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A Guide to the Dagum Distributions*

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Abstract

In a series of papers in the 1970s, Camilo Dagum proposed several variants of a new model for the size distribution of personal income. This Chapter traces the genesis of the Dagum distributions in applied economics and points out parallel developments in several branches of the applied statistics literature. It also provides interrelations with other statistical distributions as well as aspects that are of special interest in the income distribution field, including Lorenz curves and the Lorenz order and inequality measures. The Chapter ends with a survey of empirical applications of the Dagum distributions, many published in Romance language periodicals.

1 Introduction

In the 1970s, Camilo Dagum embarked on a quest for a statistical distribution closely fitting empirical income and wealth distributions. Not satisfied with the classical distributions used to summarize such data – the Pareto distribution (developed by the Italian economist and sociologist Vilfredo Pareto in the late 19th century) and the lognormal distribution (popularized by the French engineer Robert Gibrat (1931)) – he looked for a model accommodating the heavy tails present in empirical income and wealth distributions as well as permitting an interior mode. The former aspect is well captured by the Pareto but not by the lognormal distribution, the latter by the lognormal but not the Pareto distribution. Experimenting with a shifted log-logistic distribution (Dagum 1975), a generalization of a distribution previously considered by Fisk (1961), he quickly realized that a further parameter was needed. This led to the Dagum type I distribution, a three-parameter distribution, and two four-parameter generalizations (Dagum 1977, 1980).

It took more than a decade until Dagum's proposal began to appear in the English-language economic and econometric literature. The first paper in a major econometrics journal

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utilizing the Dagum distribution appears to be by Majumder and Chakravarty (1990). In the statistical literature, the situation is more favorable, in that the renowned *Encyclopedia of Statistical Sciences* contains, in Vol. 4 (Kotz, Johnson and Read, 1983), an entry on income distribution models, unsurprisingly authored by Camilo Dagum (Dagum 1983). In retrospect, the reason for this long delay is fairly obvious: Dagum's 1977 paper was published in *Economie Appliquée*, a French journal with only occasional English-language contributions and fairly limited circulation in English-language countries. In contrast, the paper introducing the more widely known Singh-Maddala (1976) distribution was published in *Econometrica*, just one year before Dagum's contribution. It slowly emerged that the Dagum distribution is, nonetheless, often preferable to the Singh-Maddala distribution in applications to income data.

This Chapter provides a brief survey of the Dagum distributions, including interrelations with several more widely known distributions as well as basic statistical properties and inferential aspects. It also revisits one of the first data sets considered by Dagum and presents a survey of applications in economics.

2 Genesis and interrelations

Dagum (1977) motivates his model from the empirical observation that the income elasticity $\eta(F,x)$ of the cumulative distribution function (CDF) F of income is a decreasing and bounded function of F. Starting from the differential equation

$$\eta(F,x) = \frac{d\log F(x)}{d\log x} = ap\{1-[F(x)]^{1/p}\}, \quad x \geq 0, \tag{1}$$

subject to $p > 0$ and $ap > 0$, one obtains

$$F(x) = [1+(x/b)^{-a}]^{-p}, \quad x > 0. \tag{2}$$

This approach was further developed in a series of papers on generating systems for income distributions (Dagum 1980b, 1980c, 1983, 1990). Recall that the well-known Pearson system is a general-purpose system not derived from observed stable regularities in a given area of application. D'Addario's (1949) system is a translation system with flexible so-called generating and transformation functions built to encompass as many income distributions as possible; see e.g. Kleiber and Kotz (2003) for further details. In contrast, the system specified by Dagum starts from characteristic properties of empirical income and wealth distributions and leads to a generating system specified in terms of

$$\frac{d\log\{F(x)-\delta\}}{d\log x} = \vartheta(x)\phi(F) \leq k, \quad 0 \leq x_0 < x < \infty, \tag{3}$$

where $k > 0$, $\vartheta(x) > 0$, $\phi(x) > 0$, $\delta < 1$, and $d\{\vartheta(x)\phi(F)\}/dx < 0$. These constraints ensure that the income elasticity of the CDF is a positive, decreasing and bounded function of F, and therefore of x. Table 1 provides a selection of models that can be deduced from Dagum's system for certain specifications of the functions ϑ and ϕ, more extensive versions

Table 1: Dagum's generalized logistic system of income distributions

Distribution	$\vartheta(x)$	$\phi(F)$	(δ, β)	Support
Pareto (I)	α	$(1-F)/F$	$(0,0)$	$0 < x_0 \leq x < \infty$
Fisk	α	$1-F$	$(0,0)$	$0 \leq x < \infty$
Singh-Maddala	α	$\frac{1-(1-F)^{\beta}}{F(1-F)^{-1}}$	$(0,+)$	$0 \leq x < \infty$
Dagum(I)	α	$1-F^{1/\beta}$	$(0,+)$	$0 \leq x < \infty$
Dagum(II)	α	$1-\left(\frac{F-\delta}{1-\delta}\right)^{1/\beta}$	$(+,+)$	$0 \leq x < \infty$
Dagum(III)	α	$1-\left(\frac{F-\delta}{1-\delta}\right)^{1/\beta}$	$(-,+)$	$0 < x_0 \leq x < \infty$

are available in Dagum (1990, 1996). The parameter denoted as α is Pareto's alpha, it depends on the parameters of the underlying distribution and equals a for the Dagum and Fisk distributions and aq in the Singh-Maddala case (see below). The parameter denoted as β also depends on the underlying distribution and equals p in the Dagum case. In addition, signs or values of the parameters β and δ consistent with the constraints of equation (3) are indicated. Among the models specified in Table 1 the Dagum type II and III distributions are mainly used as models of wealth distribution.

Dagum (1983) refers to his system as the *generalized logistic-Burr system*. This is due to the fact that the Dagum distribution with $p = 1$ is also known as the log-logistic distribution (the model Dagum 1975 experimented with). In addition, generalized (log-) logistic distributions arise naturally in Burr's (1942) system of distributions, hence the name. The most widely known Burr distributions are the Burr XII distribution – often just called the Burr distribution, especially in the actuarial literature – with CDF

$$F(x) = 1 - (1 + x^a)^{-q}, \quad x > 0,$$

and the Burr III distribution with CDF

$$F(x) = (1 + x^{-a})^{-p}, \quad x > 0.$$

In economics, these distributions are more widely known, after introduction of an additional scale parameter, as the Singh-Maddala and Dagum distributions. Thus the Dagum distribution is a Burr III distribution with an additional scale parameter and therefore a rediscovery of a distribution that had been known for some 30 years prior to its introduction in economics. However, it is not the only rediscovery of this distribution: Mielke (1973), in a meteorological application, arrives at a three-parameter distribution he calls the kappa distribution. It amounts to the Dagum distribution in a different parametrization. Mielke and Johnson (1974) refer to it as the Beta-K distribution. Even in the income distribution literature there is a parallel development: Fattorini and Lemmi (1979), starting from Mielke's kappa distribution but apparently unaware of Dagum (1977), propose (2) as an income distribution and fit it to several data sets, mostly from Italy.

Not surprisingly, this multi-discovered distribution has been considered in several parameterizations: Mielke (1973) and later Fattorini and Lemmi (1979) use $(\alpha, \beta, \theta) := (1/p, bp^{1/a}, ap)$, whereas Dagum (1977) employs $(\beta, \delta, \lambda) := (p, a, b^a)$. The parametrization used here follows McDonald (1984), because both the Dagum/Burr III and the Singh-Maddala/Burr XII distributions can be nested within a four-parameter generalized beta distribution of the second kind (hereafter: GB2) with density

$$f(x) = \frac{a\, x^{ap-1}}{b^{ap} B(p,q)[1 + (x/b)^a]^{p+q}}, \quad x > 0,$$

where $a, b, p, q > 0$. Specifically, the Singh-Maddala is a GB2 distribution with shape parameter $p = 1$, while the Dagum distribution is a GB2 with $q = 1$ and thus its density is

$$f(x) = \frac{ap\, x^{ap-1}}{b^{ap}[1 + (x/b)^a]^{p+1}}, \quad x > 0. \tag{4}$$

It is also worth noting that the Dagum distribution (D) and the Singh-Maddala distribution (SM) are intimately connected, specifically

$$X \sim D(a, b, p) \quad \Longleftrightarrow \quad \frac{1}{X} \sim SM(a, 1/b, p) \tag{5}$$

This relationship permits to translate several results pertaining to the Singh-Maddala family into corresponding results for the Dagum distributions, it is also the reason for the name *inverse Burr distribution* often found in the actuarial literature for the Dagum distribution (e.g., Panjer 2006).

Dagum (1977, 1980) introduces two further variants of his distribution, hence the previously discussed standard version will be referred to as the Dagum type I distribution in what follows. The Dagum type II distribution has the CDF

$$F(x) = \delta + (1 - \delta)[1 + (x/b)^{-a}]^{-p}, \quad x \geq 0,$$

where as before $a, b, p > 0$ and $\delta \in (0, 1)$. Clearly, this is a mixture of a point mass at the origin with a Dagum (type I) distribution over the positive halfline. The type II distribution was proposed as a model for income distributions with null and negative incomes, but more particularly to fit wealth data, which frequently presents a large number of economic units with null gross assets and with null and negative net assets.

There is also a Dagum type III distribution, like type II defined as

$$F(x) = \delta + (1 - \delta)[1 + (x/b)^{-a}]^{-p},$$

with $a, b, p > 0$. However, here $\delta < 0$. Consequently, the support of this variant is now $[x_0, \infty)$, $x_0 > 0$, where $x_0 = \{b[(1 - 1/a)^{1/p} - 1]\}^{-1/a}$ is determined implicitly from the constraint $F(x) \geq 0$.

As mentioned above, both the Dagum type II and the type III are members of Dagum's generalized logistic-Burr system.

Investigating the relation between the functional and the personal distribution of income, Dagum (1999) also obtained the following bivariate CDF when modeling the joint distribution of human capital and wealth

$$F(x_1, x_2) = (1 + b_1 x_1^{-a_1} + b_2 x_2^{-a_2} + b_3 x_1^{-a_1} x_2^{-a_2})^{-p}, \quad x_i > 0,\ i = 1, 2.$$

If $b_3 = b_1 b_2$,

$$F(x_1, x_2) = (1 + b_1 x_1^{-a_1})^{-p} (1 + b_2 x_2^{-a_2})^{-p},$$

hence the marginals are independent. There do not appear to be any empirical applications of this multivariate Dagum distribution at present.
The remainder of this paper will mainly discuss the Dagum type I distribution.

3 Basic properties

The parameter b of the Dagum distribution is a scale while the remaining two parameters a and p are shape parameters. Nonetheless, these two parameters are not on an equal footing: This is perhaps most transparent from the expression for the distribution of $Y := \log X$, a generalized logistic distribution with PDF

$$f(y) = \frac{ap\, e^{ap(y - \log b)}}{[1 + e^{a(y - \log b)}]^{p+1}}, \quad -\infty < y < \infty.$$

Here, only p is a shape (or skewness) parameter while a and $\log b$ are scale and location parameters, respectively.
Figure 1 illustrates the effect of variations of the shape parameters: for $ap < 1$, the density exhibits a pole at the origin, for $ap = 1$, $0 < f(0) < \infty$, and for $ap > 1$ there exists an interior mode. In the latter case, this mode is at

$$x_{mode} = b \left(\frac{ap - 1}{a + 1} \right)^{1/a}.$$

This built-in flexibility is an attractive feature in that the model can approximate income distributions, which are usually unimodal, and wealth distributions, which are zeromodal. It should be noted that ap and a determine the rate of increase (decrease) from (to) zero for $x \to 0$ ($x \to \infty$), and thus the probability mass in the tails. It should also be emphasized that, in contrast to several popular distributions used to approximate income data, notably the lognormal, gamma and GB2 distributions, the Dagum permits a closed-form expression for the CDF. This is also true of the quantile function,

$$F^{-1}(u) = b[u^{-1/p} - 1]^{-1/a}, \quad \text{for } 0 < u < 1, \tag{6}$$

hence random numbers from a Dagum distribution are easily generated via the inversion method.

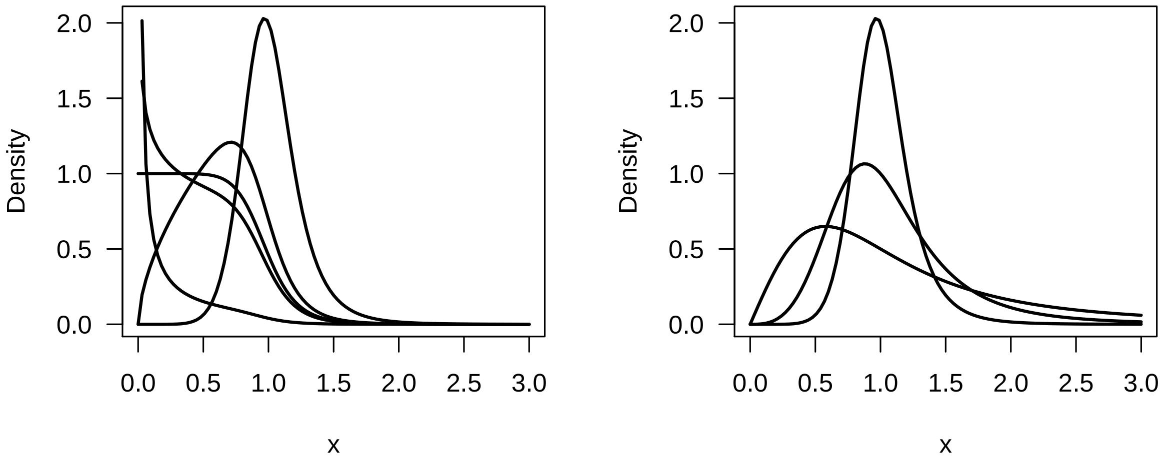

Figure 1: Shapes of Dagum distributions. Left panel: variation of p ($a = 8$, $p = 0.01, 0.1, 0.125, 0.2, 1$, from top left to bottom left). Right panel: variation of a ($p = 1$, $a = 2, 4, 8$, from left to right).

The kth moment exists for $-ap < k < a$ and equals

$$E(X^k) = \frac{b^k B(p + k/a, 1 - k/a)}{B(p, 1)} = \frac{b^k \Gamma(p + k/a)\Gamma(1 - k/a)}{\Gamma(p)}, \tag{7}$$

where $\Gamma()$ and $B()$ denote the gamma and beta functions. Specifically,

$$E(X) = \frac{b\Gamma(p + 1/a)\Gamma(1 - 1/a)}{\Gamma(p)}$$

and

$$Var(X) = \frac{b^2\{\Gamma(p)\Gamma(p + 2/a)\Gamma(1 - 2/a) - \Gamma^2(p + 1/a)\Gamma^2(1 - 1/a)\}}{\Gamma^2(p)}.$$

Moment-ratio diagrams of the Dagum and the closely related Singh-Maddala distributions, presented by Rodriguez (1983) and Tadikamalla (1980) under the names of Burr III and Burr XII distributions, reveal that both models allow for various degrees of positive skewness and leptokurtosis, and even for a considerable degree of negative skewness although this feature does not seem to be of particular interest in applications to income data. (A notable exception is an example of faculty salary distributions presented by Pocock, McDonald and Pope (2003).) Tadikamalla (1980, p. 342) observes “that although the Burr III

[= Dagum] distribution covers all of the region ... as covered by the Burr XII [= Singh-Maddala] distribution and more, much attention has not been paid to this distribution." Kleiber (1996) notes that, ironically, the same has happened independently in the econometrics literature.

An interesting aspect of Dagum's model is that it admits a mixture representation in terms of generalized gamma (GG) and Weibull (Wei) distributions. Recall that the generalized gamma and Weibull distributions have PDFs

$$f_{GG}(x) = \frac{a}{\theta^{ap}\Gamma(p)} x^{ap-1} e^{-(x/\theta)^a}, \quad x > 0,$$

and

$$f_{Wei}(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} e^{-(x/b)^a}, \quad x > 0,$$

respectively. The Dagum distribution can be obtained as a compound generalized gamma distribution whose scale parameter follows an inverse Weibull distribution (i.e., the distribution of $1/X$ for $X \sim Wei(a,b)$), symbolically

$$GG(a,\theta,p) \bigwedge_{\theta} InvWei(a,b) = D(a,b,p).$$

Note that the shape parameters a must be identical. Such representations are useful in proofs (see, e.g., Kleiber 1999), they also admit an interpretation in terms of unobserved heterogeneity.

Further distributional properties are presented in Kleiber and Kotz (2003). In addition, a rather detailed study of the hazard rate is available in Domma (2002).

4 Measuring inequality using Dagum distributions

The most widely used tool for analyzing and visualizing income inequality is the Lorenz curve (Lorenz 1905; see also Kleiber 2008 for a recent survey), and several indices of income inequality are directly related to this curve, most notably the Gini index (Gini, 1914).

Since the quantile function of the Dagum distribution is available in closed form, its normalized integral, the Lorenz curve

$$L(u) = \frac{1}{E(X)} \int_0^u F^{-1}(t)dt, \quad u \in [0,1],$$

is also of a comparatively simple form, namely (Dagum, 1977)

$$L(u) = I_z(p + 1/a, 1 - 1/a), \quad 0 \le u \le 1, \tag{8}$$

where $z = u^{1/p}$ and $I_z(x,y)$ denotes the incomplete beta function ratio. Clearly, the curve exists iff $a > 1$.

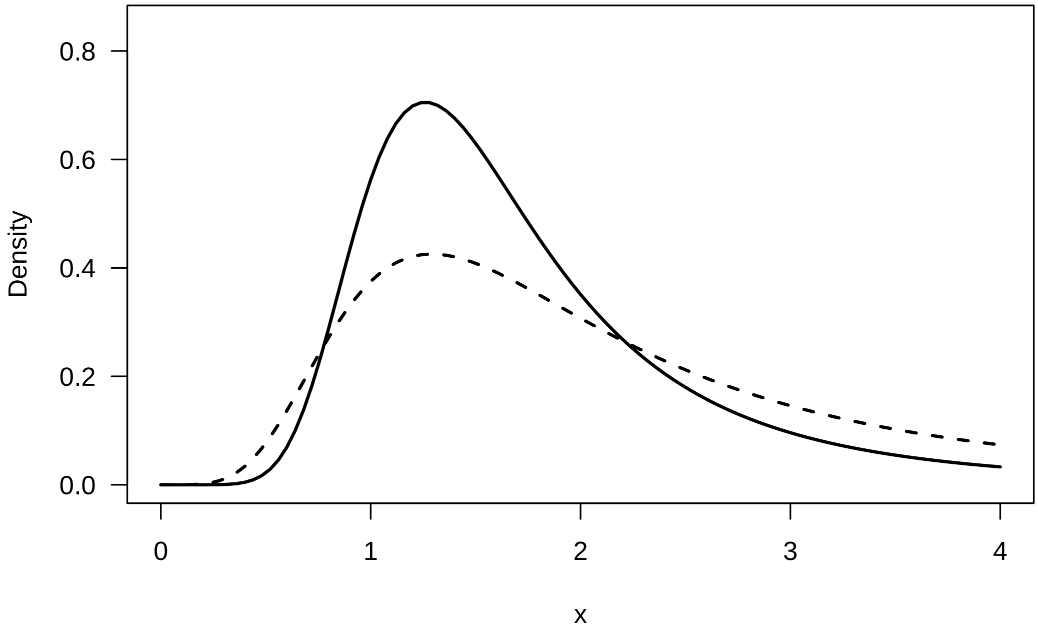

Figure 2: Tails and the Lorenz order for two Dagum distributions: $X_1 \sim D(2,1,3)$ (dashed), $X_2 \sim D(3,1,3)$ (solid), hence $F_1 \geq_L F_2$.

For the comparison of estimated income distributions it is of interest to know the parameter constellations for which Lorenz curves do or do not intersect. The corresponding stochastic order, the Lorenz order, is defined as

$$F_1 \geq_L F_2 \quad \Longleftrightarrow \quad L_1(u) \leq L_2(u) \quad \text{for all } u \in [0,1].$$

First results were obtained by Dancelli (1986) who found that inequality is decreasing to zero (i.e., the curve approaches the diagonal of the unit square) if $a \to \infty$ or $p \to \infty$ and increasing to one if $a \to 1$ or $p \to 0$, respectively, keeping the other parameter fixed. A complete analytical characterization is of more recent date. Suppose $F_i \sim \mathrm{D}(a_i, b_i, p_i)$, $i = 1, 2$. The necessary and sufficient conditions for Lorenz dominance are

$$L_1 \leq L_2 \quad \Longleftrightarrow \quad a_1 p_1 \leq a_2 p_2 \quad \text{and} \quad a_1 \leq a_2. \tag{9}$$

This shows that the less unequal distribution (in the Lorenz sense) always exhibits lighter tails. This was derived by Kleiber (1996) from the corresponding result for the Singh-Maddala distribution using (5), for a different approach see Kleiber (1999). Figure 2 provides an illustration of (9).

Apart from the Lorenz order, stochastic dominance of various degrees has been used when ranking income distributions, hence it is of interest to study conditions on the parameters implying such orderings. A distribution F_1 first-order stochastically dominates F_2, denoted as $F_1 \geq_{FSD} F_2$, iff $F_1 \leq F_2$. This criterion was suggested by Saposnik (1981) as a ranking criterion for income distributions. Klonner (2000) presents necessary as well as sufficient conditions for first-order stochastic dominance within the Dagum family. The conditions $a_1 \geq a_2$, $a_1 p_1 \leq a_2 p_2$ and $b_1 \geq b_2$ are sufficient for $F_1 \geq_{FSD} F_2$, whereas the conditions $a_1 \geq a_2$ and $a_1 p_1 \leq a_2 p_2$ are necessary.
As regards scalar measures of inequality, the most widely used of all such indices, the Gini coefficient, takes the form (Dagum, 1977)

$$G = \frac{\Gamma(p)\Gamma(2p + 1/a)}{\Gamma(2p)\Gamma(p + 1/a)} - 1. \tag{10}$$

For generalized Gini indices see Kleiber and Kotz (2003). From (7), the coefficient of variation (CV) is

$$CV = \sqrt{\frac{\Gamma(p)\Gamma(p + 2/a)\Gamma(1 - 2/a)}{\Gamma^2(p + 1/a)\Gamma^2(1 - 1/a)} - 1}. \tag{11}$$

Recall that the coefficient of variation is a monotonic transformation of a measure contained in the generalized entropy class of inequality measures (e.g., Kleiber and Kotz, 2003). All these measures are functions of the moments and thus easily derived from (7). The resulting expressions are somewhat involved, however, as are expressions for the Atkinson (1970) measures of inequality. Recently, Jenkins (2007) provided formulae for the generalized entropy measures for the more general GB2 distributions, from which the Dagum versions are also easily obtained.
Some 20 years ago, an alternative to the Lorenz curve emerged in the Italian language literature. Like the Lorenz curve the Zenga curve (Zenga, 1984) can be introduced via the first-moment distribution

$$F_{(1)}(x) = \frac{\int_0^x t f(t) dt}{E(X)}, \quad x \geq 0,$$

thus it exists iff $E(X) < \infty$. The Zenga curve is now defined in terms of the quantiles $F^{-1}(u)$ of the income distribution itself and of those of the corresponding first-moment distribution, $F_{(1)}^{-1}(u)$: for

$$Z(u) = \frac{F_{(1)}^{-1}(u) - F^{-1}(u)}{F_{(1)}^{-1}(u)} = 1 - \frac{F^{-1}(u)}{F_{(1)}^{-1}(u)}, \quad 0 < u < 1, \tag{12}$$

the set $\{(u, Z(u)) | u \in (0, 1)\}$ is the Zenga concentration curve. Note that $F_{(1)} \leq F$ implies $F^{-1} \leq F_{(1)}^{-1}$, hence the Zenga curve belongs to the unit square. It follows from (12) that the curve is scale-free.

It is then natural to call a distribution F_2 less concentrated than another distribution F_1 if its Zenga curve is nowhere above the Zenga curve associated with F_1 and thus to define an ordering via

$$F_1 \geq_Z F_2 \qquad :\Longleftrightarrow \qquad Z_1(u) \geq Z_2(u) \text{ for all } u \in (0,1).$$

Zenga ordering within the family of Dagum distributions was studied by Polisicchio (1990) who found that $a_1 \leq a_2$ implies $F_1 \geq_Z F_2$, for a fixed p, and analogously that $p_1 \leq p_2$ implies $F_1 \geq_Z F_2$, for a fixed a. Under these conditions it follows from (9) that the distributions are also Lorenz ordered, specifically $F_1 \geq_L F_2$. Recent work of Kleiber (2007) shows that the conditions for Zenga ordering coincide with those for Lorenz dominance within the class of Dagum distributions.

5 Estimation and inference

Dagum (1977), in a period when individual data were rarely available, minimized

$$\sum_{i=1}^{n} \{F_n(x_i) - [1 + (x_i/b)^{-a}]^{-p}\}^2,$$

a non-linear least-squares criterion based on the distance between the empirical CDF F_n and the CDF of a Dagum approximation. A further regression-type estimator utilizing the elasticity (1) was later considered by Stoppa (1995).
Most researchers nowadays employ maximum likelihood (ML) estimation. Two cases need to be distinguished, grouped data and individual data. Until fairly recently, only grouped data were available, and here the likelihood $L(\theta)$, where $\theta = (a, b, p)^\top$, is a multinomial likelihood with (assuming independent data)

$$L(\theta) = \prod_{j=1}^{m} \{F(x_j) - F(x_{j-1})\}, \qquad x_0 = 0, x_m = \infty.$$

By construction this likelihood is always bounded from above.
In view of the 30th anniversary of Dagum's contribution it seems appropriate to revisit one of his early empirical examples, the US family incomes for the year 1969. The data are given in Dagum (1980, p. 360). Figure 3 plots the corresponding histogram along with a Dagum type I approximation estimated via grouped maximum likelihood. The resulting estimates are $\hat{a} = 4.273$, $\hat{b} = 14.28$ and $\hat{p} = 0.36$, and are in good agreement with the values estimated by Dagum via nonlinear least squares.
With the increasing availability of microdata, likelihood estimation from individual observations attracts increasing attention, and here the situation is more involved: the log-likelihood $\ell(\theta) \equiv \log L(\theta)$ for a complete random sample of size n is

$$\ell(a,b,p) = n \log a + n \log p + (ap-1) \sum_{i=1}^{n} \log x_i - nap \log b - (p+1) \sum_{i=1}^{n} \log\{1 + (x_i/b)^a\} \quad (13)$$

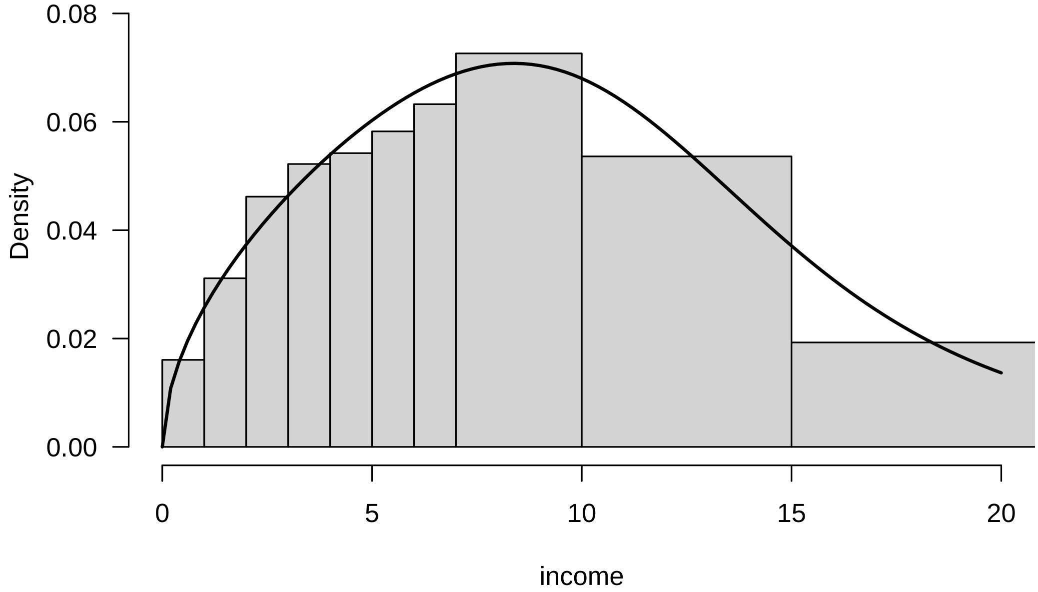

Figure 3: : Dagum distribution fitted to the 1969 US family incomes.

yielding the likelihood equations

$$\frac{n}{a} + p\sum_{i=1}^{n} \log(x_i/b) = (p+1)\sum_{i=1}^{n} \frac{\log(x_i/b)}{1+(b/x_i)^a}, \tag{14}$$

$$np = (p+1)\sum_{i=1}^{n} \frac{1}{1+(b/x_i)^a}, \tag{15}$$

$$\frac{n}{p} + a\sum_{i=1}^{n} \log(x_i/b) = \sum_{i=1}^{n} \log\{1+(x_i/b)^a\} \tag{16}$$

which must be solved numerically. However, likelihood estimation in this family is not without problems: considering the distribution of $\log X$, a generalized logistic distribution, Shao (2002) shows that the MLE may not exist, and if it does not, the so-called embedded model problem occurs. That is, letting certain parameters tend to their boundary values, a distribution with fewer parameters emerges. Implications are that the behavior of the likelihood should be carefully checked in empirical work. It would be interesting to determine to what extent this complication arises in applications to income data where the full flexibility of the Dagum family is not needed.

Apparently unaware of these problems, Domański and Jedrzejczak (1998) provide a simulation study for the performance of the MLEs. It turns out that rather large samples are required until estimates of the shape parameters a, p can be considered as unbiased, while reliable estimation of the scale parameter seems to require even larger samples.
The Fisher information matrix

$$I(\theta) = \left[-E\left(\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j} \right)_{i,j} \right] =: \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

takes the form

$$\begin{aligned}
I_{11} &= \frac{1}{a^2(2+p)} \left[p[\{\psi(p) - \psi(1) - 1\}^2 + \psi'(p) + \psi'(1)] + 2\{\psi(p) - \psi(1)\} \right] \\
I_{21} = I_{12} &= \frac{p - 1 - p\{\psi(p) - \psi(1)\}}{b(2+p)} \\
I_{22} &= \frac{a^2 p}{b^2(2+p)} \\
I_{23} = I_{32} &= \frac{a}{b(1+p)} \\
I_{31} = I_{13} &= \frac{\psi(2) - \psi(p)}{a(1+p)} \\
I_{33} &= \frac{1}{p^2}
\end{aligned}$$

where ψ is the digamma function.
It should be noted that there are several derivations of the Fisher information in the statistical literature, a detailed one using Dagum's parameterization due to Latorre (1988) and a second one due to Zelterman (1987). The latter article considers the distribution of $\log X$, a generalized logistic distribution, using the parameterization $(\theta, \sigma, \alpha) = (\log b, 1/a, p)$.
As regards alternative estimators, an inspection of the scores (14)–(16) reveals that $\sup_x \|\partial\ell/\partial\theta\| = \infty$, where $\|.\|$ stands for the Euclidean norm, thus the score function is unbounded in the Dagum case. This implies that the MLE is rather sensitive to single observations located sufficiently far away from the majority of the data. There appears, therefore, to be some interest in more robust procedures. For a robust approach to the estimation of the Dagum model parameters using an optimal B-robust estimator (OBRE) see Victoria-Feser (1995, 2000).
Income distributions have always been popular with Italian authors, and the Dagum distribution is no exception. Cheli, Lemmi and Spera (1995) study mixtures of Dagum distributions and their estimation via the EM algorithm. Distributions of the sample median and the sample range were obtained by Domma (1997). In addition, Latorre (1988) provides delta-method standard errors for several inequality measures derived from MLEs for the Dagum model.

6 Software

As regards available software, Camilo Dagum started to develop routines for fitting his distributions fairly early. A stand-alone package named "EPID" (Econometric Package for Income Distribution) (Dagum and Chiu, 1991) written in FORTRAN was available from the Time Series Research and Analysis Division of Statistics Canada for some time. The program fitted Dagum type I–III distributions and computed a number of associated statistics such as Lorenz and Zenga curves, the Gini coefficient and various goodness of fit measures. More recently, Jenkins (1999) provided Stata routines for fitting Dagum and Singh-Maddala distributions by (individual) maximum likelihood (current versions are available from the usual repositories), while Jenkins and Jäntti (2005, Appendix) present Stata code for estimating Dagum mixtures. Yee (2006) developed a rather large R (R Development Core Team, 2007) package named VGAM (for "vector generalized additive models") that permits fitting nearly all of the distributions discussed in Kleiber and Kotz (2003) – notably the Dagum type I – conditional on covariates by means of flexible regression methods. The computations for Figure 3 were also carried out in R, Version 2.5.1, but along different lines, namely via modifying the `fitdistr()` function from the MASS package, the package accompanying Venables and Ripley (2002).

7 Applications of Dagum distributions

Although the Dagum distribution was virtually unknown in the major English language economics and econometrics journals until well into the 1990s there are several early applications to income and wealth data, most of which appeared in French, Italian and Latin American publications. Examples include Fattorini and Lemmi (1979) who consider Italian data, Espinguet and Terraza (1983) who study French earnings and Falcão Carneiro (1982) with an application to Portuguese data. Even after 1990 there is a noticable bias towards Romance language contributions. Fairly recent examples include Blayac and Serra (1997), Dagum, Guibbaud-Seyte and Terraza (1995) and Martín Reyes, Fernández Morales and Bárcena Martín (2001).

Table 2 lists selected applications of Dagum distributions to some 30 countries. Only works containing parameter estimates are included. There exist several further studies mainly concerned with goodness of fit that do not provide such information. A recent example is Azzalini, dal Cappello and Kotz (2003) who fit the distribution to the 1997 data for 13 countries from the European Community Household Panel.

Of special interest are papers fitting several distributions to the same data, with an eye on relative performance. From comparative studies such as McDonald and Xu (1995), Bordley, McDonald and Mantrala (1996), Bandourian, McDonald and Turley (2003) and Azzalini, dal Capello and Kotz (2003) it emerges that the Dagum distribution typically outperforms its competitors, apart from the GB2 which has an extra parameter. Bandourian, McDonald and Turley (2003), find that, in a study utilizing 82 data sets, the Dagum is the best 3-parameter model in no less than 84% of the cases. From all these studies it would seem that

Table 2: Selected applications of Dagum distributions

Country	Source
Argentina	Dagum (1977), Botargues and Petrecolla (1997, 1999)
Australia	Bandourian, McDonald and Turley (2003)
Belgium	Bandourian, McDonald and Turley (2003)
Canada	Dagum (1977, 1985), Dagum and Chiu (1991), Bandourian, McDonald and Turley (2003), Chotikapanich and Griffiths (2006)
Czech Republik	Bandourian, McDonald and Turley (2003)
Denmark	Bandourian, McDonald and Turley (2003)
Finland	Bandourian, McDonald and Turley (2003), Jenkins and Jäntti (2005)
France	Espinguet and Terraza (1983), Dagum, Guibbaud-Seyte and Terraza (1995), Bandourian, McDonald and Turley (2003)
Germany	Bandourian, McDonald and Turley (2003)
Hungary	Bandourian, McDonald and Turley (2003)
Ireland	Bandourian, McDonald and Turley (2003)
Israel	Bandourian, McDonald and Turley (2003)
Italy	Fattorini and Lemmi (1979), Dagum and Lemmi (1989), Bandourian, McDonald and Turley (2003)
Mexico	Bandourian, McDonald and Turley (2003)
Netherlands	Bandourian, McDonald and Turley (2003)
Norway	Bandourian, McDonald and Turley (2003)
Philippines	Bantilan et al. (1995)
Poland	Domański and Jedrzejczak (2002), Bandourian, McDonald and Turley (2003), Łukasiewicz and Orłowski (2004)
Portugal	Falcão Carneiro (1982)
Russia	Bandourian, McDonald and Turley (2003)
Slovakia	Bandourian, McDonald and Turley (2003)
Spain	Bandourian, McDonald and Turley (2003)
Sri Lanka	Dagum (1977)
Sweden	Fattorini and Lemmi (1979), Bandourian, McDonald and Turley (2003)
Switzerland	Bandourian, McDonald and Turley (2003)
Taiwan	Bandourian, McDonald and Turley (2003)
United Kingdom	Victoria-Feser (1995, 2000)
USA	Dagum (1977, 1980, 1983), Fattorini and Lemmi (1979), Majumder and Chakravarty (1990), Campano (1991), McDonald and Mantrala (1995), McDonald and Xu (1995), Bandourian, McDonald and Turley (2003)

empirically relevant values of the Dagum shape parameters are $a \in [2, 7]$ and $p \in [0.1, 1]$, approximately. Hence the implied income distributions are heavy-tailed admitting moments $E(X^k)$ for $k \leq 7$ while negative moments may exist up to order 7 in some examples.
For reasons currently not fully understood, the Dagum often provides a better fit to income data than the closely related Singh-Maddala distribution. Kleiber (1996) provides a heuristic explanation arguing that in the Dagum case the upper tail is determined by the parameter a while the lower tail is governed by the product ap, for the Singh-Maddala distribution the situation is reversed. Thus the Dagum distibution has one extra parameter in the region where the majority of the data are, an aspect that may to some extent explain the excellent fit of this model.
The previously mentioned works typically consider large populations, say households of particular countries. In an interesting contribution, Pocock, McDonald and Pope (2003) estimate salary distributions for different professions (specifically, the salaries of statistics professors at different levels) from sparse data utilizing the Dagum distribution (under the name of Burr III). This is of interest for competitive salary offers as well as for determining financial incentives for retaining valued employees. One of the few applications to wealth data, and at the same time one of the few applications of the Dagum type III distributions, is provided by Jenkins and Jäntti (2005) who estimate mixtures of Dagum distributions using wealth data for Finland.
Researchers have also begun to model conditional distributions in a regression framework, recent examples are Biewen and Jenkins (2005) and Quintano and D'Agostino (2006).
During the last decade, Camilo Dagum furthermore attempted to obtain information on the distribution of human capital, an example utilizing US data is Dagum and Slottje (2000) while the paper by Martín Reyes, Fernández Morales and Bárcena Martín (2001) mentioned above considers Spanish data.
In addition to all these empirical applications, the excellent fit provided by the distribution has also led to an increasing use in simulation studies. Recent examples include Hasegawa and Kozumi (2003), who consider Bayesian estimation of Lorenz curves, and Cowell and Victoria-Feser (2006), who study the effects of trimming on distributional dominance, both groups of authors utilize Dagum samples for illustrations. Also, Palmitesta, Provasi and Spera (1999, 2000) investigate improved finite-sample confidence intervals for inequality measures using Gram-Charlier series and bootstrap methods, respectively. Their methods are illustrated using Dagum samples. There even exist occasional illustrations in economic theory such as Glomm and Ravikumar (1998). Finally, there are numerous applications of this multi-discovered distribution in many fields of science and engineering (typically under the name of Burr III distribution), a fairly recent example from geophysics explicitly citing Dagum (1977) is Clark, Cox and Laslett (1999).

8 Concluding remarks

This Chapter has provided a brief introduction to the Dagum distributions and their applications in economics. Given that the distribution only began to appear in the English-language literature in the 1990s, it is safe to predict that there will be many further applications. On the methodological side, there are still some unresolved issues including aspects of likelihood inference. When the distribution celebrates its golden jubilee in economics, these problems no doubt will be solved.

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