up to order k. The 'market price of risk' is defined via the function

$$\theta(y) := \frac{b(y) - r(y)}{\sigma(y)},$$

and we will assume that $\sigma(\cdot) \geq \sigma_0 > 0$ for some constant σ_0. The assumption of time-independent coefficients is for notational convenience only and can easily be relaxed.

In most economic situations, investors typically face *model uncertainty* in the sense that the dynamics of the relevant quantities are not precisely known. One common approach to coping with model uncertainty is to allow in principle all probability models corresponding to probability measures $Q \ll \mathbb{P}$ and to penalize each such model with a penalty $\gamma(Q)$. To define $\gamma(Q)$, we assume henceforth that everything is modeled on the canonical path space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ of $W = (W^1, W^2)$. Then every probability measure $Q \ll \mathbb{P}$ admits a progressively measurable process $\eta = (\eta_1, \eta_2)$ such that

$$\frac{dQ}{d\mathbb{P}} = \mathcal{E}\Big(\int \eta_{1t}\,dW_t^1 + \int \eta_{2t}\,dW_t^2\Big)_T \qquad Q\text{-a.s.},$$

where $\mathcal{E}(M)_t = \exp(M_t - \langle M\rangle_t/2)$ denotes the Doleans-Dade exponential of a continuous semimartingale M; see Lemma 3.1 below. Such a measure Q will receive a penalty

$$\gamma(Q) := E_Q\Big[\int_0^T h(\eta_t)\,dt\Big], \tag{4}$$

where $h : \mathbb{R}^2 \to [0,\infty]$ is convex and lower semicontinuous. For simplicity, we will suppose $h(0) = 0$ so that $\gamma(\mathbb{P}) = 0$. We will also assume that h is continuously differentiable on its effective domain $\mathrm{dom}\, h := \{\eta \in \mathbb{R}^2 \,|\, h(\eta) < \infty\}$ and satisfies the coercivity condition

$$h(x) \geq \kappa_1|x|^2 - \kappa_2 \qquad \text{for some constants } \kappa_1,\, \kappa_2 > 0. \tag{5}$$

The choice $h(x) = |x|^2/2$ corresponds to the entropic penalty function considered in Hansen and Sargent [11] and Bordigoni et al. [2]; see Remark 2.6 below. Again, our assumption that h does not depend on time is for notational convenience only.

Let $\mathcal{A}$ denote the set of all progressively measurable process π such that $\int_0^T \pi_s^2\,ds < \infty$ $\mathbb{P}$-a.s. For $\pi \in \mathcal{A}$ we define

$$X_t^{x,\pi} := x \cdot \exp\Big(\int_0^t \pi_s\sigma(Y_s)\,dW_s^1 + \int_0^t \Big[r(Y_s) + \pi_s\big(b(Y_s) - r(Y_s)\big) - \frac{1}{2}\sigma^2(Y_s)\pi_s^2\Big]\,ds\Big). \tag{6}$$

Then $X^{x,\pi}$ satisfies

$$X_t^{x,\pi} = x + \int_0^t \frac{X_s^{x,\pi}(1-\pi_s)}{S_s^0}\,dS_s^0 + \int_0^t \frac{X_s^{x,\pi}\pi_s}{S_s}\,dS_s$$

and thus describes the evolution of the wealth process $X^{x,\pi}$ of an investor with initial endowment $X_0^{x,\pi} = x > 0$ investing the fraction π_s of the current wealth into the risky asset at time $s \in [0,T]$.

The objective of the investor consists in

$$\text{maximizing} \quad \inf_{Q \ll \mathbb{P}} \big(E_Q[\,U(X_T^{x,\pi})\,] + \gamma(Q)\big) \quad \text{over} \quad \pi \in \mathcal{A}, \tag{7}$$

where the utility function $U : (0,\infty) \to \mathbb{R}$ will be specified in the sequel as a HARA utility function with risk aversion parameter $\alpha = 0$, i.e.,

$$U(x) = \log x. \tag{8}$$

Our goal is to characterize the value function

$$u(x) := \sup_{\pi\in\mathcal{A}} \inf_{Q\ll\mathbb{P}} \big(E_Q[\,\log X_T^{x,\pi}\,] + \gamma(Q)\big)$$

of the robust utility maximization problem (7) in terms of the solution v of the quasi-linear parabolic initial value problem

$$\begin{cases} v_t = \frac{1}{2}v_{yy} + \phi(v_y) + gv_y + r \\ v(0,\cdot) = 0, \end{cases} \tag{9}$$

where the nonlinearity $\phi(v_y) = \phi(y, v_y(t,y))$ is given by

$$\phi(y,z) := \psi(y,(\rho,\overline{\rho})z) \qquad y,z \in \mathbb{R}.$$

for the function

$$\psi(y,x) := \inf_{\eta\in\mathbb{R}^2}\Big\{\eta\cdot x + \frac{1}{2}(\eta_1+\theta(y))^2 + h(\eta)\Big\}, \qquad y\in\mathbb{R},\ x\in\mathbb{R}^2.$$

Here, $\eta\cdot x$ denotes the inner product of η and x. The easy case is the one in which the effective domain of h is compact:

Theorem 2.1 *Suppose that* dom h *is compact. Then the value function u of the robust utility maximization problem satisfies*

$$u(x) = \log x + v(T, Y_0),$$

where $v : [0,T]\times\mathbb{R}\to\mathbb{R}$ is the unique classical solution to (9) *within the class of functions in $C^{1,2}((0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$ satisfying a polynomial growth condition.*

Suppose furthermore that $\eta^* : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that $\eta^*(t,y)$ belongs to the supergradient of the concave function $x\mapsto\psi(y,x)$ at $x=(\rho,\overline{\rho})v_y(t,y)$. Then an optimal strategy $\widehat{\pi}$ for the robust problem can be obtained by letting

$$\widehat{\pi}_t = \frac{\eta_1^*(T-t,Y_t)+\theta(Y_t)}{\sigma(Y_t)}, \qquad 0\le t\le T.$$

Moreover, by defining a measure $\widehat{Q}\sim\mathbb{P}$ via

$$\frac{d\widehat{Q}}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta^*(T-t,Y_t)\,dW_t\Big)_T, \tag{10}$$

we obtain a saddlepoint $(\widehat{\pi},\widehat{Q})$ for the maximin problem (7).

The regularity of the value function obtained in the preceding theorem is important, because it justifies the use of standard numerical methods for solving the PDE (9). In Example 2.7, we will use such methods in illustrating some qualitative properties of the optimal strategy.

Remark 2.2 The proof of Theorem 2.1 will show that the probability measure P^* with density

$$\frac{dP^*}{d\mathbb{P}} = \mathcal{E}\Big(-\int\theta(Y_s)\,dW_s^1 + \int\eta_2^*(T-s,Y_s)\,dW_s^2\Big)_T$$

is a least favorable martingale measure in the sense of Föllmer and Gundel [8]. This will also be true in the setting of Theorems 2.3 and 2.5.

The problem becomes more difficult when dom h is noncompact, because then we can no longer apply standard theorems on the existence of classical solutions to (9). Other difficulties appear when dom h is not only noncompact but also unbounded. For instance, we may have $\gamma(Q) < \infty$ even if Q is not equivalent but merely absolutely continuous with respect to $\mathbb{P}$, and this leads to difficulties when one tries to work directly on the primal problem; see Remark 4.2. Moreover, since the optimal η^* takes values in the unbounded set dom h, one needs an additional argument to ensure that the stochastic exponential in (10) is a true martingale and hence defines a probability measure $\widehat{Q} \ll \mathbb{P}$. Our strategy to get the necessary integrability of the process $\eta_1^*(T-t, Y_t)$ is to use qualitative properties of solutions v to (9) as to control the growth of the gradient v_y. In doing so, we have to eliminate the possible competition between the linear term gv_y and the nonlinear term $\phi(v_y)$ by imposing a growth condition on ϕ.

Theorem 2.3 *Suppose that g is bounded and that there exists some $\varepsilon > 0$ such that*

$$\liminf_{|p|\to\infty} \left|\frac{\phi(y,p)}{p}\right| \geq \varepsilon + |g(y)|. \tag{11}$$

Then the value function u of the robust utility maximization problem satisfies $u(x) = \log x + v(T, Y_0)$ where v is the unique classical solution of (9) within the class of functions in $C^{1,2}((0,T)\times\mathbb{R}) \cap C([0,T]\times\mathbb{R})$ with bounded gradient v_y. Under these conditions, also the conclusions on the optimal strategy $\widehat{\pi}$ and the measure $\widehat{Q}$ in Theorem 2.1 remain true.

The most interesting case is the one in which both dom h and the function g are unbounded. Here we need an additional condition on the shape of the function ψ. Note that g is unbounded if, e.g., Y is an Ornstein-Uhlenbeck process.

Definition 2.4 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an upper semicontinuous concave function. We will say that f satisfies a *radial growth condition in direction $x \in \mathbb{R}^2$* if there exist positive constants p_0 and C such that

$$\max\left\{ |z| \,\middle|\, z \in \partial f(px) \right\} \leq C\big(1 + |\partial_p^+ f(px)| \vee |\partial_p^- f(px)|\big) \qquad \text{for } p \in \mathbb{R},\ |p| \geq p_0,$$

where $\partial f(px)$ denotes the supergradient of f in px and $\partial_p^+ f(px)$ and $\partial_p^- f(px)$ are the right-hand and left-hand derivatives of the concave function $p \mapsto f(px)$.

Note that if f is of the form $f(x) = f_0(|x|)$ for some convex increasing function f_0, then the radial growth condition is satisfied in any direction $x \neq 0$ with constant $C = 1$.

Theorem 2.5 *Suppose that $|\phi(y,p)/p)| \to \infty$ as $|p| \to \infty$ and assume that $\psi(y,\cdot)$ satisfies a radial growth condition in direction $(\rho, \overline{\rho})$, uniformly in y. Then the value function u of the robust utility maximization problem satisfies $u(x) = \log x + v(T, Y_0)$ where v is the unique classical solution of (9) within the class of polynomially growing functions in $C^{1,2}((0,T)\times\mathbb{R}) \cap C([0,T]\times\mathbb{R})$ whose gradient satisfies a growth condition of the form*

$$\left|\partial_p^- \phi\big(y; v_y(t,y)\big)\right| \vee \left|\partial_p^+ \phi\big(y; v_y(t,y)\big)\right| \leq C_1(1+|y|)$$

for some constant C_1. Under these conditions, also the conclusions on the optimal strategy $\widehat{\pi}$ and the measure $\widehat{Q}$ in Theorem 2.1 remain true.

Remark 2.6 For $q > 0$, the choice $h(x) = \frac{1}{2q}|x|^2$ corresponds to the penalty function $\gamma(Q) = \frac{1}{q}H(Q|\mathbb{P})$, where

$$H(Q|\mathbb{P}) = \int \frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}}\, d\mathbb{P} = \sup_{Y \in L^\infty} \big(E_Q[\,Y\,] - \log \mathbb{E}[\,e^Y\,]\big)$$

is the relative entropy of Q with respect to $\mathbb{P}$. Due to the classical duality formula

$$\log \mathbb{E}[\,e^X\,] = \sup_{Q \in \mathcal{Q}} \big(E_Q[\,X\,] - H(Q|\mathbb{P})\big),$$

the above choices correspond to the utility functional

$$\inf_{Q \ll \mathbb{P}} \big(E_Q[\,\log X\,] + \gamma(Q)\big) = -\frac{1}{q} \log \mathbb{E}\big[\,e^{-q \log X}\,\big] = -\frac{1}{q} \log \mathbb{E}[\,X^{-q}\,].$$

In this case, the robust utility maximization problem (7) is equivalent to the maximization of the standard expected utility $\mathbb{E}[\,U(X_T^{x,\pi})\,]$ for the HARA utility function $U(x) = -x^{-q}$. This standard utility maximization problem is covered as a special case of Theorem 2.5. Indeed, the function ψ has the quadratic form

$$\psi(y,x) = -\frac{1}{2}\Big(\frac{q}{1+q}(x_1 + \theta(y))^2 + q x_2^2 - \theta(y)^2\Big),$$

and it is easily checked that it satisfies the radial growth condition in any direction.

Example 2.7 Here we will give some numerical results for the case in which $\mathbb{P}$ is such that Y is an Ornstein-Uhlenbeck process with $g(y) = 100 - y$ and S follows the SDE

$$dS_t = S_t\Big(\frac{1}{2}\, dW_t^1 + \frac{1}{10} Y_t\, dt\Big).$$

We suppose that $r = 0$. Then θ is given by $\theta(y) = y/5$. Let us first consider the 'coherent' case

$$h_1(\eta) = \begin{cases} 0 & \text{if } |\eta_1| \le 20 \text{ and } \eta_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

The corresponding penalty function $\gamma_1(Q)$ takes only the values 0 and ∞. If $\rho = 0$ then the optimal η^* is given by $\eta_2^*(t,y) = 0$ and

$$\eta_1^*(t,y) = \begin{cases} -\theta(y) & \text{if } |y| \le 100, \\ -20\,\mathrm{sign}(y) & \text{otherwise.} \end{cases}$$

In particular, our formula for $\widehat{\pi}$ shows that there will be no investment into the risky asset as long as the factor process Y stays in the interval $[-100, 100]$. This corresponds to the fact that S has a local martingale dynamic under the 'worst-case measure' $\widehat{Q}$ as long as $-100 \le Y_t \le 100$.

A nonzero correlation factor ρ, however, can change the picture. This is illustrated in Figure 1, which shows the function v for the 'coherent' penalty function h_1 but with nonzero correlation $\rho = 1/2$. This figure clearly exhibits a nonvanishing gradient of

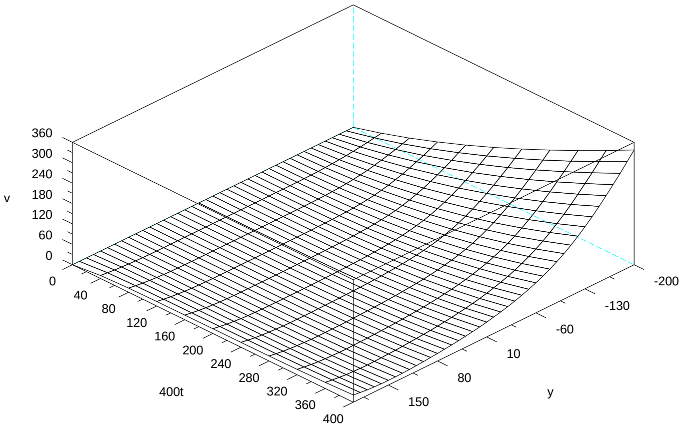

Figure 1: The function v for the choice h_1.

v, which leads in turn to a nontrivial investment into the risky asset—despite the fact that for $-100 \leq Y_t \leq 100$ we can still turn S locally into a martingale by choosing an appropriate probability measure Q with $\gamma_1(Q) = 0$. This effect occurs as a tradeoff between the tendencies of minimizing asset returns and driving Y further away from 'favorable regions' under the 'worst-case measure' $\widehat{Q}$.

Figure 2 shows the function v for the case in which we add to the relative entropy $H(Q|\mathbb{P})$ to the penalty function γ_1. That is, we use the function

$$h_2(\eta) = \begin{cases} \frac{1}{2}\eta_1^2 & \text{if } |\eta_1| \leq 20 \text{ and } \eta_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

It can be compared to the value function for the standard utility maximization problem with subjective measure $\mathbb{P}$, which is plotted in Figure 3.

3 Control processes associated with absolutely continuous measure changes

The following lemma is well known, but we include it here since its statement and the arguments employed in the proof will be important in the sequel.

Lemma 3.1 *For any $Q \ll \mathbb{P}$ there exists a progressive process $\eta = (\eta_1, \eta_2)$ such that*

$$\int_0^t |\eta_s|^2 \, ds < \infty \qquad Q\text{-a.s. for all } t \tag{12}$$

and

$$\frac{dQ}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}\Big(\int_0 \eta_{1s} \, dW_s^1 + \int_0 \eta_{2s} \, dW_s^2\Big)_t \qquad Q\text{-a.s.} \tag{13}$$

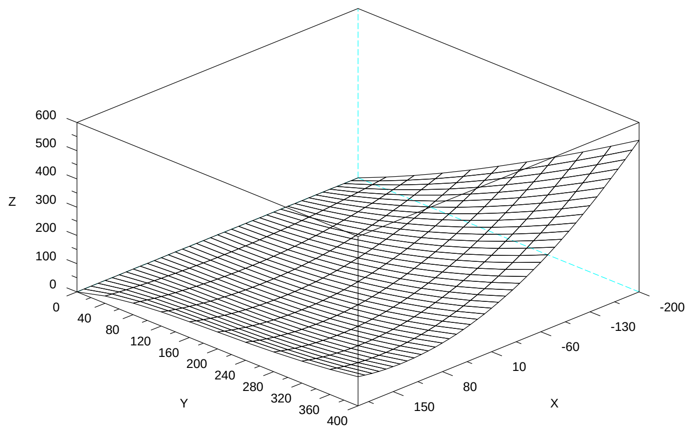

Figure 2: The function v for h_2.

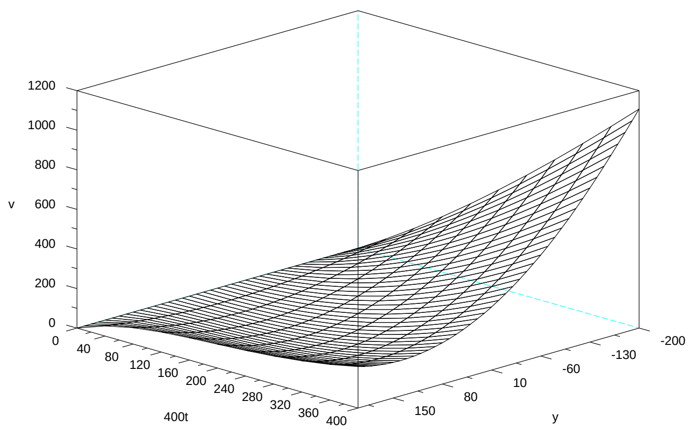

Figure 3: The function v for the choice $h(\eta) = \infty \mathbb{I}_{\{\eta \neq 0\}}$.

Proof: If $Q \ll \mathbb{P}$ is given, we let $D_t := dQ/d\mathbb{P}|_{\mathcal{F}_t}$ and define $\tau_n := \inf\{t \geq 0 \,|\, D_t \leq 1/n\}$. By representing the local $\mathbb{P}$-martingale $\int_0^{t\wedge\tau_n} D_s^{-1}\,dD_s$ as a stochastic integral with respect to $W = (W^1, W^2)$, we obtain the existence of a progressive process $\eta_s^{(n)}$, $s \leq \tau_n$, such that $\int_0^{t\wedge\tau_n} |\eta_s^{(n)}|^2\,ds < \infty$ and

$$D_{t\wedge\tau_n} = \mathcal{E}\Big(\int_0 \eta_s^{(n)}\,dW_s\Big)_{t\wedge\tau_n}$$

$\mathbb{P}$-a.s. for all t. Consistency requires that $\eta_t^{(n)} = \eta_t^{(n+1)}$ $dt \otimes d\mathbb{P}$-a.e. on $\{t \leq \tau_n\}$. Using that $\tau_n \nearrow \infty$ Q-a.s., we obtain a Q-a.s. defined process η, which is as desired. $\square$

The following concept of a localized martingale measure is related to the extended martingale measures recently introduced by Föllmer and Gundel [8]. We define as usual $\mathcal{F}_\infty := \sigma\big(\bigcup_{t\geq 0}\mathcal{F}_t\big)$.

Definition 3.2 A *localized martingale measure* is a probability measure $\widehat{P}$ on $\mathcal{F}_\infty$ for which there exists an increasing sequence (τ_n) of stopping times, called a *localizing sequence for* $\widehat{P}$, such that $\tau_n(\omega) \nearrow \infty$ for all $\omega \in \Omega$ and such that $\widehat{P}$ is equivalent to $\mathbb{P}$ on each $\mathcal{F}_{\tau_n}$. Moreover, the density process

$$Z_t = \frac{d\widehat{P}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}, \qquad 0 \leq t,$$

understood in the sense of the Lebesgue decomposition, is supposed to be $\mathbb{P}$-a.s. strictly positive and such that ZS/S^0 is a local $\mathbb{P}$-martingale. By $\widehat{\mathcal{P}}$ we will denote the set of all localized martingale measures.

In the sequel, $\mathcal{M}$ will denote the set of all progressive processes ν such that $\int_0^t \nu_s^2(\omega)\,ds < \infty$ for all t and ω. Recall that we assume that everything is modeled on the canonical path space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ of $W = (W^1, W^2)$. In the remainder of this section, all Radon-Nikodym densities will be understood in the sense of the Lebesgue decomposition unless otherwise mentioned.

Lemma 3.3 *For every $\widehat{P} \in \widehat{\mathcal{P}}$ there exists some $\nu \in \mathcal{M}$ such that*

$$\tau_n = \inf\Big\{t \geq 0 \,\Big|\, \int_0^t \nu_s^2\,ds = n\Big\}, \qquad n \in \mathbb{N}, \tag{14}$$

is a localizing sequence for $\widehat{P}$, and the positive local $\mathbb{P}$-martingale

$$Z_t^\nu := \mathcal{E}\Big(-\int \theta(Y_s)\,dW_s^1 - \int \nu_s\,dW_s^2\Big)_t \tag{15}$$

is $\mathbb{P}$-a.s. equal to the density $d\widehat{P}/d\mathbb{P}|_{\mathcal{F}_t}$. Conversely, let $\nu \in \mathcal{M}$ be given. Then there exists a localized martingale measure $\widehat{P}$ such that $d\widehat{P}/d\mathbb{P}|_{\mathcal{F}_t}$ is $\mathbb{P}$-a.s. equal to Z_t^ν and such that τ_n defined via (14) *is a localizing sequence.*

Proof: For $\widehat{P} \in \widehat{\mathcal{P}}$ the density process Z_t, understood in the sense of the Lebesgue decomposition, is a strictly positive local $\mathbb{P}$-martingale and can be represented $\mathbb{P}$-a.s. as $Z_t = \mathcal{E}(-\int \mu_s \, dW_s^1 - \int \nu_s \, dW_s^2)$ for certain $\mu, \nu \in \mathcal{M}$. This follows as in the proof of Lemma 3.1. The condition that ZS/S^0 is a local $\mathbb{P}$-martingale determines μ_t as $\theta(Y_t)$ $dt \otimes d\mathbb{P}$-a.s. and in turn yields $Z = Z^\nu$. Hence, when defining τ_n as in (14), Z_{τ_n} is $\mathbb{P}$-a.s. strictly positive and satisfies $\mathbb{E}[\, Z_{\tau_n}\,] = 1$, and this implies that $\widehat{P} \sim \mathbb{P}$ on $\mathcal{F}_{\tau_n}$. Therefore, (τ_n) is a localizing sequence for $\widehat{P}$.

Conversely, let $\nu \in \mathcal{M}$ be given, and define Z^ν as in the assertion. Then Z^ν is a strictly positive local $\mathbb{P}$-martingale and $Z^\nu S/S^0$ is a local $\mathbb{P}$-martingale. Defining τ_n as in (14) yields that $Z^\nu_{\tau_n}$ is the density of a probability measure $\widehat{P}^n$, which is equivalent to $\mathbb{P}$. Moreover, $(Z^\nu_{\tau_n})_{n=1,2,\dots}$ is a discrete-time $\mathbb{P}$-martingale, and so $\widehat{P}^{n+1}$ coincides with $\widehat{P}^n$ on $\mathcal{F}_{\tau_n}$. Hence, the Kolmogorov consistency theorem yields the existence of a probability measure $\widehat{P}$ on $\mathcal{F}_\infty = \sigma\big(\bigcup_n \mathcal{F}_{\tau_n}\big)$ whose restrictions to $\mathcal{F}_{\tau_n}$ are equal to $\widehat{P}^n$. Finally, it is well known that $\lim_n Z^\nu_{t\wedge\tau_n} = \lim_n d\widehat{P}/d\mathbb{P}|_{\mathcal{F}_{t\wedge\tau_n}}$ is $\mathbb{P}$-a.s. equal to the density $d\widehat{P}/d\mathbb{P}$ on $\sigma(\bigcup_n \mathcal{F}_{t\wedge\tau_n}) = \mathcal{F}_t$, that is, $Z^\nu_t = d\widehat{P}/d\mathbb{P}|_{cF_t}$. □

Let

$$H_{\mathcal{G}}(Q|P) := \begin{cases} E_Q\Big[\,\log \frac{dQ}{dP}\Big|_{\mathcal{G}}\,\Big] & \text{if } Q \ll P \text{ on } \mathcal{G} \\ +\infty & \text{otherwise,} \end{cases}$$

denote the relative entropy of Q with respect to P on a σ-algebra $\mathcal{G} \subset \mathcal{F}$.

Lemma 3.4 *Let $\widehat{P}$ be a localized martingale measure associated with $\nu \in \mathcal{M}$, define Z^ν as in* (15)*, and suppose that η is a progressive processes corresponding to some $Q \ll \mathbb{P}$ with density process $D_t = dQ/d\mathbb{P}|_{\mathcal{F}_t}$. Then $Q \ll \widehat{P}$ on $\mathcal{F}_t$ and*

$$E_Q\Big[\,\log \frac{D_t}{Z^\nu_t}\,\Big] = H_{\mathcal{F}_t}(Q|\widehat{P}) = \frac{1}{2} E_Q\Big[\int_0^t (\eta_{1s} + \theta(Y_s^y))^2 + (\eta_{2s} + \nu_s)^2 \, ds\Big].$$

Proof: Since $Z^\nu_t = d\widehat{P}/d\mathbb{P}|_{\mathcal{F}_t} > 0$ $\mathbb{P}$-a.s., it follows that $Q \ll \mathbb{P} \ll \widehat{P}$ of $\mathcal{F}_t$. Moreover,

$$\frac{dQ}{d\widehat{P}}\Big|_{\mathcal{F}_t} = \lim_{n\uparrow\infty} \frac{dQ}{d\widehat{P}}\Big|_{\mathcal{F}_{t\wedge\tau_n}} = \lim_{n\uparrow\infty} \frac{D_{\tau_n\wedge t}}{Z^\nu_{\tau_n\wedge t}} = \frac{D_t}{Z^\nu_t} \qquad Q\text{-a.s.}$$

Hence,

$$E_Q\Big[\,\log \frac{D_t}{Z^\nu_t}\,\Big] = E_Q\Big[\,\log \frac{dQ}{d\widehat{P}}\Big|_{\mathcal{F}_t}\,\Big] = H_{\mathcal{F}_t}(Q|\widehat{P})$$

follows. Moreover, standard arguments based on uniform integrability show that

$$E_Q\Big[\,\log \frac{D_t}{Z^\nu_t}\,\Big] = \sup_{n\uparrow\infty} E_Q\Big[\,\log \frac{D_{\sigma_n\wedge t}}{Z^\nu_{\sigma_n\wedge t}}\,\Big] = \sup_{n\uparrow\infty} H_{\mathcal{F}_{\sigma_n\wedge t}}(Q|\widehat{P}), \tag{16}$$

whenever (σ_n) is a sequence of stopping times that increases Q-a.s. to infinity. Now take

$$\sigma_n := \inf\Big\{\, t \geq 0 \,\Big|\, \int_0^t (\nu_s^2 + |\eta_s|^2)\, ds \geq n \Big\}.$$

Then $\sigma_n \nearrow \infty$ Q-a.s., and a straightforward computation shows that

$$E_Q\Big[\,\log\frac{D_{\sigma_n\wedge t}}{Z^{\nu}_{\sigma_n\wedge t}}\,\Big]=\frac{1}{2}E_Q\Big[\int_0^{t\wedge\sigma_n}(\eta_{1s}+\theta(Y^y_s))^2+(\eta_{2s}+\nu_s)^2\,ds\Big].$$

Combining this fact with (16) and monotone convergence yields the second identity in the assertion. □

4 Formulation of the dual problem

In this section, we will first apply results from Schied [22] in preparation for the application of stochastic control techniques. To check for the applicability of the results in [22], note first that our utility function (8) belongs to C^1, is increasing and strictly concave, and satisfies the Inada conditions $U'(0+)=\infty$ and $U'(\infty-)=0$. It also has asymptotic elasticity $AE(U)=\limsup_{x\uparrow\infty}xU'(x)/U(x)=0<1$. The following lemma states that the penalty function γ satisfies [22, Assumption 2.1], which is needed for the applicability of the duality results in [22].

Lemma 4.1 *The penalty function $\gamma(Q)$ defined in (4) is the minimal penalty function of the convex risk measure*

$$\rho(X):=\sup_{Q\ll\mathbb{P}}\big(E_Q[\,-X\,]-\gamma(Q)\big),$$

that is, γ satisfies the biduality relation

$$\gamma(Q)=\inf_{X\in L^\infty}\big(E_Q[\,-X\,]-\rho(X)\big),\qquad Q\ll\mathbb{P}.$$

Moreover, ρ is continuous from below on $L^\infty(\mathbb{P})$.

Proof: By the biduality theorem and the general representation theory for convex risk measures on $L^\infty(\mathbb{P})$ as described in [9], γ will be identified as the minimal penalty function of ρ once we have shown that it is convex and lower semicontinuous for the strong (and hence the weak) topology on $L^1(\mathbb{P})$.

We first show convexity. Take $Q,\widetilde{Q}\ll\mathbb{P}$ such that both $\gamma(Q)$ and $\gamma(\widetilde{Q})$ are finite and let $Q^\lambda:=\lambda Q+(1-\lambda)\widetilde{Q}$ for $\lambda\in[0,1]$. To this end, suppose that η and $\widetilde{\eta}$ are two progressive processes associated via (12) and (13) with Q and $\widetilde{Q}$, respectively. Let D_t and $\widetilde{D}_t$ denote the corresponding density processes. Since

$$\infty>\gamma(Q)\ge\kappa_1\mathbb{E}\Big[\int_0^T D_t|\eta_t|^2\,dt\Big]-T\kappa_2$$

due to (5), we have $D_t|\eta_t|<\infty$ $dt\otimes d\mathbb{P}$-a.e., and so we can define the process

$$\xi_t:=\frac{\lambda D_t\eta_t+(1-\lambda)\widetilde{D}_t\widetilde{\eta}_t}{\lambda D_t+(1-\lambda)\widetilde{D}_t}\cdot\mathbb{I}_{\{\lambda D_t+(1-\lambda)\widetilde{D}_t>0\}}.$$

We use next that $(x, y) \mapsto xh(y/x)$ is a convex function on $(0, \infty) \times [0, \infty)$; see, e.g., [23, Equation (21)]. Hence,

$$E_{Q^\lambda}\Big[\int_0^T h(\xi_t)\,dt\Big] \le \lambda\gamma(Q) + (1-\lambda)\gamma(\widetilde{Q}) < \infty,$$

where we have used that $D^\lambda := \lambda D + (1-\lambda)\widetilde{D}$ is the density process of Q^λ with respect to $\mathbb{P}$. In particular, we get $\int_0^T |\xi_t|^2\,dt < \infty$ Q^λ-a.s. Moreover, one easily checks that D^λ satisfies $dD^\lambda_t = D^\lambda_t \xi_t\,dW_t$, and we obtain the identity $\gamma(Q^\lambda) = E_{Q^\lambda}\big[\int_0^T h(\xi_t)\,dt\big]$. This proves the convexity of γ.

Next, we will show the lower semicontinuity of γ for L^1-convergence. To this end, let $D^n_T := dQ_n/d\mathbb{P}$ be a sequence of probability densities converging to $D_T := dQ/d\mathbb{P}$ in $L^1(\mathbb{P})$. Then $\sup_{t\le T}|D^n_t - D_t| \to 0$ in $\mathbb{P}$-probability. A localization argument then shows that $\int \frac{1}{D^n_t}\,dD^n_t$ converges to $\int \frac{1}{D_t}\,dD_t$ uniformly in Q-probability. It follows that the sequence (η^n) associated to (Q_n) via (12) and (13) converges in $dt \otimes dQ$-measure to the process η associated with Q. Fatou's lemma now yields $\liminf_n \gamma(Q_n) \le \gamma(Q)$.

Let us now show that ρ is continuous from below. Due to our coercivity assumption (5), we have $\gamma(Q)+\kappa_2 \ge 2\kappa_1 H(Q|\mathbb{P}) = 2\kappa_1\mathbb{E}\big[\frac{dQ}{d\mathbb{P}}\log\frac{dQ}{d\mathbb{P}}\big]$ for $Q \ll \mathbb{P}$. Hence, the level sets $\big\{\frac{dQ}{d\mathbb{P}} \,|\, \gamma(Q) \le c\big\}$ are uniformly integrable. Therefore continuity from below follows from [17, Lemma 2] together with [9, Corollary 4.35] and the Dunford-Pettis theorem. □

Remark 4.2 Given the preceding lemma, it follows from [22, Theorem 2.4] that the value function u of the primal problem satisfies

$$u(x) = \sup_{\pi\in\mathcal{A}}\inf_{Q\ll\mathbb{P}}\big(E_Q[\log X^{x,\pi}_T] + \gamma(Q)\big) = \inf_{Q\ll\mathbb{P}}\sup_{\pi\in\mathcal{A}}\big(E_Q[\log X^{x,\pi}_T] + \gamma(Q)\big).$$

Due to (6), one might thus guess that

$$\sup_{\pi\in\mathcal{A}} E_Q[\log X^{x,\pi}_T] = \log x + \frac{1}{2}\int_0^T E_Q[(\eta_{1t} + \theta(Y_t))^2 + r(Y_t)]\,dt, \tag{17}$$

if η is associated with $Q \ll \mathbb{P}$ via (12) and (13). Moreover, this argument suggests that the optimal strategy for Q is given by

$$\pi^Q_t = \frac{\eta_{1t} + \theta(Y_t)}{\sigma(Y_t)}. \tag{18}$$

Minimizing over $Q \ll \mathbb{P}$ would then formally yield the HJB equation (9) for our value function. There are, however, some subtleties associated with this approach. First of all, one needs a proper localization argument to justify (17). While this localization argument can be carried out via similar arguments as those in Lemma 3.4, another difficulty arises from the fact that the strategy in (18) is defined Q-a.s. only. Therefore one would have to check whether it can be extended to a $\mathbb{P}$-a.s. defined strategy in $\mathcal{A}$. In fact, if a strategy is admissible under some $Q \ll \mathbb{P}$ but not under $\mathbb{P}$ itself it may be an arbitrage opportunity in the model Q; see [23, Example 2.5]. For these reasons, we do not pursue further the control approach on the primal problem and work on the dual problem instead.

It follows from [22, Theorems 2.4 and 2.6] that the dual value function of the robust utility maximization problem is given as

$$\widetilde{u}(\lambda) := \inf_{\nu\in\mathcal{M}} \inf_{Q\ll\mathbb{P}} \Big(\mathbb{E}\Big[\, D_T^Q \widetilde{U}\Big(\frac{\lambda Z_T^\nu}{D_T^Q S_T^0}\Big)\,\Big] + \gamma(Q)\Big), \tag{19}$$

where $\widetilde{U}(z) = \sup_{x\geq 0}(U(x) - zx)$ is the Fenchel-Legendre transform of the convex function $-U(-x)$. Due to [22, Theorem 2.4], the primal value function

$$u(x) = \sup_{\pi\in\mathcal{A}} \inf_{Q\ll\mathbb{P}} \big(E_Q[\log X_T^{x,\pi}\,] + \gamma(Q)\big)$$

can then be obtained as

$$u(x) = \min_{\lambda>0}(\widetilde{u}(\lambda) + \lambda x). \tag{20}$$

In our specific setting (8), we have $\widetilde{U}(z) = -\log z - 1$. Thus, we can simplify the duality formula (20) as follows. First, the expectation in (19) can be computed as

$$\mathbb{E}\Big[\, D_T \widetilde{U}\Big(\frac{\lambda Z_T^\nu}{D_T S_T^0}\Big)\,\Big] = \mathbb{E}\Big[\, D_T \log \frac{D_T S_T^0}{Z_T^\nu}\,\Big] - \log\lambda - 1 =: \Lambda_{Q,\nu} - \log\lambda - 1.$$

Hence,

$$u(x) = \log x + \inf_{Q\ll\mathbb{P}} \inf_{\nu\in\mathcal{M}} (\Lambda_{Q,\nu} + \gamma(Q))$$

Lemma 4.3 *For $Q\sim\mathbb{P}$ such that $\gamma(Q)<\infty$, we have $\Lambda_{Q,0}<\infty$. In particular, condition (12) in [22] is satisfied.*

Proof: Our conditions on h yield that $\kappa_1 H(Q|\mathbb{P}) \leq \gamma(Q) + \kappa_2 < \infty$. Let P^* be the equivalent local martingale measure defined by $dP^*/d\mathbb{P} = Z_T^0$. Then

$$\begin{aligned}
\mathbb{E}\Big[\, D_T \log\frac{D_T}{Z_T^0}\,\Big] &= H(Q|P^*) = H(Q|\mathbb{P}) + E_Q\Big[\,\log\frac{d\mathbb{P}}{dP^*}\,\Big] \\
&= H(Q|\mathbb{P}) + E_Q\Big[\int_0^T \theta(Y_t)\,dW_t^1 + \frac{1}{2}\int_0^T \theta(Y_t)^2\,dt\Big] \\
&= H(Q|\mathbb{P}) + E_Q\Big[\int_0^T \Big(\theta(Y_t)\eta_{1t} + \frac{1}{2}\theta(Y_t)^2\Big)\,dt\Big].
\end{aligned}$$

Using again $\gamma(Q)<\infty$ one sees that the last term is finite, and this implies the assertion. □

Due to the preceding lemma, we may now apply [22, Theorem 2.6]. It yields that, if the pair $(\widehat{Q},\widehat{\nu})$ minimizes (19), then there exists an optimal strategy $\widehat{\pi}\in\mathcal{A}$, whose terminal wealth is given by

$$X_T^{x,\widehat{\pi}} = I\Big(\frac{\widehat{\lambda} Z_T^{\widehat{\nu}}}{D_T^{\widehat{Q}} S_T^0}\Big), \tag{21}$$

where $I(y) := -\widetilde{U}'(y) = \log y + 1$ and $\widehat{\lambda}>0$ minimizes (20).

5 HJB approach to the dual problem

In this section, we will describe the dual problem by stochastic control techniques. Our aim is to maximize $\Lambda_{Q,\nu}$ over $Q \in \mathcal{Q}$ and $\nu \in \mathcal{M}$. Let us first heuristically derive the HJB equation for the dual problem; a rigorous argument will be provided at a later stage. To this end, we will use Lemma 3.1 to write the density process of $Q \ll \mathbb{P}$ as

$$D_t^\eta = \mathcal{E}\Big(\int \eta_s \, dW_s \Big)_t \qquad Q\text{-a.s.}$$

and denote by $\mathcal{N}$ the set of all processes η arising in this way. Note that the stochastic exponential need not be not defined under $\mathbb{P}$ if Q is not equivalent to $\mathbb{P}$. We will use both $\eta \in \mathcal{N}$ and $\nu \in \mathcal{M}$ as control processes. Let us write $(Y_t^y)_{t\geq 0}$ to indicate the starting point $y = Y_0^y$ of the solution to the SDE (3). We then introduce the function

$$J(t, y, \eta, \nu) := \mathbb{E}\Big[D_t^\eta \log \frac{D_t^\eta S_t^0}{Z_t^\nu} \Big] + \mathbb{E}\Big[D_t^\eta \int_0^t h(\eta_s) \, ds \Big],$$

where Z^ν and S^0 depend on y via Y^y. In particular, $J(T, Y_0, \eta, \nu) = \Lambda_{Q,\nu} + \gamma(Q)$. Our aim is to study the value function

$$V(t, y) := \inf_{\eta\in\mathcal{N}} \inf_{\nu\in\mathcal{M}} J(t, y, \eta, \nu).$$

Remark 5.1 The process $dW^{(\eta)} := dW_t + \eta_t \, dt$ is a two-dimensional Q-Brownian motion. Hence, if the processes η and ν are sufficiently bounded, then their stochastic integrals with respect to $W^{(\eta)}$ are Q-martingales, and we get

$$J(t, y, \eta, \nu) = E_Q\Big[\int_0^t \Big(\frac{1}{2}|\eta_s|^2 + r(Y_s^y) + \theta(Y_s^y)\eta_{1s} + \nu_s \eta_{2s} + \frac{1}{2}\big(\theta^2(Y_s^y) + \nu_s^2\big) + h(\eta_s) \Big) \, ds \Big].$$

Under Q, the process Y^y follows an SDE of the form

$$dY_t^y = g(Y_t^y) \, dt + \rho\eta_{1t} \, dt + \overline{\rho}\eta_{2t} \, dt + d\widetilde{W}_t^{(\eta)}, \tag{22}$$

where $\widetilde{W}^{(\eta)}$ is a one-dimensional Q-Brownian motion. Standard control theory now suggests that the function V is (formally) a solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$v_t = \frac{1}{2}v_{yy} + g v_y + r + \inf_{\nu\in\mathbb{R}} \inf_{\eta\in\mathbb{R}^2} \Big(\big[\rho\eta_1 + \overline{\rho}\eta_2\big] v_y + \frac{1}{2}(\eta_2 + \nu)^2 + \frac{1}{2}(\eta_1 + \theta)^2 + h(\eta) \Big)$$

with initial condition

$$v(0, y) = 0. \tag{23}$$

Eliminating the control parameter ν by taking $\nu = -\eta_2$ yields the reduced equation

$$\begin{aligned} v_t &= \frac{1}{2}v_{yy} + g v_y + r + \inf_{\eta\in\mathbb{R}^2} \Big(\big[\rho\eta_1 + \overline{\rho}\eta_2\big] v_y + \frac{1}{2}(\eta_1 + \theta)^2 + h(\eta) \Big) \\ &= \frac{1}{2}v_{yy} + g v_y + r + \phi(v_y). \end{aligned} \tag{24}$$

The preceding heuristic argument is made precise by the following verification result.

Proposition 5.2 (Verification result) *Suppose the PDE* (24)–(23) *admits a classical solution* $v \in C^{1,2}((0,T)\times\mathbb{R}) \cap C([0,T]\times\mathbb{R})$ *satisfying a polynomial growth condition in* y *and suppose that one of the following three conditions is satisfied:*

(a) dom h *is bounded;*

(b) v_y *is bounded;*

(c) $\psi(y,\cdot)$ *satisfies a radial growth condition in direction* $(\rho,\overline{\rho})$*, uniformly in* y*, and* v_y *satisfies*

$$\big|\partial_p^- \phi\prime\big(y; v_y(t,y)\big)\big| \vee \big|\partial_p^+ \phi\big(y; v_y(t,y)\big)\big| \le C_1(1+|y|)$$

for some constant C_1.

Then $v = V$*. Suppose furthermore that* $\eta^* : [0,T]\times\mathbb{R}\to\mathbb{R}$ *is a measurable function realizing the infimum in* (24)*. Then* $\widehat{\eta}_t := \eta^*(T-t, Y_t)$ *belongs to the set* $\mathcal{N}$*,* $\widehat{\nu}_t := -\widehat{\eta}_{2t}$ *belongs to* $\mathcal{M}$*, and we have* $V(T,y) = J(T,y,\widehat{\eta},\widehat{\nu})$*.*

Proof: Let $\nu \in \mathcal{M}$ and $\eta \in \mathcal{N}$ be control processes such that $J(t,y,\eta,\nu) < \infty$ and consider the localized martingale measure $\widehat{P}$ associated with ν and let $\eta \in \mathcal{N}$ be associated to $Q \ll \mathbb{P}$. Then

$$J(t,y,\eta,\nu) = E_Q\Big[\,\log\frac{D_t^\eta}{Z_t^\nu}\,\Big] + E_Q[\log S_t^0\,] + E_Q\Big[\int_0^t h(\eta_s)\,ds\Big].$$

The control process ν occurs only in the first term on the right, which according to Lemma 3.4 is given by

$$E_Q\Big[\,\log\frac{D_t^\eta}{Z_t^\nu}\,\Big] = \frac{1}{2}E_Q\Big[\int_0^t (\eta_{1s}+\theta(Y_s^y))^2 + (\eta_{2s}+\nu_s)^2\,ds\Big].$$

This term is minimized by taking $\nu_s(\omega) := -\eta_{2s}(\omega)$ for $s \le t$ and $\omega \in \{\int_0^t \eta_{2s}^2\,ds < \infty\}$ and $\nu_s(\omega) := 0$ otherwise. Thus, we arrive at

$$\widetilde{J}(t,y,\eta) := \inf_{\nu\in\mathcal{M}} J(t,y,\eta,\nu) = E_Q\Big[\int_0^t \frac{1}{2}(\eta_{1s}+\theta(Y_s^y))^2 + r(Y_s^y) + h(\eta_s)\,ds\Big]. \tag{25}$$

Due to (22), we have under Q that

$$\begin{aligned} dv(u-t,Y_t^y) &= v_y(u-t,Y_t^y)\,d\widetilde{W}_t^{(\eta)} \\ &\quad + \Big\{v_y(u-t,Y_t^y)\big(g(Y_t^y)+\rho\eta_{1t}+\overline{\rho}\eta_{2t}\big) - v_t(u-t,Y_t^y) + \frac{1}{2}v_{yy}(u-t,Y_t^y)\Big\}\,dt \\ &\ge v_y(u-t,Y_t^y)\,d\widetilde{W}_t^{(\eta)} - \Big\{\frac{1}{2}\big(\eta_{1t}+\theta(Y_t^y)\big)^2 + r(Y_t^y) + h(\eta_t)\Big\}\,dt, \end{aligned} \tag{26}$$

where we have used (24) in the latter inequality. Letting $\sigma_n := \inf\{t \ge 0\,|\,|Y_t^y| \ge n\}$, by the continuity of v_y and the boundedness of the process Y_t^y for $0 \le t \le \sigma_n$, we get

$$v(u,y) \le E_Q\Big[\int_0^{u\wedge\sigma_n} \frac{1}{2}\big(\eta_{1t}+\theta(Y_t^y)\big)^2 + r(Y_t^y) + h(\eta_t)\,dt + v(u-u\wedge\sigma_n, Y^y_{u\wedge\sigma_n})\Big]. \tag{27}$$

Since $\sigma_n \uparrow \infty$ Q-a.s., we obtain $v(u, y) \leq \widetilde{J}(u, y, \eta)$ and in turn $v \leq V$. Here we have also used the initial condition $v(0, \cdot) = 0$, the fact that r is bounded, and the assumption that v satisfies a polynomial growth condition in y together with dominated convergence and Theorem 4.7 in [19], which states that

$$\sup_{0 \leq t \leq T} \mathbb{E}\big[\, \exp\{\delta |Y_t|^2\} \,\big] < \infty \qquad \text{for some } \delta > 0. \tag{28}$$

Now we shall prove the reverse inequality. The coercivity condition (5) and the lower semicontinuity of $h(\cdot)$ imply that for each t and y there exists

$$\eta^*(t, y) \in \operatorname*{arg\,min}_{\eta \in \mathbb{R}^2} \Big[(\rho\eta_1 + \overline{\rho}\eta_2) v_y(t, y) + \frac{1}{2}(\eta_1 + \theta(y))^2 + h(\eta) \Big].$$

By a measurable selection argument, $\eta^*(t, y)$ can be chosen as a measurable function of t and y. To prove that $\hat{\eta}_s := \eta^*(u - s, Y_s)$ is an admissible Markov control, i.e., $\hat{\eta} \in \mathcal{N}$, we need to verify that

$$D_t^{\hat{\eta}} := \mathcal{E}\Big(\int \hat{\eta}_{1s}\, dW_s^1 + \int \hat{\eta}_{2s}\, dW_s^2 \Big)_t, \qquad 0 \leq t \leq u,$$

is a $\mathbb{P}$-martingale. Once this has been proved, we get an equality in (26) and hence in (27).

According to Liptser and Shiryayev [19], p.220, $D^{\hat{\eta}}$ is a martingale if we can show that for some $\varepsilon > 0$

$$\sup_{0 \leq t \leq u} \mathbb{E}\big[\, \exp\{\varepsilon |\hat{\eta}_t|^2\} \,\big] < \infty. \tag{29}$$

This is clear when dom h is bounded or when v_y is bounded, i.e., under conditions (a) or (b). Assuming condition (c), note that $\eta^*(t, y)$ belongs in fact to the supergradient of $x \mapsto \psi(y, x)$ at $x = (\rho, \overline{\rho}) v_y(t, y)$. Hence, the radial growth condition together with the estimate on $\partial_p^{\pm}\phi(y; v_y(t, y))$ implies that $|\eta^*(t, y)| \leq c(1 + |y|)$ for some constant c. Therefore (29) now follows from (28). □

Proof of Theorem 2.1: If dom h is compact, we can restrict the infimum in (24) to controls η in the compact set dom h, and Theorem IV.4.3 and Remark IV.3.3 in [7] yield the existence of a classical solution v to the PDE (24)–(23) satisfying a polynomial growth condition. Thus, condition (a) of Proposition 5.2 is satisfied, and we get the identification $v = V$. The form of the optimal strategy $\widehat{\pi}$ and the fact that $(\widehat{Q}, \widehat{\pi})$ is a saddlepoint follow immediately from (6), (21), and the results in [22]. □

6 Existence of a classical solution for a noncompact control domain

In this section, we will derive existence results for the PDE (24)–(23) in case of a non-compact effective domain dom h. We will need the following estimate.

Lemma 6.1 *For $\delta > 0$, the value function V satisfies*

$$K_- \le \frac{V(t+\delta, y) - V(t,y)}{\delta} \le K_+,$$

where

$$K_- = -\|r^-\|_\infty \qquad \text{and} \qquad K_+ = \frac{1}{2}\|\theta\|_\infty^2 + \|r^+\|_\infty.$$

In particular, we have $tK_- \le V(t,y) \le tK_+$.

Proof: To obtain the lower bound, note that by (25)

$$\begin{aligned} V(t+\delta, y) - V(t,y) &\ge \inf_\eta \left(\widetilde{J}(t+\delta, y, \eta) - \widetilde{J}(t,y,\eta)\right) \\ &= \inf_\eta \mathbb{E}\left[D^\eta_{t+\delta} \int_t^{t+\delta} \left(\frac{1}{2}\big(\eta_{1s} + \theta(Y^y_s)\big)^2 + r(Y^y_s) + h(\eta_s)\right) ds \right] \\ &\ge -\|r^-\|_\infty \delta. \end{aligned}$$

To prove the upper bound, take a process $\widehat{\eta}$ such that $V(t,y) \ge \varepsilon\delta + \widetilde{J}(t,y,\widehat{\eta})$ and $\widehat{\eta}_s = 0$ for $s \in [t, t+\delta]$. It follows that

$$V(t+\delta, y) - V(t,y) - \varepsilon h \le \widetilde{J}(t+\delta, y, \widehat{\eta}) - \widetilde{J}(t,y,\widehat{\eta}) \le \mathbb{E}\left[D^{\widehat{\eta}}_{t+\delta} \int_t^{t+\delta} \left(\frac{1}{2}\theta(Y^y_s)^2 + r(Y^y_s)\right) ds \right],$$

which gives the upper bound. □

For $n \in \mathbb{N}$, let us introduce the auxiliary functions

$$h_n(\eta) := \begin{cases} h(\eta) & \text{if } h(\eta) \le n, \\ +\infty & \text{otherwise.} \end{cases}$$

Then h_n also satisfies the assumptions made on h, and its effective domain dom h_n is compact. Thus, according to Theorem 2.1 and its proof, the value function V^n obtained by replacing h with h_n coincides with the unique bounded classical solution v^n of the corresponding HJB equation. Based in the preceding lemma we now deduce an estimate on the growth of the gradients v^n_y.

Lemma 6.2 *Suppose first that $p \mapsto \phi(y;p)$ has superlinear growth. Then for every $R > 0$ there exist $C_R > 0$ and $n_0 \in \mathbb{N}$, both depending only on R, T, and the model parameters, such that $|v^n_y(t,y)| \le C_R$ whenever $n \ge n_0$, $|y| \le R$, and $0 < t < T$.*

If, alternatively, g is bounded and (11) holds, then n_0 can be chosen independently of R, and $v^n_y(t,y)$ can be bounded uniformly for $n \ge n_0$, $t \in (0,T)$, and $y \in \mathbb{R}$.

Proof: Let $R > 0$ be given. Recall from Lemma 6.1 that $-K \leq v^n(t,y) \leq K$ for some constant K depending only on the model parameters and T. Therefore, due to the mean value theorem, there exist $y_+^n \in (R, R+1)$ and $y_-^n \in (-R-1, -R)$ such that

$$|v_y^n(t, y_\pm^n)| \leq 2K.$$

If $|v_y^n(t,\cdot)|$ exceeds $2K$ in (y_-^n, y_+^n), and hence in $[-R, R]$, this implies the existence of a local maximum of the continuous function $|v_y^n(t,\cdot)|$. Hence, it is enough to estimate $|v_y^n(t,y)|$ in critical points y of $v_y^n(t,\cdot)$, which are located in $[-R-1, R+1]$. In such points y, v^n satisfies the equation

$$v_t^n = \phi^n(v_y^n) + g v_y^n + r, \tag{30}$$

where ϕ^n corresponds to h_n. Due to Lemma 6.1, the left-hand side is bounded in absolute value by $K_+ - K_-$. Next, let c_R be an upper bound for $|g(y)|$ when $|y| \leq R+1$. Due to the superlinear growth assumption on $p \mapsto \phi(y;p)$, there exists some n_0 such that

$$\liminf_{|p|\to\infty} \left| \frac{\phi^n(y;p)}{p} \right| \geq c_R + 1 \qquad \text{for } n \geq n_0 \text{ and all } y. \tag{31}$$

But in view of (30) and the uniform bound on v_t^n, this clearly implies a uniform bound of the form $|v_y^n(t,y)| \leq c_0$ whenever $n \geq n_0$, $0 \leq t \leq T$, and y is a critical point of v_y^n with $|y| \leq R+1$. Taking $C_R := c_0 \vee (2K)$ yields the first part of the result.

If g is bounded and (11), then c_R can be chosen independently of R, and (31) holds with $c_R + \varepsilon/2$ instead of $c_R + 1$, where ε is taken from (11). □

Note that the functions v^n decrease pointwise to a function v, which also satisfies the bounds

$$tK_- \leq v(t,y) \leq tK_+.$$

Lemma 6.3 *Suppose that $p \mapsto \phi(y;p)$ has superlinear growth or g is bounded and* (11) *holds. Then, for any $\beta \in (0,1)$, $v(t,y) = \lim_n v^n(t,y)$ is a bounded classical solution in $C^{1+\frac{\beta}{2},2+\beta}((0,T)\times\mathbb{R}) \cap C([0,T]\times\mathbb{R})$ to the Cauchy problem*

$$\begin{cases} v_t = \frac{1}{2} v_{yy} + \phi(v_y) + g v_y + r \\ v(0,\cdot) = 0. \end{cases} \tag{32}$$

Moreover, $|v_t(t,y)| \leq K_+ - K_-$.

Proof: Take $R > 0$ and let C_R be as in Lemma 6.2. Note then that for $|p| \leq C_R$ there exists some n_1 such that $\phi^n(y;p) = \phi(y;p)$ for $n \geq n_1$ and all y. Hence, Lemma 6.2 yields that for $n \geq n_0 \vee n_1$ and $|y| \leq R$

$$v_t^n = \frac{1}{2} v_{yy}^n + \phi(v_y^n) + g v_y^n + r. \tag{33}$$

Since the terms v_t^n, $\phi(v_y^n)$, $g v_y^n$, and r are uniformly bounded, the same must be true of v_{yy}^n. Hence, the Arzela-Ascoli theorem yields the existence of a subsequence (n_k) such that

$v_y^{n_k}$ converges locally uniformly in $C([0,T]\times\mathbb{R})$ to some function w, and the pointwise convergence of $v^{n_k}(t,y)$ implies that w is equal to the y-derivative v_y of v. Moreover, the uniform bounds on v_t^n and v_{yy}^n imply that v_y is locally Lipschitz continuous on $[0,T]\times\mathbb{R}$. In particular, v belongs to $W^{1,1}_{2,loc}([0,T]\times\mathbb{R})$ and satisfies $|v_t(t,y)|\le K_+-K_-$.

Taking $\varphi\in C^\infty((0,T)\times\mathbb{R})$ and writing (33) in integral form yields

$$\int\int\Big(v^{n_k}\varphi_t-\frac{1}{2}v_y^{n_k}\varphi_y+\big[gv_y^{n_k}+r+\phi(v_y^{n_k})\big]\varphi\Big)\,dy\,dt=0.$$

Taking the limit when $k\uparrow\infty$ it follows that v is a generalized solution of the parabolic equation

$$v_t=\frac{1}{2}v_{yy}+gv_y+f,$$

where the free term f is equal to $\phi(v_y)+r$ and satisfies a local Lipschitz condition in $[0,T]\times\mathbb{R}$. In view of standard regularity results for parabolic equations (see, for instance, Theorem 12.2 in Chapter III of [18]) we conclude that $v\in C^{1+\frac{\beta}{2},2+\beta}((0,T)\times\mathbb{R})$ for any $\beta\in(0,1)$. The fact that v solves (32) is now obvious. □

Proof of Theorem 2.3: It follows from Lemma 6.2, Lemma 6.3, and its proof that there exists a bounded classical solution v with a bounded gradient v_y. Hence, Proposition 5.2 (b) applies, and the first part of Theorem 2.3 follows. The part on $\widehat{Q}$ and $\widehat{\pi}$ follows as in Theorem 2.1. □

The application of our verification result in Proposition 5.2 requires a growth condition on the gradient of v.

Lemma 6.4 *Suppose that $p\mapsto\phi(y;p)$ has superlinear growth. Then there exists a constant C_1, depending only on T and the model parameters, such that*

$$\big|\partial_p^-\phi\big(y;v_y(t,y)\big)\big|\vee\big|\partial_p^+\phi\big(y;v_y(t,y)\big)\big|\le C_1(1+|y|).$$

Proof: The C^2-function $y\mapsto v(t,y)$ is bounded from above and below by the two constants TK_+ and TK_-, which are independent of $t\le T$. Therefore, the function $y\mapsto|v_y(t,y)|$ cannot increase to its supremum, and we conclude that it is enough to estimate $|v_y(t,y)|$ in such points y that are critical points of $v_y(t,\cdot)$. In these points y, $v_{yy}(t,y)$ vanishes, and we obtain $v_t=\phi(v_y)+gv_y+r$. Dividing by $|v_y|$, we hence get that for $|v_y|\ge 1$

$$\Big|\frac{\phi\big(y;v_y(t,y)\big)}{v_y(t,y)}\Big|\le K_+-K_-+|g(0)|+\|g'\|_\infty|y|+\|r\|_\infty.$$

and the right-hand side can be bounded by $c_1(1+|y|)$ for an appropriate constant c_1.

The coercivity condition (5) implies that the concave function $p\mapsto\phi(y;p)$ grows at most quadratically as $|p|\to\infty$. Hence, there are constants $p_0,c_2\ge 1$ such that

$$|\partial_p^+\phi(p)|\vee|\partial_p^-\phi(p)|\le c_2|\partial_p^+\phi(p/2)|\vee|\partial_p^-\phi(p/2)|\qquad\text{for }|p|\ge p_0.$$

Next, choose p_1 such that $\phi(y;p) \leq 0$ and $\partial_p^- \phi(y;p) \leq 0$ for $p \geq p_1/2$. Such a p_1 exists due to concavity. Then we obtain that for $p \geq p_0 \vee p_1$

$$\Big|\frac{1}{p}\phi(p)\Big| \geq \frac{-1}{p}(\phi(p) - \phi(p/2)) \geq \frac{1}{2}|\partial_p^+ \phi(p/2)| \geq \frac{1}{2c_2}|\partial_p^+ \phi(p)|.$$

An analogous inequality holds for p less than some $p_2 \leq 0$ and $\partial_p^- \phi$. Putting everything together yields the assertion. □

Proof of Theorem 2.5: From Lemma 6.3 we know that there exists a classical solution v to the equation (13). Lemma 6.4 gives the conditions to apply part (c) of Proposition 5.2. This proposition then implies the uniqueness of v, while the rest of the theorem follows as before. □

Acknowledgement: Part of this work was done while the first author visited INRIA Rocquencourt, its support trough the project Mathfi is greatly appreciated as well as its hospitality. The second author is grateful for support and hospitality during a research visit at the National Center for Theoretical Sciences at Tsing-Hua University and the Academia Sinica in Taipei, where part of the research for this paper was conducted. He is indebted to Shuenn-Jyi Sheu and Ching-Tang Wu for many stimulating discussions. Finally, both authors wish to thank Lars Putzig for providing the numerical simulations in Example 2.7.

References

[1] Barrieu, P., El Karoui, N. *Optimal Derivatives Design under Dynamic Risk Measures.* In: Mathematics of finance, 13–25, Contemp. Math., 351, Amer. Math. Soc., Providence, RI, 2004.

[2] Bordigoni, G. Matoussi, A. Schweizer, M. *A stochastic control approach to a robust utility maximization problem.* To appear in Proceedings of Abel Symposium 2005, Springer.

[3] Burgert, C., Rüschendorf, L. *Optimal consumption strategies under model uncertainty.* Statist. Decisions 23 (2005), no. 1, 1–14.

[4] Castañeda-Leyva, N., Hernández-Hernández D. *Optimal investment in incomplete financial markets with stochastic volatility.* Contemp. Math. 336, 119–136 (2003).

[5] Castañeda-Leyva, N., Hernández-Hernández D. *Optimal consumption-investment problems in incomplete markets with stochastic coefficients.* SIAM J. Control Optim. 44, No. 4, pp. 1322–1344 (2005).

[6] Fleming, W., Hernández-Hernández, D. *Tradeoff between consumption and investments in incomplete financial markets.* Appl. Math. Optim. 52 (2005), 219–235.

[7] Fleming, W., Soner, M. *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York, 1993.

[8] Föllmer, H., Gundel, A. *Robust projections in the class of martingale measures.* To appear in Illinois J. Math.

[9] Föllmer, H., Schied, A. *Stochastic Finance: An Introduction in Discrete Time.* Berlin: de Gruyter Studies in Mathematics 27 (2002). Second edition (2004).

[10] Gilboa, I., Schmeidler, D. *Maximin expected utility with non-unique prior.* J. Math. Econ. 18, 141-153 (1989).

[11] Hansen, L., Sargent, T. *Robust control and model uncertainty.* American Economic Review 91, 60-66 (2001).

[12] Hernández-Hernández D., Schied, A. *Robust utility maximization in a stochastic factor model.* Stat. Decisions 24 (2006).

[13] E. Jouini, W. Schachermayer, N. Touzi: *Optimal risk sharing with law-invariant monetary utility functions.* Working paper, Université Paris Dauphine (2005).

[14] Klöppel, S., Schweizer, M. *Dynamic utility indifference valuation via convex risk measures.* NCCR FINRISK working paper No. 209, ETH Zurich (2005).

[15] Klöppel, S., Schweizer, M. *Dynamic indifference valuation via convex risk measures.* To appear in Math. Finance (2006).

[16] Maccheroni, F., Marinacci, M., Rustichini, A. *Ambiguity aversion, malevolent nature, and the variational representation of preferences.* Preprint 2004.

[17] Krätschmer, V. *Robust representation of convex risk measures by probability measures.* Stoch. Finance (to appear).

[18] O.A. Ladyzenskaya, V.A. Solonikov and N.N. Uralseva, *Linear and Quasilinear Equations of Parabolic Type*, AMS Transl. of Math. Monographs, Providence, RI, 1968.

[19] Liptser, R.S., Shiryayev A.N. *Statistics of Random Processes I: General theory*, Springer Verlag, New York, 1977.

[20] Quenez, M.-C. *Optimal portfolio in a multiple-priors model.* Seminar on Stochastic Analysis, Random Fields and Applications IV, 291–321, Progr. Probab., 58, Birkhäuser, Basel, 2004.

[21] Schied, A. *Optimal investments for robust utility functionals in complete market models.* Math. Oper. Research. 30, No. 3, 750-764 (2005).

[22] Schied, A. *Optimal investments for risk- and ambiguity-averse preferences: a duality approach.* Preprint, TU Berlin (2005).

[23] Schied, A., Wu, C.-T. *Duality theory for optimal investments under model uncertainty.* Stat. Decisions 23, No. 3, 199-217 (2005).

[24] Schmeidler, D. *Subjective probability and expected utility without additivity.* Econometrica 57, no. 3, 571-587 (1989).

[25] Talay, D., Zheng, Z. *Worst case model risk management.* Finance Stochast. 6, 517-537 (2002).

Alexander Schied
Institute of Mathematics, MA 7-4
Berlin University of Technology
Strasse des 17. Juni 136
10623 Berlin, Germany
schied@math.tu-berlin.de

Daniel Hernández–Hernández
Centro de Investigación en Matemáticas
Apartado Postal 402
Guanajuato, Gto. C.P. 36000
México
dher@cimat.mx

SFB 649 Discussion Paper Series 2006

For a complete list of Discussion Papers published by the SFB 649, please visit http://sfb649.wiwi.hu-berlin.de.

001 "Calibration Risk for Exotic Options" by Kai Detlefsen and Wolfgang K. Härdle, January 2006.
002 "Calibration Design of Implied Volatility Surfaces" by Kai Detlefsen and Wolfgang K. Härdle, January 2006.
003 "On the Appropriateness of Inappropriate VaR Models" by Wolfgang Härdle, Zdeněk Hlávka and Gerhard Stahl, January 2006.
004 "Regional Labor Markets, Network Externalities and Migration: The Case of German Reunification" by Harald Uhlig, January/February 2006.
005 "British Interest Rate Convergence between the US and Europe: A Recursive Cointegration Analysis" by Enzo Weber, January 2006.
006 "A Combined Approach for Segment-Specific Analysis of Market Basket Data" by Yasemin Boztuğ and Thomas Reutterer, January 2006.
007 "Robust utility maximization in a stochastic factor model" by Daniel Hernández–Hernández and Alexander Schied, January 2006.
008 "Economic Growth of Agglomerations and Geographic Concentration of Industries - Evidence for Germany" by Kurt Geppert, Martin Gornig and Axel Werwatz, January 2006.
009 "Institutions, Bargaining Power and Labor Shares" by Benjamin Bental and Dominique Demougin, January 2006.
010 "Common Functional Principal Components" by Michal Benko, Wolfgang Härdle and Alois Kneip, Jauary 2006.
011 "VAR Modeling for Dynamic Semiparametric Factors of Volatility Strings" by Ralf Brüggemann, Wolfgang Härdle, Julius Mungo and Carsten Trenkler, February 2006.
012 "Bootstrapping Systems Cointegration Tests with a Prior Adjustment for Deterministic Terms" by Carsten Trenkler, February 2006.
013 "Penalties and Optimality in Financial Contracts: Taking Stock" by Michel A. Robe, Eva-Maria Steiger and Pierre-Armand Michel, February 2006.
014 "Core Labour Standards and FDI: Friends or Foes? The Case of Child Labour" by Sebastian Braun, February 2006.
015 "Graphical Data Representation in Bankruptcy Analysis" by Wolfgang Härdle, Rouslan Moro and Dorothea Schäfer, February 2006.
016 "Fiscal Policy Effects in the European Union" by Andreas Thams, February 2006.
017 "Estimation with the Nested Logit Model: Specifications and Software Particularities" by Nadja Silberhorn, Yasemin Boztuğ and Lutz Hildebrandt, March 2006.
018 "The Bologna Process: How student mobility affects multi-cultural skills and educational quality" by Lydia Mechtenberg and Roland Strausz, March 2006.
019 "Cheap Talk in the Classroom" by Lydia Mechtenberg, March 2006.
020 "Time Dependent Relative Risk Aversion" by Enzo Giacomini, Michael Handel and Wolfgang Härdle, March 2006.
021 "Finite Sample Properties of Impulse Response Intervals in SVECMs with Long-Run Identifying Restrictions" by Ralf Brüggemann, March 2006.
022 "Barrier Option Hedging under Constraints: A Viscosity Approach" by Imen Bentahar and Bruno Bouchard, March 2006.

SFB 649, Spandauer Straße 1, D-10178 Berlin
http://sfb649.wiwi.hu-berlin.de

This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

023 "How Far Are We From The Slippery Slope? The Laffer Curve Revisited" by Mathias Trabandt and Harald Uhlig, April 2006.
024 "e-Learning Statistics – A Selective Review" by Wolfgang Härdle, Sigbert Klinke and Uwe Ziegenhagen, April 2006.
025 "Macroeconomic Regime Switches and Speculative Attacks" by Bartosz Maćkowiak, April 2006.
026 "External Shocks, U.S. Monetary Policy and Macroeconomic Fluctuations in Emerging Markets" by Bartosz Maćkowiak, April 2006.
027 "Institutional Competition, Political Process and Holdup" by Bruno Deffains and Dominique Demougin, April 2006.
028 "Technological Choice under Organizational Diseconomies of Scale" by Dominique Demougin and Anja Schöttner, April 2006.
029 "Tail Conditional Expectation for vector-valued Risks" by Imen Bentahar, April 2006.
030 "Approximate Solutions to Dynamic Models – Linear Methods" by Harald Uhlig, April 2006.
031 "Exploratory Graphics of a Financial Dataset" by Antony Unwin, Martin Theus and Wolfgang Härdle, April 2006.
032 "When did the 2001 recession *really* start?" by Jörg Polzehl, Vladimir Spokoiny and Cătălin Stărică, April 2006.
033 "Varying coefficient GARCH versus local constant volatility modeling. Comparison of the predictive power" by Jörg Polzehl and Vladimir Spokoiny, April 2006.
034 "Spectral calibration of exponential Lévy Models [1]" by Denis Belomestny and Markus Reiß, April 2006.
035 "Spectral calibration of exponential Lévy Models [2]" by Denis Belomestny and Markus Reiß, April 2006.
036 "Spatial aggregation of local likelihood estimates with applications to classification" by Denis Belomestny and Vladimir Spokoiny, April 2006.
037 "A jump-diffusion Libor model and its robust calibration" by Denis Belomestny and John Schoenmakers, April 2006.
038 "Adaptive Simulation Algorithms for Pricing American and Bermudan Options by Local Analysis of Financial Market" by Denis Belomestny and Grigori N. Milstein, April 2006.
039 "Macroeconomic Integration in Asia Pacific: Common Stochastic Trends and Business Cycle Coherence" by Enzo Weber, May 2006.
040 "In Search of Non-Gaussian Components of a High-Dimensional Distribution" by Gilles Blanchard, Motoaki Kawanabe, Masashi Sugiyama, Vladimir Spokoiny and Klaus-Robert Müller, May 2006.
041 "Forward and reverse representations for Markov chains" by Grigori N. Milstein, John G. M. Schoenmakers and Vladimir Spokoiny, May 2006.
042 "Discussion of 'The Source of Historical Economic Fluctuations: An Analysis using Long-Run Restrictions' by Neville Francis and Valerie A. Ramey" by Harald Uhlig, May 2006.
043 "An Iteration Procedure for Solving Integral Equations Related to Optimal Stopping Problems" by Denis Belomestny and Pavel V. Gapeev, May 2006.
044 "East Germany's Wage Gap: A non-parametric decomposition based on establishment characteristics" by Bernd Görzig, Martin Gornig and Axel Werwatz, May 2006.
045 "Firm Specific Wage Spread in Germany - Decomposition of regional differences in inter firm wage dispersion" by Bernd Görzig, Martin Gornig and Axel Werwatz, May 2006.

This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

046 "Produktdiversifizierung: Haben die ostdeutschen Unternehmen den Anschluss an den Westen geschafft? – Eine vergleichende Analyse mit Mikrodaten der amtlichen Statistik" by Bernd Görzig, Martin Gornig and Axel Werwatz, May 2006.

047 "The Division of Ownership in New Ventures" by Dominique Demougin and Oliver Fabel, June 2006.

048 "The Anglo-German Industrial Productivity Paradox, 1895-1938: A Restatement and a Possible Resolution" by Albrecht Ritschl, May 2006.

049 "The Influence of Information Costs on the Integration of Financial Markets: Northern Europe, 1350-1560" by Oliver Volckart, May 2006.

050 "Robust Econometrics" by Pavel Čížek and Wolfgang Härdle, June 2006.

051 "Regression methods in pricing American and Bermudan options using consumption processes" by Denis Belomestny, Grigori N. Milstein and Vladimir Spokoiny, July 2006.

052 "Forecasting the Term Structure of Variance Swaps" by Kai Detlefsen and Wolfgang Härdle, July 2006.

053 "Governance: Who Controls Matters" by Bruno Deffains and Dominique Demougin, July 2006.

054 "On the Coexistence of Banks and Markets" by Hans Gersbach and Harald Uhlig, August 2006.

055 "Reassessing Intergenerational Mobility in Germany and the United States: The Impact of Differences in Lifecycle Earnings Patterns" by Thorsten Vogel, September 2006.

056 "The Euro and the Transatlantic Capital Market Leadership: A Recursive Cointegration Analysis" by Enzo Weber, September 2006.

057 "Discounted Optimal Stopping for Maxima in Diffusion Models with Finite Horizon" by Pavel V. Gapeev, September 2006.

058 "Perpetual Barrier Options in Jump-Diffusion Models" by Pavel V. Gapeev, September 2006.

059 "Discounted Optimal Stopping for Maxima of some Jump-Diffusion Processes" by Pavel V. Gapeev, September 2006.

060 "On Maximal Inequalities for some Jump Processes" by Pavel V. Gapeev, September 2006.

061 "A Control Approach to Robust Utility Maximization with Logarithmic Utility and Time-Consistent Penalties" by Daniel Hernández–Hernández and Alexander Schied, September 2006.

SFB 649, Spandauer Straße 1, D-10178 Berlin
http://sfb649.wiwi.hu-berlin.de

This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".