ASSET PRICING IN HETEROGENEOUS ECONOMIES I. WEAK HETEROGENEITY*

YVAN LENGWILER† SEMYON MALAMUD‡
EUGENE TRUBOWITZ§

January 24, 2006

Abstract

We extend the Lucas asset pricing tree economy to a heterogeneous population. Perturbative methods are applied to explicitly calculate the second order response of asset returns to heterogeneity. We discover that there exists a unique "best homogeneous approximation" to a weakly heterogeneous economy. We determine the status of various stylized facts. For example, we find that the equity premium always varies counter cyclically, the equity returns are always predictable and price dividend ratios are positively autocorrelated. Moreover, sufficiently positive correlation between risk aversion and patience increases the risk premium and decreases the interest rate, thus giving another perspective on the equity premium and the risk-free rate puzzles. This motivates us to make a concrete social prediction. Finally, there is an open region of weakly heterogeneous economies where several other stylized facts hold. *JEL classifications: D91, E43, G12.*

Keywords: heterogeneity, perturbation theory, equity premium puzzle, stylized facts.

*The authors thank Rajnish Mehra and Leonid Kogan for useful comments and remarks

†University of Basel, Dept. of Economics (WWZ), Petersgraben 51, CH–4003 Basel, Switzerland (yvan.lengwiler@unibas.ch)

‡ETHZ, D-MATH, HG G 33.1, Rämistrasse 101, CH–8092 Zürich, Switzerland (semka@math.ethz.ch)

§ETHZ, D-MATH, HG G 34.1 Rämistrasse 101, CH–8092 Zürich, Switzerland (trub@math.ethz.ch).

CONTENTS

1 INTRODUCTION

We extend the classical model of Lucas (1978) to accommodate heterogeneous consumers. In some respects, our model is quite simple. It is assumed that all agents have constant relative risk aversion (CRRA), discount exponentially, and that markets are complete. In other respects, the model is more elaborate, agents may simultaneously differ with respect to their coefficients of risk aversion, their discount factors, and they are also subject to idiosyncratic endowment risk. Moreover, we allow for an infinite horizon. See, Appendix A for a complete and detailed formulation of the model that we investigate.

This is the first of two papers devoted to the analysis of this class of models. In this paper we analyze the case of small (weak) heterogeneity. In the second paper Malamud and Trubowitz (2006), we give a complete description of the long (infinite) horizon behavior of asset returns in the presence of strong heterogeneity. Lengwiler, Malamud, and Trubowitz (2005) contains both parts of analysis. All proofs and detailed derivations of all mathematical results are also contained in Lengwiler, Malamud, and Trubowitz (2005).

We have developed, in collaboration with W. P. Petersen (see, Malamud, Petersen, and Trubowitz (2006)), a refined code that evaluates the economic indicators for strongly heterogeneous economies at intermediate horizon. The weakly heterogeneous expansions and infinite horizon asymptotics are essential for determining the validity of the code. Precisely, the code reproduces to high precision the predictions of the weakly heterogeneous expansions and infinite horizon asymptotics.

There are several natural ways to introduce heterogeneity. For example, heterogeneous information (Ross (1989), Wang (1993), Wang (1994)), heterogeneous uninsurable income processes (see Constantinides and Duffie (1996), Brav, Constantinides, and Geczy (2002)) and heterogeneous preferences. In this paper we consider only heterogeneous preferences.

A special case of the model we analyze is considered by Wang (1996) in a continuous time setting. He considered the case of not more than four consumers with identical discount factors and very special choices of risk aversion and studied Pareto efficient allocations. He also obtained an interesting result on the infinite horizon end of the yield curve. Here, by contrast, we study an arbitrary number of agents with arbitrary (but weak) heterogeneity and determine the behavior of the yield curve.

Kogan, Makarov, and Uppal (2003) analyze an economy with two agents who have different risk aversion and face borrowing constraints. They show that their economy exhibits simultaneously a high Sharpe ratio for the equity and a low risk-free rate.

Dumas (1989) made numerical simulations of a two agent production economy. Lengwiler (2005) considers an economy with heterogeneous discount factors. Benninga and Mayshar (2000) consider a one-period economy with heterogeneous risk aversions and discount factors and show that the conventional representative agent has a decreasing risk-aversion. They study implications of this phenomenon for option pricing. Gollier and Zeckhauser (2005) analyze Pareto-efficient allocations in heterogeneous economies with general utility functions and discover some interesting aggregation phenomena. There are also aggregation results (see Rubinstein (1974), Mehra and Prescott (1980) and Constantinides (1982)) justifying the conventional representative agent approach.

In this paper, perturbative methods are applied to calculate the second order response of prices to heterogeneity and to determine the status of various stylized facts in weakly heterogeneous economies. Of course, the model generates, among other economic indicators, equity returns, price dividend ratios, and long term risk free rates.

Perturbation theory was used by Kogan and Uppal (2001) to analyze weakly heterogeneous economies. In a subsequent paper Chan and Kogan (2002) introduced a model with "catching up with the Joneses" preferences and identical discount factors. They used perturbative methods to identify the effects of heterogeneous risk aversion. But they (just as Wang (1996)) considered a social planner problem and hence did not consider the dependence on individual endowment processes. They perturbed around the homogeneous economy with logarithmic preferences and discovered that heterogeneity has unexpected consequences. They showed that heterogeneous risk aversion, together with habit formation, is able to explain several empirically observed properties of stock returns such as negative autocorrelation and negative correlation of returns and volatility changes.

The main point of this paper is that joint heterogeneity in discount factors and risk aversion gives a much richer and more complete picture than heterogeneous risk aversion alone. We show that in our model the cross-sectional variance of risk aversion by itself simultaneously decreases the return on equity and the risk-free rate and therefore is unable to give any insight into the equity premium / risk free rate puzzle. Indeed, we are able to determine the status of many well known stylized facts. The moral is, "one needs more heterogeneity". All of the "economic insights" into the relationship between standard stylized facts and heterogeneity that we have extracted from weakly heterogeneous models are distilled in Theorem 10.4 and Remarks 11.1 - 11.12. In the essentially self contained Theorem 11.13 it is shown that there is an explicitly given open set of weakly heterogeneous economies for which the "real" economic behavior (stylized facts) of equity returns, price dividend ratios, and risk free rates, coincides with the behavior of those generated by the model.

It turns out that there is a concrete region in the space of discount factors and risk aversions, such that the effect of joint heterogeneity simultaneously increases the equity premium and decreases the risk-free rate. This result suggests the social prediction that real world risk aversions and discount factors lie in this region. In particular, risk aversions and discount factors are sufficiently positively correlated (see, Remark 11.4)! Interestingly enough, there are social experiments (Van Praag and Booji, 2003) that indicate a strong positive correlation. So, we invert the equity premium puzzle: the asset returns are used to predict the nature of heterogeneity in the real world.

A prerequisite for any discussion of asset pricing in heterogeneous economies is the existence and uniqueness of heterogeneous equilibria. There is a large, abstract literature that provides various sufficient conditions for the existence and uniqueness of equilibria in the presence of infinite state spaces (see e.g. Mas-Colell and Zame (1991), Dana (1993a), Dana (1993b), Karatzas, Lehoczky, and Shreve (1990) and references therein). In Lengwiler, Malamud, and Trubowitz (2005) we will exploit the special structure of constant relative risk aversion preferences to obtain a *necessary and sufficient* condition for the existence of equilibria.

Precisely, our economy has n different agents. Agent $i = 1,\ldots,n$ has a utility function

$$E\left[\sum_{t=0}^{\infty} \delta_i^t \frac{c_t^{1-\gamma_i}-1}{1-\gamma_i}\right] \tag{2.1}$$

and an endowment process w_{it}, $t = 0,\cdots,\infty$. Here, $0 < \delta_i < 1$ is the patience, or discount factor, of agent i and $\gamma_i > 0$ is his/her relative risk aversion. Also, c_t is the amount of the single nondurable good consumed by agent i at time t.

THEOREM 2.1 *Let γ_i, δ_i and w_{it}, $t \geqslant 0$, be the risk aversion, patience/discount factor and individual endowment process of agent $i = 1,\cdots,n$. Let*

$$W_t = \sum_{i=1}^{n} w_{it}$$

be the aggregate endowment process. An equilibrium exists if and only if the aggregate endowment does not yield infinite utility for any agent. That is,

$$E\left[\sum_{t=0}^{\infty} \delta_i^t W_t^{1-\gamma_i}\right] < \infty \tag{2.2}$$

for all $i = 1,\cdots,n$. Remarkably, (2.2) *is equivalent to the condition that the price of equity is always finite.*

In general, there may be more than one equilibrium[1]. In Lengwiler, Malamud, and Trubowitz (2005) we show that for weakly heterogeneous economies the equilibrium is always unique and smooth. More precisely,

THEOREM 2.2 *If condition* (2.2) *holds for all (γ_1,δ_1) in a small neighborhood $\mathscr{B}$ of (γ,δ) then the equilibrium is unique for any choice of $(\gamma_1,\delta_1),\cdots,(\gamma_n,\delta_n) \in \mathscr{B}$ and the equity and bond prices are infinitely differentiable functions of risk aversion and patience.*

This result is a necessary prerequisite for using Taylor expansions to analyze asset returns.

[1]See, Lengwiler, Malamud, and Trubowitz (2005) for a concrete example of an economy with multiple equilibria

We calculate the response of prices to weak heterogeneity. Precisely, we start with a homogeneous economy with parameters (γ,δ) and depart from it in a direction $(\Gamma,\Delta)\in\mathbb{R}^{2n}$. In other words, we study the weakly heterogeneous economy given by

$$(\gamma_i,\delta_i) = (\gamma+\varepsilon\Gamma_i,\delta+\varepsilon\Delta_i), \qquad i=1,\ldots,n. \tag{3.1}$$

and initial heterogeneous individual endowments. The complete details of these calculations can be found in Lengwiler, Malamud, and Trubowitz (2005).

An essential prerequisite is to prove *smoothness and global uniqueness* of equilibrium prices in the weakly heterogeneous case [2]. This result is very important. It is the basis of our perturbation calculations. It allows us to compute first and second order effects of weak heterogeneity and to obtain an estimate for the error term. See, Lengwiler, Malamud, and Trubowitz (2005).

The equilibrium state-price densities (SPD) M_t, $t=0,\ldots,\infty$, are expanded in power series in ε (see, (A.1) for the definition of the SPD). We compute the first three terms. They enable us to estimate how the different kinds of heterogeneity interact and quantitatively effect the state price densities (see, Lengwiler, Malamud, and Trubowitz (2005), Theorem E.1).

We introduce a measure η on the set of all agents. Concretely, we put the weight η_i on the agent i, equal to

$$\eta_i := \frac{E\left[\sum_{t\in\mathbf{T}}\delta^t\,W_t^{-\gamma}\,w_{i\,t}\right]}{E\left[\sum_{t\in\mathbf{T}}\delta^t\,W_t^{1-\gamma}\right]} \tag{3.2}$$

where,

$$W_t = \sum_{i=1}^{n} w_{i\,t}$$

is the aggregate endowment process. By construction, η_i is the fraction of the aggregate, inter temporal wealth belonging to agent i in the homogeneous economy (γ,δ)[3]. Expectations with respect to this measure are denoted by $\mathscr{E}$ and are referred to as wealth weighted averages. For example,

$$\mathscr{E}(\Delta) = \sum_{i=1}^{n}\eta_i\,\Delta_i. \tag{3.3}$$

[2]In a separate paper, we also study several concrete examples of economies that do exhibit more than one equilibria. It is quite difficult to generate multiple equilibria with reasonable choices of parameters if one stays within the additively separable CRRA framework. This at least tentatively suggests that multiplicity might not be such a great problem after all for practical applications. We do not know, however, if multiplicity is possible with a reasonable calibration of the model if we allow for many agents and aggregate uncertainty. See, Lengwiler, Malamud, and Trubowitz (2005) for an example of non uniqueness

[3]This measure is similar to that, introduced by Kogan and Uppal (2001)

Observe that the dependence of wealth weighted averages $\mathscr{E}$ on the homogeneous economy (γ, δ) and the individual endowments has been consciously suppressed.

We also introduce (see, Lengwiler, Malamud, and Trubowitz (2005), formula (E.3)) the quantities ζ_i and ξ_i, $i = 1, \ldots, n$. The quantity ζ_i measures correlation between the aggregate endowment process and the endowment process of agent i. The quantity ξ_i measures the growth rate of agent i's wealth. The individual endowments appear in the perturbation expansion of the state price densities (up to second order in ε) only through the vectors

$$
\begin{aligned}
\eta &:= (\eta_1, \ldots, \eta_n) \\
\zeta &:= (\zeta_1, \ldots, \zeta_n) \\
\xi &:= (\xi_1, \ldots, \xi_n)
\end{aligned}
$$

Set

$$
\mathrm{g}_t(s) := \left(\frac{W_t(s)}{W_0}\right)^{1/t}
$$

and let

$$
\rho_t(s) := \log \mathrm{g}_t(s) = t^{-1} \log\left(\frac{W_t(s)}{W_0}\right) \tag{3.4}
$$

be the normalized growth rate of the aggregate endowment. The perturbation expansion for the state-price densities (SPD) is written as

$$
M_t(s, \varepsilon) = \delta^t \, e^{-t\gamma\rho_t(s)} \left[1 + \varepsilon \, t \, M_t^{(1)}(s) + \varepsilon^2 \, t \, M_t^{(2)}(s) \right] + O(\varepsilon^3) \tag{3.5}
$$

where

$$
\begin{aligned}
M_t^{(1)}(s) &:= \delta^{-1}\mathscr{E}(\Delta) - \mathscr{E}(\Gamma)\,\rho_t(s), \\
M_t^{(2)}(s) &:= Y_1(t)\,\mathrm{var}_\eta(\Delta) + Y_2(t,s)\,\mathrm{var}_\eta(\Gamma) + Y_3(t,s)\,\mathrm{cov}_\eta(\Gamma,\Delta) \\
&\quad + Y_4(t,s)\,\mathscr{E}(\Delta)\,\mathrm{cov}_\eta(\Gamma,\xi) + Y_5(t,s)\,\mathscr{E}(\Gamma)\,\mathrm{cov}_\eta(\Gamma,\zeta) \\
&\quad + Y_6\,\mathscr{E}(\Delta)\,\mathrm{cov}_\eta(\Delta,\xi) + Y_7\,\mathscr{E}(\Gamma)\,\mathrm{cov}_\eta(\Delta,\zeta) + H
\end{aligned} \tag{3.6}
$$

and the coefficients $Y_1, \ldots, Y_7$ are given in Lengwiler, Malamud, and Trubowitz (2005), Theorem E.1. Here the variances, covariances and expectations are with respect to the wealth weighted average η. The coefficients $Y_1, \ldots, Y_7$ do not depend on Γ or Δ. As the notation indicates, the coefficient Y_1 depends only on t and Y_6, Y_7 do not depend on either s or t. The last term H is the second order response to a perturbation in the homogeneous direction $(\mathscr{E}(\Gamma), \mathscr{E}(\Delta))$. It is a quadratic polynomial in $\mathscr{E}(\Gamma)$ and $\mathscr{E}(\Delta)$ that vanishes when $\mathscr{E}(\Gamma) = \mathscr{E}(\Delta) = 0$. Why display such an apparently complicated formula? For a very good reason. Recall, that the state price densities for n agents with identical risk aversions and discount factors (i.e., a homogeneous economy) are independent of the individual endowments. Indeed, for co-linear individual endowments the equilibrium separates into n completely uncoupled single agent economies. The expression above makes explicit the interaction, or coupling, to second order, between the heterogeneous agents that is hidden in the abstract concept of an equilibrium. The variances and covariances appearing in (3.6) are explicit realizations of the inter agent interactions. The coefficients $Y_1, \ldots, Y_7$ are "coupling constants" that exhibit the sign and strength of these interactions. Loosely speaking, one can see with the naked eye how heterogeneous agents "conspire" to keep the economy in equilibrium.

To interpret and apply (3.5), it is essential to remember that the error term $O(\varepsilon^3)$ depends on the period t and on the state s. The error depends on t through the infimum of W_t. It is uniform in the state s when the infimum is strictly bigger than zero. We typically fix a horizon T and make ε so small that the error $O(\varepsilon^3)$ is uniformly small for all t between 0 and T. The error term becomes $O(K^t \varepsilon^3)$, when the aggregate endowment is a geometric random walk. If K is bigger than 1, notice that the exponentially large K^t must be balanced by the polynomially small ε^3. The constant K is strictly bigger than 1 when the probability of recession is positive.

Observe that the zeroth order term $\delta^t \, e^{-t\gamma\rho_t(s)}$ appearing in (3.5) is the state price density of the homogeneous economy with parameters (γ, δ). It is also crucial to observe that the first order term is the response of the homogeneous system to a perturbation in the *homogeneous* direction

$$(\bar{\gamma}, \bar{\delta}) = (\mathscr{E}(\Gamma), \mathscr{E}(\Delta))$$

and therefore gives no insight into heterogeneity. We will show (see, Proposition 8.6) that for any weakly heterogeneous economy there exists a unique "best homogeneous approximation" (see, Definition 8.3) for which $\mathscr{E}(\Delta)$ and $\mathscr{E}(\Gamma)$ are zero. In this case,

$$M_t(s, \varepsilon) \;=\; \delta^t \, e^{-t\gamma\rho_t(s)} \left[\, 1 \;+\; \varepsilon^2 \, t \, M_t^{(2)}(s) \,\right] \;+\; O(\varepsilon^3) \tag{3.7}$$

where

$$M_t^{(2)}(s) := Y_1(t) \, \mathrm{var}_\eta(\Delta) \;+\; Y_2(t,s) \, \mathrm{var}_\eta(\Gamma) \;+\; Y_3(t,s) \, \mathrm{cov}_\eta(\Gamma, \Delta) \tag{3.8}$$

A great deal of information can be extracted from the second order term (3.6). As a first rough indication, consider the effect of "the interaction" $\mathrm{cov}_\eta(\Gamma, \zeta)$. Intuitively, more risk averse people (large γ_i) will make "choices" that leave them less exposed to aggregate risk (small ζ_i). For this reason, one would expect $\mathrm{cov}_\eta(\Gamma, \zeta) < 0$. If $\mathscr{E}(\Gamma) > 0$ and $\mathrm{cov}_\eta(\Gamma, \zeta) < 0$, then it follows directly from (3.6) and Lengwiler, Malamud, and Trubowitz (2005), Theorem E.1 that the SPD decreases in good states (i.e. with high $\rho_t(s)$) and increases in bad states (i.e. with low $\rho_t(s)$), generating a larger market risk premium.

4 ECONOMIC INDICATORS

In this paper we analyze the Lucas tree asset, whose dividend process coincides with the aggregate endowment. From now on, the Lucas tree asset is referred to as equity. We also analyze the return on general risk free bonds. That is, the interest rate term structure.

Fix an infinite filtration. The price P_t of equity at time $t \geqslant 0$, relative to the filtration is given by

$$P_t \; = \; E_t \left[\sum_{\tau=1}^{\infty} \frac{M_{t+\tau}}{M_t} \, W_{t+\tau} \right] \tag{4.1}$$

Similarly, the price $\beta^{\mathrm{F}}(t_1, t_2)$ at time t_1 of the risk free bond that matures at time t_2 is given by

$$\beta^{\mathrm{F}}(t_1, t_2) \; = \; E_{t_1} \left[\frac{M_{t_2}}{M_{t_1}} \right] \tag{4.2}$$

For each period $t \geqslant 1$ we define the random variable

$$\mathrm{r}^{\mathrm{E}}_t := \frac{P_t + W_t}{P_{t-1}} \; = \; \frac{P_t \, W_t^{-1} + 1}{P_{t-1} \, W_{t-1}^{-1}} \, \frac{W_t}{W_{t-1}} \tag{4.3}$$

to be the return on equity. The quotient

$$P_t \, W_t^{-1}$$

is the price dividend ratio.

For all $t_1 < t_2$ let

$$\mathrm{r}^{\mathrm{F}}(t_1, t_2) := \left(\beta^{\mathrm{F}}(t_1, t_2)\right)^{-1}. \tag{4.4}$$

be the return on the risk free bond. We also introduce the final wealth, or cumulative return,

$$\mathrm{r}^{\mathrm{E}}(t_1, t_2) := \mathrm{r}^{\mathrm{E}}_{t_1+1} \, \mathrm{r}^{\mathrm{E}}_{t_1+2} \cdots \mathrm{r}^{\mathrm{E}}_{t_2} \tag{4.5}$$

at time $t_2 > t_1$. It is the return on reinvesting all dividends in equity at each period during the interval $[t_1, t_2 - 1]$. One can invest solely in equity, then only in risk free bonds, then again only in equity and so on. Concretely, let $t_1 < t_2 < \ldots < t_k$ be a sequence of times and let $\mathrm{I}_j \in \{\mathrm{E}, \mathrm{F}\}$, $j = 1, \ldots, k-1$. Set,

$$C(t_1, \mathrm{I}_1 \ldots, t_{k-1}, \mathrm{I}_{k-1}, t_k) := E_{t_1} \left[\mathrm{r}^{\mathrm{I}_1}(t_1, t_2) \, \mathrm{r}^{\mathrm{I}_2}(t_2, t_3) \cdots \mathrm{r}^{\mathrm{I}_{k-1}}(t_{k-1}, t_k) \right]. \tag{4.6}$$

These expectations contain information about pairwise correlations of equity returns and interest rates.

Many important empirically observed quantities can be defined in terms of the correlations introduced in the last paragraph. For example, the expected cumulative return

$$\mathrm{R}^{\mathrm{E}}(t_1, t_2) := E_{t_1} \left[\mathrm{r}^{\mathrm{E}}(t_1, t_2) \right] \tag{4.7}$$

is the extreme case of investment solely in equity. Similarly,

$$\mathrm{R}^{\mathrm{F}}(t_1, t_2) := E_{t_1}\left[\mathrm{r}^{\mathrm{F}}(t_1, t_1+1)\,\mathrm{r}^{\mathrm{F}}(t_1+1, t_1+2)\cdots\mathrm{r}^{\mathrm{F}}(t_2-1, t_2)\right]. \tag{4.8}$$

is the opposite extreme in which the only investment is in one period risk free bonds. Observe that

$$\mathrm{r}^{\mathrm{F}}(t_1, t_2) = E_{t_1}[\mathrm{r}^{\mathrm{F}}(t_1, t_2)]$$

is also a correlation of the same form.

The returns $\mathrm{R}^{\mathrm{E}}(t_1, t_2)$, $\mathrm{R}^{\mathrm{F}}(t_1, t_2)$ and $\mathrm{r}^{\mathrm{F}}(t_1, t_2)$ grow exponentially when the interval $\tau = t_2 - t_1$ becomes large. It is therefore natural to consider the corresponding normalized growth rates

$$\tau^{-1}\log\mathrm{R}^{\mathrm{E}}(t_1, t_2) \qquad \tau^{-1}\log\mathrm{R}^{\mathrm{F}}(t_1, t_2) \qquad \tau^{-1}\log\mathrm{r}^{\mathrm{F}}(t_1, t_2) \tag{4.9}$$

We shall work exclusively with normalized growth rates. Economists refer to the sequence

$$\tau^{-1}\log\mathrm{r}^{\mathrm{F}}(t, t+\tau)\ ,\ \tau = 1, 2, \cdots$$

as the yield curve at time t. By definition, the difference

$$\tau^{-1}\log\mathrm{R}^{\mathrm{E}}(t_1, t_2) - \tau^{-1}\log\mathrm{R}^{\mathrm{F}}(t_1, t_2) = \tau^{-1}\log\frac{\mathrm{R}^{\mathrm{E}}(t_1, t_2)}{\mathrm{R}^{\mathrm{F}}(t_1, t_2)} \tag{4.10}$$

is the cumulative log equity premium relative to short term bonds. Similarly, the cumulative log equity premium relative to long maturity bonds is

$$\tau^{-1}\log\mathrm{R}^{\mathrm{E}}(t_1, t_2) - \tau^{-1}\log\mathrm{r}^{\mathrm{F}}(t_1, t_2) = \tau^{-1}\log\frac{\mathrm{R}^{\mathrm{E}}(t_1, t_2)}{\mathrm{r}^{\mathrm{F}}(t_1, t_2)} \tag{4.11}$$

These "economic indicators" are pertinent for a discussion of the so called "equity premium / risk free rate puzzle" and other stylized facts.

5 STYLIZED FACTS

It is essential to "test" our model, to understand its strengths and limitations. This is a prerequisite for any attempt to isolate the social mechanisms responsible for the observed behavior of returns. For this purpose, we shall determine the status of a series of important "stylized facts" (see, Campbell (2003), Campbell and Cochrane (1999), Duffee (forth.)). A stylized fact is simply an observed property of real market data.

Here is a collection of eleven representative stylized facts with mathematical interpretations.

(F1) Risk free rates

$$\tau^{-1} \log \mathrm{R}^{\mathrm{F}}(t_1, t_2) \quad , \quad \tau^{-1} \log \mathrm{r}^{\mathrm{F}}(t_1, t_2)$$

are "rather small" for sufficiently large τ. A related "puzzle" is that risk free rates are "much smaller" than the risk free rates of the "standard" homogeneous economy (see, Remark 7.8).

We will interpret the "risk free rate puzzle", relative to a fixed homogeneous economy (γ, δ), as the concrete problem of determining all heterogeneous economies, "well approximated by the homogeneous economy" (γ, δ) with risk free rates that are smaller than those of (γ, δ) for sufficiently large τ. This gives us insight into the relationship between heterogeneity and risk free rates.

(F2) Per-period returns on long term bonds are "on average larger" than the return on short term bonds[4].

We interpret this "fact" as the statement that the normalized log term premium

$$\tau^{-1} \log \mathrm{r}^{\mathrm{F}}(t_1, t_2) - \tau^{-1} \log \mathrm{R}^{\mathrm{F}}(t_1, t_2)$$

is positive for all sufficiently large τ.

(F3) Per period equity returns are "much larger" than the per period returns on risk free bonds. In fact, the equity premia

$$\tau^{-1} \log \frac{\mathrm{R}^{\mathrm{E}}(t_1, t_2)}{\mathrm{R}^{\mathrm{F}}(t_1, t_2)} \quad , \quad \tau^{-1} \log \frac{\mathrm{R}^{\mathrm{E}}(t_1, t_2)}{\mathrm{r}^{\mathrm{F}}(t_1, t_2)}$$

are "surprisingly large" for sufficiently large τ. A related "puzzle" is that equity premia are "much larger" than the equity premia of the "standard" homogeneous economy.

We will interpret the "equity premium puzzle" in the same manner as the "risk free rate puzzle".

(F4) The equity premium relative to short term bonds varies counter cyclically.

We interpret this "fact" in the idealized technical sense that

$$\operatorname{cov}\left(\log \frac{\mathrm{R}^{\mathrm{E}}(t, t+1)}{\mathrm{R}^{\mathrm{F}}(t, t+1)}, \log W_t \right) < 0$$

for all sufficiently large time periods t.

(F5) The price dividend ratio varies pro cyclically.

That is,

$$\operatorname{cov}\big(\log (P_t W_t^{-1}), \log W_t \big) > 0$$

for all sufficiently large t.

[4] the only exception is Italy, where the term premium is negative

(F6) Price dividend ratios are positively autocorrelated.
That is,

$$\operatorname{cov}\bigl(\log(P_t\,W_t^{-1})\,,\,\log(P_{t+1}\,W_{t+1}^{-1})\bigr)\;>\;0$$

for all sufficiently large t.

(F7) Price dividend ratios and equity returns are negatively correlated.
That is,

$$\operatorname{cov}\bigl(\log(P_t\,W_t^{-1})\,,\,\log \mathrm{R}^{\mathrm{E}}(t+j\,,\,t+j+1)\bigr)\;<\;0$$

for $j = 0, \ldots, 6$ and all sufficiently large t.

(F8) Equity returns are negatively autocorrelated.
That is[5],

$$\operatorname{cov}\bigl(\mathrm{r}^{\mathrm{E}}(t_1,\,t_2)\,,\,\mathrm{r}^{\mathrm{E}}(t_2,\,t_3)\bigr)\;<\;0$$

for all sufficiently large t.

(F9) The conditional variance of equity returns at time t

$$\operatorname{var}_t(\mathrm{r}^{\mathrm{E}}_{t+1})\;:=\;E_t\left[\,(\mathrm{r}^{\mathrm{E}}_{t+1})^2\,\right]-\bigl(E_t\left[\,\mathrm{r}^{\mathrm{E}}_{t+1}\,\right]\bigr)^2$$

varies counter cyclically.
That is,

$$\operatorname{cov}\bigl(\operatorname{var}_t(\mathrm{r}^{\mathrm{E}}_{t+1})\,,\,\log W_t\bigr)<0$$

for all sufficiently large t.

(F10) The conditional correlation of equity returns with the consumption growth

$$\operatorname{corr}_t(\mathrm{r}^{\mathrm{E}}_{t+1}\,,\,W_{t+1}\,W_t^{-1})\;:=\;\frac{E_t\left[\,\mathrm{r}^{\mathrm{E}}_{t+1}\,W_{t+1}\,W_t^{-1}\,\right]-E_t\left[\,\mathrm{r}^{\mathrm{E}}_{t+1}\,\right]E_t\left[\,W_{t+1}\,W_t^{-1}\,\right]}{\Bigl(E_t\left[\,(\mathrm{r}^{\mathrm{E}}_{t+1})^2\,\right]-\bigl(E_t\left[\,\mathrm{r}^{\mathrm{E}}_{t+1}\,\right]\bigr)^2\Bigr)^{1/2}}$$

varies procyclically.
That is,

$$\operatorname{cov}\bigl(\operatorname{corr}_t(\mathrm{r}^{\mathrm{E}}_{t+1}\,,\,W_{t+1}\,W_t^{-1})\,,\,\log W_t\bigr)\;>\;0$$

for all sufficiently large t.

[5]Observe that

$$\operatorname{cov}(\mathrm{r}^{\mathrm{E}}(t_1,\,t_2),\,\mathrm{r}^{\mathrm{E}}(t_2,\,t_3))\;=\;E[\mathrm{R}^{\mathrm{E}}(t_1,\,t_3)]\;-\;E[\mathrm{R}^{\mathrm{E}}(t_1,\,t_2)]\cdot E[\mathrm{R}^{\mathrm{E}}(t_2,\,t_3)]$$

which again confirms the statement made above that many important quantities can be written in terms of the basic correlations.

(F11) The changes in price dividend ratios are much larger in recessions than in booms, so they vary counter cyclically. That is,

$$\mathrm{cov}(\log(P_{t+1}W_{t+1}^{-1}) - \log(P_t W_t^{-1}), \log W_t) < 0$$

for all sufficiently large t.

REMARK 5.1 *In* (F1) *and* (F3), *the precise meaning of the phrase "well approximated by the homogeneous economy" must be explained. See, Proposition* 8.4 below.

One of our main results is

THEOREM 5.2 *There exists an open set of weakly heterogeneous economies such that the stylized facts (F4), (F5), (F6), (F7), (F8), (F9) are simultaneously valid for sufficiently long horizon for all economies in this set. Moreover, for the economies in this open set, equity premia are larger and risk free rates are smaller than those of the best homogeneous approximation. See, Theorem* 11.13 for the complete description of this set.

6 THE GEOMETRIC RANDOM WALK AGGREGATE ENDOWMENT PROCESS

To assess the status, in our model, of the facts listed above it is necessary to obtain more refined consequences of heterogeneity. For this refined analysis, we consistently specialize the aggregate endowment process to the geometric random walk.

DEFINITION 6.1 *The geometric random walk* W_t, $t \geq 0$ *that steps up* u *with probability* $\mathrm{p} \in (0,1)$ *and steps down* d *with probability* $\mathrm{q} := 1 - \mathrm{p}$ *is the unique process satisfying*

$$W_{t+1} = \begin{cases} \mathrm{u}\, W_t\,, & \textit{with probability}\ \mathrm{p} \\ \mathrm{d}\, W_t\,, & \textit{with probability}\ \mathrm{q} \end{cases}$$

We make the

ASSUMPTION 1 $\mathrm{u} > \mathrm{d} > 0$

Eventually, u, d and p will be chosen so that the "growth" W_{t+1}/W_t of the aggregate endowment process has the "observed" mean and volatility.[6]

We also make the

[6] The results extend to arbitrary geometric random walks (i.e., W_{t+1}/W_t are i.i.d). That is, the discretization of the geometric Brownian motion. It is taken as the standard model in the continuous time literature (see, e.g. (Wang, 1996), (Chan and Kogan, 2002)). The results for continuous time economies can be obtained by passing to the limit. One could also generalize the results to Markov ergodic processes with serial correlation (for example, the one in (Mehra and Prescott, 1985)).

ASSUMPTION 2 *The expected growth rate*

$$\ell(p) = p \log u + q \log d$$

is strictly positive (see, Definition 7.1 below). There is a positive probability for recession, that is $d < 1$.

REMARK 6.2 *By convexity,*

$$\log(pu + qd) > \ell(p)$$

The existence criterion (2.2) takes the simple form

$$\delta(pu^{1-\gamma} + qd^{1-\gamma}) < 1 \tag{6.1}$$

REMARK 6.3 *The existence criterion*

$$\delta(pu^{1-\gamma} + qd^{1-\gamma}) < 1$$

imposes an important constraint on the pair γ, δ. *For each* $0 < \delta < 1$ *there exist, by convexity, unique* $0 \leqslant \gamma_*(\delta) < \gamma^*(\delta) < \infty$ *such that the existence criterion is fulfilled if and only if* $\gamma \in (\gamma_*, \gamma^*)$. *By construction,* $\gamma_*(\delta)$ *is an increasing function of* δ *and* $\gamma^*(\delta)$ *is a decreasing function of* δ. *The existence criterion clearly implies that* γ *cannot be arbitrarily large.*

By Remark 6.2, Assumption 2 implies that $pu + qd > 1$. Roughly speaking, this technical condition means that, on average, the economy is growing. In this case, $\gamma_*(1) = 1$. Otherwise, $\gamma_*(1) = 0$. Under this assumption, $\gamma^*(1) = 1$ if and only if $\ell(p) = 0$, and

$$\gamma_*(\delta) < \gamma_*(1) = 1 \leqslant \gamma^*(1) < \gamma^*(\delta)$$

The geometric random walk, aggregate endowment process is calibrated by choosing $p = 0.5$, $u = 1.054$ and $d = 0.982$ (see, Mehra and Prescott (1985)). It is customary to assume that "on average" δ is bigger than 0.9. Mehra and Prescott (1985) take 0.99 for δ. It follows that in the calibrated model

$$\big(\gamma_*(\delta), \gamma^*(\delta)\big) \subset \big(\gamma_*(0.9), \gamma^*(0.9)\big) \approx (0, 42)$$

and

$$\gamma_*(0.99) \approx 0.42 < \gamma_*(1) = 1 < \gamma^*(1) \approx 34 < \gamma^*(0.99) \approx 35$$

We will often make the

Calibration Hypothesis *The geometric random walk, aggregate endowment process is calibrated by choosing* $p = 0.5$, $u = 1.054$ *and* $d = 0.982$

One of our main objectives is to obtain expansions for the normalized growth rates (4.9) and the logarithmic price dividend ratio

$$\log\left(P_t\, W_t^{-1}\right)$$

By itself, the existence of expansions is not very interesting. The "art" of perturbative calculations is to "write" the second order terms in a form that directly exhibits economic information. These expansions are subsequently used to determine the status of the stylized facts introduced above.

DEFINITION 7.1 *Consciously suppressing* u *and* d*, set*

$$\ell(\mathrm{p}) := \log(\mathrm{u}^{\mathrm{p}}\, \mathrm{d}^{1-\mathrm{p}})$$

and

$$\mathrm{p}' \;=\; \frac{\mathrm{p\,u}}{\mathrm{p\,u} + \mathrm{q\,d}}$$

for any $\mathrm{p} \in (0,1)$.

We always have

$$\mathrm{p}' \,>\, \mathrm{p}$$

for all $\mathrm{p} \in (0,1)$ and consequently,

$$\ell(\mathrm{p}') \,>\, \ell(\mathrm{p})$$

It is surprising that $\ell(\mathrm{p})$ and $\ell(\mathrm{p}')$ figure prominently in *both* the perturbative analysis of returns and in the non perturbative infinite horizon behavior of returns. The non zero difference $\ell(\mathrm{p}') - \ell(\mathrm{p}) > 0$ generates many interesting economic phenomena.

As above, set

$$\rho_t(s) \;:=\; t^{-1} \log\left(\frac{W_t(s)}{W_0}\right)$$

Note, that $E[\rho_t] = \ell(\mathrm{p})$ is independent of t.

THEOREM 7.2 (Expansion of the normalized, equity growth rates) *Let W_t be the geometric random walk introduced above and*

$$\tau^{-1} \log \mathrm{R}^{\mathrm{E}}_{\mathrm{h}} \;=\; \tau^{-1} \log\left(\delta^{-1} \frac{\mathrm{pu+qd}}{\mathrm{pu}^{1-\gamma} + \mathrm{qd}^{1-\gamma}}\right)^{\tau} \;=\; \log\left(\delta^{-1} \frac{\mathrm{pu+qd}}{\mathrm{pu}^{1-\gamma} + \mathrm{qd}^{1-\gamma}}\right)$$

the equity growth rate for the homogeneous economy with parameters (γ, δ). *Then,*

$$\begin{aligned} \tau^{-1} \log \mathrm{R}^{\mathrm{E}}(t_1, t_2) \;=\; & \tau^{-1} \log \mathrm{R}^{\mathrm{E}}_{\mathrm{h}} \\ & + \; \varepsilon \; [\mathrm{A}^{\mathrm{E}}_1 \mathscr{E}(\Delta) + \mathrm{A}^{\mathrm{E}}_2 \mathscr{E}(\Gamma)] \\ & - \; \varepsilon^2 [\, t_1 \mathrm{B}^{\mathrm{E}}_1(s) + \tau \mathrm{B}^{\mathrm{E}}_2 + \mathrm{B}^{\mathrm{E}}_3 \,] \; + O(\varepsilon^3) \end{aligned} \tag{7.1}$$

Here

$$\mathrm{A}^{\mathrm{E}}_1 \;=\; \frac{\partial}{\partial \delta}\left(\tau^{-1} \log \mathrm{R}^{\mathrm{E}}_{\mathrm{h}}\right) \;=\; -\frac{1}{\delta}, \tag{7.2}$$

$$\mathrm{A}^{\mathrm{E}}_2 \;=\; \frac{\partial}{\partial \gamma}\left(\tau^{-1} \log \mathrm{R}^{\mathrm{E}}_{\mathrm{h}}\right) \;=\; \left(\frac{\mathrm{p}\,\mathrm{u}^{1-\gamma} \log \mathrm{u} + \mathrm{q}\,\mathrm{d}^{1-\gamma} \log \mathrm{d}}{\mathrm{p}\,\mathrm{u}^{1-\gamma} + \mathrm{q}\,\mathrm{d}^{1-\gamma}}\right) \tag{7.3}$$

and

$$\begin{aligned} \gamma \mathrm{B}^{\mathrm{E}}_1 \;=\; & -\frac{\mathrm{A}^{\mathrm{E}}_1}{\delta} \mathrm{var}_\eta(\Delta) + \rho_{\,t_1}(s)\, \mathrm{A}^{\mathrm{E}}_2 \,\mathrm{var}_\eta(\Gamma) \\ & + \left(\rho_{\,t_1}(s) \mathrm{A}^{\mathrm{E}}_1 - \frac{\mathrm{A}^{\mathrm{E}}_2}{\delta}\right) \mathrm{cov}_\eta(\Gamma, \Delta) \\ 2\gamma \mathrm{B}^{\mathrm{E}}_2 \;=\; & -\frac{\mathrm{A}^{\mathrm{E}}_1}{\delta} \mathrm{var}_\eta(\Delta) + \ell(\mathrm{p}')\, \mathrm{A}^{\mathrm{E}}_2 \,\mathrm{var}_\eta(\Gamma) \\ & + \left(\ell(\mathrm{p}') \mathrm{A}^{\mathrm{E}}_1 - \frac{\mathrm{A}^{\mathrm{E}}_2}{\delta}\right) \mathrm{cov}_\eta(\Gamma, \Delta) \end{aligned} \tag{7.4}$$

Similarly,

THEOREM 7.3 (Expansion of the normalized, log risk free rates) *Let* W_t *be the geometric random walk introduced above and*

$$\tau^{-1} \log \mathrm{R}^{\mathrm{F}}_{\mathrm{h}} \;=\; \tau^{-1} \log \mathrm{r}^{\mathrm{F}}_{\mathrm{h}} \;=\; \tau^{-1} \log \left(\delta^{-1} \frac{1}{\mathrm{p}\,\mathrm{u}^{-\gamma} + \mathrm{q}\,\mathrm{d}^{-\gamma}}\right)^{\tau} \;=\; \log \left(\delta^{-1} \frac{1}{\mathrm{p}\,\mathrm{u}^{-\gamma} + \mathrm{q}\,\mathrm{d}^{-\gamma}}\right)$$

the normalized, log risk free rates for the homogeneous economy with parameters (γ, δ).

(1) *We have*

$$\begin{aligned} \tau^{-1} \log \mathrm{R}^{\mathrm{F}}(t_1, t_2) \;=\; & \tau^{-1} \log \mathrm{R}^{\mathrm{F}}_{\mathrm{h}} \\ & + \; \varepsilon \; [\mathrm{A}^{\mathrm{F}}_1 \mathscr{E}(\Delta) + \mathrm{A}^{\mathrm{F}}_2 \mathscr{E}(\Gamma)] \\ & - \; \varepsilon^2 [\, t_1 \mathrm{B}^{\mathrm{F}}_1(s) + \tau \mathrm{B}^{\mathrm{F}}_2 + \mathrm{B}^{\mathrm{F}}_3 \,] \; + O(\varepsilon^3) \end{aligned} \tag{7.5}$$

where

$$\begin{aligned} \mathrm{A}^{\mathrm{F}}_1 \;=\; & \frac{\partial}{\partial \delta}\left(\tau^{-1} \log \mathrm{R}^{\mathrm{F}}_{\mathrm{h}}\right) = -\frac{1}{\delta} \\ \mathrm{A}^{\mathrm{F}}_2 \;=\; & \frac{\partial}{\partial \gamma}\left(\tau^{-1} \log \mathrm{R}^{\mathrm{F}}_{\mathrm{h}}\right) = \left(\frac{\mathrm{p}\,\mathrm{u}^{-\gamma} \log \mathrm{u} + \mathrm{q}\,\mathrm{d}^{-\gamma} \log \mathrm{d}}{\mathrm{p}\,\mathrm{u}^{-\gamma} + \mathrm{q}\,\mathrm{d}^{-\gamma}}\right) \end{aligned} \tag{7.6}$$

and

$$\begin{aligned}
\gamma\,\mathrm{B}_1^{\mathrm{F}} &= -\frac{\mathrm{A}_1^{\mathrm{F}}}{\delta}\,\mathrm{var}_\eta(\Delta) + \rho_{t_1}(s)\,\mathrm{A}_2^{\mathrm{F}}\,\mathrm{var}_\eta(\Gamma) \\
&\quad + \left(\rho_{t_1}(s)\,\mathrm{A}_1^{\mathrm{F}} - \frac{\mathrm{A}_2^{\mathrm{F}}}{\delta}\right)\mathrm{cov}_\eta(\Gamma,\Delta) \\
2\gamma\,\mathrm{B}_2^{\mathrm{F}} &= -\frac{\mathrm{A}_1^{\mathrm{F}}}{\delta}\,\mathrm{var}_\eta(\Delta) + \ell(\mathrm{p})\,\mathrm{A}_2^{\mathrm{F}}\,\mathrm{var}_\eta(\Gamma) \\
&\quad + \left(\ell(\mathrm{p})\,\mathrm{A}_1^{\mathrm{F}} - \frac{\mathrm{A}_2^{\mathrm{F}}}{\delta}\right)\mathrm{cov}_\eta(\Gamma,\Delta)
\end{aligned} \tag{7.7}$$

(2) *We have*

$$\begin{aligned}
\tau^{-1}\log \mathrm{r}^{\mathrm{F}}(t_1,\,t_2) &= \tau^{-1}\log \mathrm{r}_{\mathrm{h}}^{\mathrm{F}} \\
&\quad + \varepsilon\ [\,\mathrm{a}_1^{\mathrm{F}}\,\mathscr{E}(\Delta) + \mathrm{a}_2^{\mathrm{F}}\,\mathscr{E}(\Gamma)\,] \\
&\quad - \varepsilon^2[\,t_1\,\mathrm{b}_1^{\mathrm{F}}(s) + \tau\,\mathrm{b}_2^{\mathrm{F}} + \mathrm{b}_3^{\mathrm{F}}\,] + O(\varepsilon^3)
\end{aligned} \tag{7.8}$$

where

$$\mathrm{a}_1^{\mathrm{F}} = \mathrm{A}_1^{\mathrm{F}},\quad \mathrm{a}_2^{\mathrm{F}} = \mathrm{A}_2^{\mathrm{F}},\quad \mathrm{b}_1^{\mathrm{F}}(s) = \mathrm{B}_1^{\mathrm{F}}(s),\quad \mathrm{b}_3^{\mathrm{F}} = \mathrm{B}_2^{\mathrm{F}} + \mathrm{B}_3^{\mathrm{F}} - \mathrm{b}_2^{\mathrm{F}} \tag{7.9}$$

and

$$2\gamma\,\mathrm{b}_2^{\mathrm{F}} = \mathrm{var}_\eta\left(\mathrm{a}_1^{\mathrm{F}}\Delta + \mathrm{a}_2^{\mathrm{F}}\,\Gamma\right) \geqslant 0. \tag{7.10}$$

Note, that $E[\mathrm{B}_1^{\mathrm{F}}] = 2\,\mathrm{B}_2^{\mathrm{F}}$.

REMARK 7.4 *The homogeneous risk free rate* $\tau^{-1}\log \mathrm{R}_{\mathrm{h}}^{\mathrm{F}} = \tau^{-1}\log \mathrm{r}_{\mathrm{h}}^{\mathrm{F}}$ *(see, Theorem* 7.3) is a strictly concave function of $\gamma > 0$ that increases until

$$\frac{\partial}{\partial\gamma}\left(\tau^{-1}\log \mathrm{R}_{\mathrm{h}}^{\mathrm{F}}\right) = \mathrm{A}_2^{\mathrm{F}} = 0$$

REMARK 7.5 *The homogeneous log equity premium*

$$\tau^{-1}\log \mathrm{R}_{\mathrm{h}}^{\mathrm{E}} - \tau^{-1}\log \mathrm{R}_{\mathrm{h}}^{\mathrm{F}} = \tau^{-1}\log \mathrm{R}_{\mathrm{h}}^{\mathrm{E}} - \tau^{-1}\log \mathrm{r}_{\mathrm{h}}^{\mathrm{F}}$$

(see, Theorem 7.2) is a strictly increasing function of $\gamma > 0$ and is independent of δ.

DEFINITION 7.6 *By Assumption 2, the expected growth rate of the aggregate endowment* $\ell(\mathrm{p}) > 0$, *and therefore* $\mathrm{A}_2^{\mathrm{F}}(\gamma)\big|_{\gamma=0} > 0$. *Since* $d < 1$, *then* $\lim_{\gamma\to\infty}\mathrm{A}_2^{\mathrm{F}}(\gamma) < 0$. *Let* $\mathfrak{G}$ *be the unique root of the strictly decreasing function* $\mathrm{A}_2^{\mathrm{F}}(\gamma)$. *That is, the unique critical point of the homogeneous risk free rate*

$$\tau^{-1}\log \mathrm{R}_{\mathrm{h}}^{\mathrm{F}}$$

REMARK 7.7 *The* **Calibration Hypothesis** *yields*

$$1 = \gamma_*(1) \; < \; \mathfrak{G} \approx 15 \; < \; \gamma^*(1) \approx 34$$

In general, for suitable u, d *we can have* $\mathfrak{G} < 1$. *On the other hand, we always have* $\mathfrak{G} < \gamma^*(1) - 1$.

REMARK 7.8 *For* $\gamma > \mathfrak{G}$, *the risk free rate is a strictly decreasing function of* γ. *In this regime, the "puzzle" formulated in* (F1) *ultimately evaporates. It follows from this observation that the "standard" homogeneous model referred to in* (F1) *implicitly has* $\gamma < \mathfrak{G}$. *For this reason, we will always assume that* $\gamma < \mathfrak{G}$.

Here, is the last expansion we will need.

THEOREM 7.9 (Expansion of the log price dividend ratio) *Let* W_t *be the geometric random walk introduced above and*

$$\log\left(\frac{P_{\mathrm{h}\,t}}{W_t}\right) \; = \; \log\left(\frac{\delta\,(\mathrm{p}\,\mathrm{u}^{1-\gamma} + \mathrm{q}\,\mathrm{d}^{1-\gamma})}{1-\delta\,(\mathrm{p}\,\mathrm{u}^{1-\gamma} + \mathrm{q}\,\mathrm{d}^{1-\gamma})}\right)$$

the log price dividend ratio for the homogeneous economy with parameters (γ, δ). *We have*

$$\begin{aligned} \log\left(\frac{P_t}{W_t}\right) \; &= \; \log P_{\mathrm{h}0} \\ &\quad + \; \varepsilon \; [\, \mathrm{A}_1^{\mathrm{P}}\,\mathscr{E}(\Delta) + \mathrm{A}_2^{\mathrm{P}}\,\mathscr{E}(\Gamma)\,] \\ &\quad - \; \varepsilon^2[\; t\,\mathrm{B}_1^{\mathrm{P}}(s) + \mathrm{B}_3^{\mathrm{P}}\,] + O(\varepsilon^3). \end{aligned} \tag{7.11}$$

Here,

$$\mathrm{A}_1^{\mathrm{P}} \; = \; \frac{\partial}{\partial\delta}\log P_{\mathrm{h}0} \; = \; \frac{1}{\delta\,(1-\delta\,(\mathrm{p}\,\mathrm{u}^{1-\gamma} + \mathrm{q}\,\mathrm{d}^{1-\gamma}))}, \tag{7.12}$$

$$\mathrm{A}_2^{\mathrm{P}} \; = \; \frac{\partial}{\partial\gamma}\log P_{\mathrm{h}0} = -\,\frac{\mathrm{p}\,\mathrm{u}^{1-\gamma}\,\log\mathrm{u} + \mathrm{q}\,\mathrm{d}^{1-\gamma}\,\log\mathrm{d}}{(\mathrm{p}\,\mathrm{u}^{1-\gamma} + \mathrm{q}\,\mathrm{d}^{1-\gamma})\,(1-\delta\,(\mathrm{p}\,\mathrm{u}^{1-\gamma} + \mathrm{q}\,\mathrm{d}^{1-\gamma}))} \tag{7.13}$$

and

$$\gamma\,\mathrm{B}_1^{\mathrm{P}} \; = \; -\frac{\mathrm{A}_1^{\mathrm{P}}}{\delta}\,\mathrm{var}_\eta(\Delta) + \rho_{\,t}(s)\,\mathrm{A}_2^{\mathrm{P}}\,\mathrm{var}_\eta(\Gamma) \; + \; \left(\mathrm{A}_1^{\mathrm{P}}\,\rho_{\,t}(s) - \frac{\mathrm{A}_2^{\mathrm{P}}}{\delta}\right)\mathrm{cov}_\eta(\Gamma,\Delta) \tag{7.14}$$

REMARK 7.10 *In a homogeneous economy the price dividend ratio is constant,*

$$P_{\mathrm{h}\,t}\,W_t^{-1} \; = \; P_{\mathrm{h}0}$$

for all $t \geqslant 0$.

REMARK 7.11 *The discussion of the error terms appearing in the expansions for the state price densities* (3.5) *applies equally to the error terms appearing in the three theorems stated just above. The justification for displaying such "complicated" formulas is the same.*

REMARK 7.12 *The coefficients* $B_3{}^{I}$, I=E,F,P, *are time and state independent. They are responsible for a shift in the asset returns. We will not analyze their influence on asset returns.*

8 THE BEST HOMOGENEOUS APPROXIMATION

A "real" economy is often "compared" to a "standard" homogeneous economy. Consciously or unconsciously, it is implicit that the homogeneous economy should be a "good approximation" to the real economy. Otherwise, it makes no sense to compare the economies. What in fact constitutes a good approximation?

From now on, we consciously suppress individual endowments almost everywhere and "sloppily" refer to $((\gamma_1,\delta_1),\cdots,(\gamma_n,\delta_n))$ as a heterogeneous economy.

DEFINITION 8.1 *An economy* $((\gamma_1,\delta_1),\cdots,(\gamma_n,\delta_n))$ *is weakly heterogeneous when the variations*

$$\max_i \gamma_i - \min_i \gamma_i \quad , \quad \max_i \delta_i - \min_i \delta_i \tag{8.1}$$

are small.

REMARK 8.2 *Observe that the economy* $((\gamma_1,\delta_1),\cdots,(\gamma_n,\delta_n))$ *is weakly heterogeneous if and only if there is a decomposition*

$$(\gamma_i , \delta_i) = (\gamma , \delta) + \varepsilon\,(\Gamma_i , \Delta_i)$$

for some (γ,δ) *and* (Γ_i,Δ_i), $i = 1,\cdots,n$ *and a small parameter* ε. *Of course, there are many possible decompositions.*

Suppose,

$$(\gamma_i , \delta_i) = (\gamma , \delta) + \varepsilon\,(\Gamma_i , \Delta_i) \qquad i = 1,\cdots,n.$$

is a weakly heterogeneous economy. Is the homogeneous economy (γ,δ) a good approximation ? In general, no !

Notice that, by Theorem 7.2, the normalized, equity growth rate

$$\tau^{-1} \log R^{E}(t_1 , t_2)$$

departs from that of the homogeneous economy (γ, δ) by

$$\begin{aligned} &\tau^{-1} \log R^{E}_{h(\gamma,\delta)} \ + \ \varepsilon \ [A^{E}_{1} \mathscr{E}(\Delta) + A^{E}_{2} \mathscr{E}(\Gamma)] \\ &= \ \tau^{-1} \log R^{E}_{h(\gamma+\varepsilon\mathscr{E}(\Gamma),\delta+\varepsilon\mathscr{E}(\Delta))} \ + \ O(\varepsilon^2) \end{aligned}$$

Here, $h(\gamma, \delta)$ stands for the homogeneous economy (γ, δ).

In other words, the first order term extracts more homogeneity from the heterogeneous economy and consequently, (γ, δ) is not a "good" homogeneous approximation to (γ_i, δ_i), $i = 1, \cdots, n$, unless

$$A^{E}_{1} \mathscr{E}(\Delta) + A^{E}_{2} \mathscr{E}(\Gamma) \ = \ 0$$

Similarly, by Theorem 7.3, (γ, δ) is not a "good" homogeneous approximation unless

$$A^{F}_{1} \mathscr{E}(\Delta) + A^{F}_{2} \mathscr{E}(\Gamma) \ = \ 0$$

We have

$$\begin{vmatrix} A^{E}_{1} & A^{E}_{2} \\ A^{F}_{1} & A^{F}_{2} \end{vmatrix} \neq \ 0$$

It follows from the discussion above that (γ, δ) can reasonably be called a "good" homogeneous approximation to the weakly heterogeneous economy (γ_i, δ_i), $i = 1, \cdots, n$, when $\mathscr{E}(\Delta) = 0$ and $\mathscr{E}(\Gamma) = 0$.

DEFINITION 8.3 *The homogeneous economy (γ, δ) is a good approximation to the weakly heterogeneous economy (γ_i, δ_i), $i = 1, \cdots, n$, when both $\mathscr{E}(\Delta) = 0$ and $\mathscr{E}(\Gamma) = 0$. We call (γ, δ) the "best" homogeneous approximation when it is the only good homogeneous approximation.*

PROPOSITION 8.4 *The homogeneous economy (γ, δ) is a good approximation to the weakly heterogeneous economy $((\gamma_1, \delta_1), \cdots, (\gamma_n, \delta_n))$ if and only if*

$$\gamma = \mathscr{E}(\gamma_1, \cdots, \gamma_n) \tag{8.2}$$

$$\delta = \mathscr{E}(\delta_1, \cdots, \delta_n) \tag{8.3}$$

In this case,

$$\min_i \gamma_i \leqslant \gamma \leqslant \max_i \gamma_i$$

$$\min_i \delta_i \leqslant \delta \leqslant \max_i \delta_i$$

REMARK 8.5 *Bear in mind that the wealth weighted average $\mathscr{E}$ depends on (γ, δ). The equations* (8.2), (8.3) *are nonlinear in (γ, δ).*

It is natural to ask whether there exists a "good" homogeneous approximation to any weakly heterogeneous economy, and whether it is unique.

THEOREM 8.6 *Let* (γ_i, δ_i), $i = 1, \cdots, n$, *be a weakly heterogeneous economy. That is,*

$$\max_i \gamma_i - \min_i \gamma_i \leqslant \varepsilon$$
$$\max_i \delta_i - \min_i \delta_i \leqslant \varepsilon$$

for some small $\varepsilon > 0$. *If* ε *is sufficiently small, there exists the best homogeneous approximation* (γ, δ) *to* (γ_i, δ_i), $i = 1, \cdots, n$.

REMARK 8.7 *By an application of the Brouwer fixed point theorem, equations* (8.2), (8.3) *have a solution* (γ, δ) *for any heterogeneous economy.*

9 SOME INEQUALITIES

We now collect a number of important inequalities. They are essential for unravelling, in our model, the relationships between the stylized facts listed above.

Recall that

$$\mathrm{A}_2^{\mathrm{E}} = \frac{\mathrm{p}\mathrm{u}^{1-\gamma}\log\mathrm{u} + \mathrm{q}\mathrm{d}^{1-\gamma}\log\mathrm{d}}{\mathrm{p}\mathrm{u}^{1-\gamma} + \mathrm{q}\mathrm{d}^{1-\gamma}}$$

and

$$\mathrm{A}_2^{\mathrm{F}} = \frac{\mathrm{p}\mathrm{u}^{-\gamma}\log\mathrm{u} + \mathrm{q}\mathrm{d}^{-\gamma}\log\mathrm{d}}{\mathrm{p}\mathrm{u}^{-\gamma} + \mathrm{q}\mathrm{d}^{-\gamma}}$$

PROPOSITION 9.1 *For all* $\mathrm{p} \in (0,1)$ *and all* $\mathrm{u} > \mathrm{d}$, *we have*

(1) *If* $\gamma > 1$, *then*

$$\mathrm{A}_2^{\mathrm{F}} < \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - (\mathrm{A}_2^{\mathrm{F}})^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}} < \mathrm{A}_2^{\mathrm{E}} < \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} < \ell(\mathrm{p}) < \ell(\mathrm{p}') \quad (9.1)$$

(2) *If* $\gamma = 1$, *then*

$$\mathrm{A}_2^{\mathrm{F}} < \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - (\mathrm{A}_2^{\mathrm{F}})^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}} < \mathrm{A}_2^{\mathrm{E}} = \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} = \ell(\mathrm{p}) < \ell(\mathrm{p}') \quad (9.2)$$

(3) If $\gamma < 1$, *then*

$$\frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - (\mathrm{A}_2^{\mathrm{F}})^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}} \;,\; \ell(\mathrm{p}) \in \left(\mathrm{A}_2^{\mathrm{F}}, \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})}\right) \quad (9.3)$$

and

$$\mathrm{A}_2^{\mathrm{F}} < \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} < \mathrm{A}_2^{\mathrm{E}} < \ell(\mathrm{p}') \quad (9.4)$$

REMARK 9.2 *It is possible to show that there exists a* $\gamma_0 \in (0,1)$ *such that*

$$\ell(\mathrm{p}) < \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - (\mathrm{A}_2^{\mathrm{F}})^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}}$$

if and only if $\gamma \in (0, \gamma_0)$.

10 PERTURBATIVE ANALYSIS OF STYLIZED FACTS

Fix a homogeneous economy (γ, δ). For each of the stylized facts listed above we shall determine all heterogeneous economies, sufficiently close to (γ, δ), in which the fact is "true". If the fact vaguely stipulates that an economic indicator is "rather small" or "rather large" then we only determine its qualitative truth. For example, that risk free rates are smaller than those of (γ, δ). On the other hand, if the fact prescribes a sign, such as the covariance between economic indicators, then we determine its absolute truth.

Concretely, we will first determine the space of all heterogeneous directions (Γ, Δ) for which the fact is "true" to second order in the "heterogeneity strength" ε. In this step we make essential use of the theorems and proposition stated just above. Then, it is shown that for any fixed horizon the fact remains "true" in the full heterogeneous economy for all sufficiently small ε.

DEFINITION 10.1 *Consciously suppressing* (γ, δ), (Γ, Δ) *and the parameters* u, d, p, *set*

$$\tau_0^{\mathrm{F}}(t) := -\frac{tE[\mathrm{B}_1^{\mathrm{F}}] + \mathrm{B}_3^{\mathrm{F}}}{\mathrm{B}_2^{\mathrm{F}}}$$

and

$$\tau_1^{\mathrm{F}}(t) := -\frac{tE[\mathrm{b}_1^{\mathrm{F}}] + \mathrm{b}_3^{\mathrm{F}}}{\mathrm{b}_2^{\mathrm{F}}}$$

Similarly,

$$\tau_0^{\mathrm{E}}(t) := -\frac{tE[\mathrm{B}_1^{\mathrm{E}} - \mathrm{B}_1^{\mathrm{F}}] + \mathrm{B}_3^{\mathrm{E}} - \mathrm{B}_2^{\mathrm{F}}}{\mathrm{B}_2^{\mathrm{E}} - \mathrm{B}_2^{\mathrm{F}}}$$

and

$$\tau_1^{\mathrm{E}}(t) := -\frac{tE[\mathrm{B}_1^{\mathrm{E}} - \mathrm{b}_1^{\mathrm{F}}] + \mathrm{B}_3^{\mathrm{E}} - \mathrm{b}_2^{\mathrm{F}}}{\mathrm{B}_2^{\mathrm{E}} - \mathrm{b}_2^{\mathrm{F}}}$$

REMARK 10.2 *The denominators* $\mathrm{B}_2^{\mathrm{F}}$, $\mathrm{b}_2^{\mathrm{F}}$, $\mathrm{B}_2^{\mathrm{E}} - \mathrm{B}_2^{\mathrm{F}}$ *and* $B_2^{\mathrm{E}} - b_2^{\mathrm{F}}$ *are independent of* s *and* t *and generically non zero. Precisely, they vanish on a real analytic subvariety of codimension one.*

REMARK 10.3 *The quantities* $E[\mathrm{B}_1^{\mathrm{F}}] = \mathrm{E}[\mathrm{b}_1^{\mathrm{F}}] = 2\mathrm{B}_2^{\mathrm{F}}$ *and* $E[\mathrm{B}_1^{\mathrm{E}}]$ *are independent of* t. *The quantities* $\mathrm{B}_3^{\mathrm{F}}$, $\mathrm{b}_3^{\mathrm{F}}$ *and* $\mathrm{B}_3^{\mathrm{E}}$ *are also independent of* s *and* t.

THEOREM 10.4 *Let* W_t *be the geometric random walk introduced above and* (γ,δ) *a fixed homogeneous economy. Suppose,* (γ,δ) *is the best homogeneous approximation to*

$$(\gamma,\delta)+\varepsilon(\Gamma_i,\Delta_i),\qquad i=1,\cdots,n$$

Then, all the statements below are valid to second order in the parameter ε.

(F1) (a) *Suppose,* $\mathrm{B}_2^{\mathrm{F}}\neq 0$. *For all* $\tau>\tau_0^{\mathrm{F}}(t_1)$, *the log short term risk free rate*

$$\tau^{-1}\log \mathrm{R}^{\mathrm{F}}(t_1,t_1+\tau)$$

is smaller in the direction (Γ,Δ) *than the corresponding rate in the homogeneous economy, if and only if* $B_2^{\mathrm{F}}>0$, *that is*

$$\frac{1}{\delta^2}\,\mathrm{var}_\eta(\Delta)+\ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}\,\mathrm{var}_\eta(\Gamma)>\frac{1}{\delta}\left(\ell(\mathrm{p})+\mathrm{A}_2^{\mathrm{F}}\right)\mathrm{cov}_\eta(\Gamma,\Delta)\quad(10.1)$$

The last inequality is always fulfilled if

$$\frac{\mathrm{var}_\eta(\Gamma)}{\mathrm{var}_\eta(\Delta)}\notin\left[\frac{1}{(\delta\,\ell(\mathrm{p}))^2},\frac{1}{(\delta\mathrm{A}_2^{\mathrm{F}})^2}\right]\quad(10.2)$$

(b) *Suppose,* $\mathrm{b}_2^{\mathrm{F}}\neq 0$. *The normalized, log long term risk free rate*

$$\tau^{-1}\log \mathrm{r}^{\mathrm{F}}(t_1,t_1+\tau)$$

is smaller in the direction (Γ,Δ) *than the corresponding rate in the homogeneous economy for all* $\tau>\tau_1^{\mathrm{F}}(t_1)$ *if and only if* $b_2^{\mathrm{F}}>0$, *that is*

$$\frac{1}{\delta^2}\,\mathrm{var}_\eta(\Delta)+\left(\mathrm{a}_2^{\mathrm{F}}\right)^2\mathrm{var}_\eta(\Gamma)-2\frac{1}{\delta}\,\mathrm{a}_2^{\mathrm{F}}\,\mathrm{cov}_\eta(\Gamma,\Delta)>0$$

The last condition is generically true. Observe, that the expression on the left hand side of the inequality is equal to

$$\mathrm{var}_\eta\left(\mathrm{a}_1^{\mathrm{F}}\Delta+\mathrm{a}_2^{\mathrm{F}}\,\Gamma\right)$$

and by convexity always greater than or equal to zero.

(F2) *The normalized, log term premium*

$$\tau^{-1}\log \mathrm{r}^{\mathrm{F}}(t_1,t_2)-\tau^{-1}\log \mathrm{R}^{\mathrm{F}}(t_1,t_2)$$

is positive for all $\tau\geqslant 2$ *if and only if* $\mathrm{B}_2^{\mathrm{F}}>\mathrm{b}_2^{\mathrm{F}}$, *that is*

$$\frac{\mathrm{cov}_\eta(\Gamma,\Delta)}{\mathrm{var}_\eta(\Gamma)}<\delta\mathrm{A}_2^{\mathrm{F}}\quad(10.3)$$

(F3) (a) *Suppose,* $\mathrm{B}_2^{\mathrm{E}} - \mathrm{B}_2^{\mathrm{F}} \neq 0$*. For all* $\tau > \tau_0^{\mathrm{E}}(t_1)$*, the normalized, log equity premium* (4.10) *relative to short term bonds is larger in the direction* (Γ, Δ) *than the corresponding equity premium in the homogeneous economy, if and only if* $\mathrm{B}_2^{\mathrm{E}} - \mathrm{B}_2^{\mathrm{F}} < 0$*, that is*

$$\frac{\mathrm{cov}_\eta(\Gamma, \Delta)}{\mathrm{var}_\eta(\Gamma)} > \delta \, \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} \tag{10.4}$$

(b) *Suppose,* $\mathrm{B}_2^{\mathrm{E}} - \mathrm{b}_2^{\mathrm{F}} \neq 0$*. For all* $\tau > \tau_1^{\mathrm{E}}(t_1)$*, the normalized, log equity premium* (4.11) *relative to long term bonds is larger in the direction* (Γ, Δ) *than the corresponding equity premium in the homogeneous economy, if and only if* $\mathrm{B}_2^{\mathrm{E}} - \mathrm{b}_2^{\mathrm{F}} < 0$*, that is*

$$\frac{\mathrm{cov}_\eta(\Gamma, \Delta)}{\mathrm{var}_\eta(\Gamma)} > \delta \, \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \left(\mathrm{A}_2^{\mathrm{F}}\right)^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}} \tag{10.5}$$

(F4) *For all* t*,*

$$\mathrm{cov}\left(\log \frac{\mathrm{R}^{\mathrm{E}}(t, t+1)}{\mathrm{R}^{\mathrm{F}}(t, t+1)}, \log W_t\right) < 0$$

if and only if

$$\mathrm{var}_\eta(\Gamma) > 0.$$

The last condition holds if and only if $\gamma_1, \cdots, \gamma_n$ *are heterogeneous.*

(F5) *For all* t*,*

$$\mathrm{cov}\left(\log(P_t W_t^{-1}), \log W_t\right) > 0$$

if and only if

$$\frac{\mathrm{cov}_\eta(\Gamma, \Delta)}{\mathrm{var}_\eta(\Gamma)} < \delta \mathrm{A}_2^{\mathrm{E}}. \tag{10.6}$$

(F6) *For all* t*,*

$$\mathrm{cov}\left(\log(P_t W_t^{-1}), \log(P_{t+1} W_{t+1}^{-1})\right) > 0$$

if and only if

$$\mathrm{var}_\eta(\Gamma) > 0.$$

Again, the last condition holds if and only if $\gamma_1, \cdots, \gamma_n$ *are heterogeneous.*

(F7) *For all* t *and all* $j \geqslant 0$

$$\operatorname{cov}\big(\log(P_t W_t^{-1}), \mathrm{R}^{\mathrm{E}}(t+j, t+j+1)\big) < 0$$

if and only if

$$\operatorname{var}_\eta(\Gamma) > 0.$$

Again, the last condition holds if and only if $\gamma_1, \cdots, \gamma_n$ *are heterogeneous.*

(F8) *For all* t,

$$\operatorname{cov}\big(\mathrm{r}^{\mathrm{E}}(t_1, t_2), \mathrm{r}^{\mathrm{E}}(t_2, t_3)\big) < 0$$

if and only if the condition (10.6) *is fulfilled.*

(F9) *For all* t,

$$\operatorname{cov}\big(\operatorname{var}_t(\mathrm{r}^{\mathrm{E}}_{t+1}), \log W_t\big) < 0$$

if and only if the condition (10.6) *is fulfilled.*

We now formalize the intuition that "statements" true to second order in ε are true to all orders.

THEOREM 10.5 *For any* $T \in \mathbb{N}$ *there exists an* $\varepsilon > 0$ *such that the statements of Theorem* 10.4 hold to all orders as soon as $t + \tau < T$. Furthermore,

(F10) The conditional correlation at time t,

$$\operatorname{corr}_t(\mathrm{r}^{\mathrm{E}}_{t+1}, W_{t+1} W_t^{-1}) = \varepsilon^2 Z + O(\varepsilon^3)$$

for some constant Z. In other words, the conditional correlation is independent of time and state to second order.

(F11) The "changes" or forward difference of the log price dividend ratios at time t,

$$\log(P_{t+1} W_{t+1}^{-1}) - \log(P_t W_t^{-1}) = \varepsilon^2 Z_1 \log(W_{t+1} W_t^{-1}) + O(\varepsilon^3)$$

are independent, identically distributed random variables independent of W_t to second order. In other words, there is no correlation between forward differences of price dividend ratios and the aggregate endowment to second order.

REMARK 10.6 *The heterogeneity strength* ε *must be chosen so small that the error of order* ε^3 *does not change the sign of the second order term. It follows that the size of* ε *depends implicitly on the direction* (Γ, Δ).

We now interpret Theorem 10.5 in a series of remarks.

REMARK 11.1 *The sequence*

$$\tau^{-1} \log \mathrm{R}^{\mathrm{F}}(t, t+\tau) \ , \ \tau = 1, 2, \cdots$$

is not only smaller (for large τ) than that of the homogeneous economy but strictly decreases in τ when the direction (Γ, Δ) *satisfies the condition* (10.1).

By contrast, the yield curve

$$\tau^{-1} \log \mathrm{r}^{\mathrm{F}}(t, t+\tau) \ , \ \tau = 1, 2, \cdots$$

is strictly decreasing in τ for a generic direction (Γ, Δ). *That is, no essential restriction on the direction is necessary.*

This is surprising. After all, the two sequences are identical and always decreasing for homogeneous risk aversion, that is

$$\gamma_1 = \cdots = \gamma_n$$

Why does the first sequence only decay for a restricted class of heterogeneous economies ? One can, a posteriori, give an "economic explanation". However, by direct inspection of the two constraints

$$\frac{1}{\delta^2} \mathrm{var}_\eta(\Delta) + \ell(\mathrm{p}) \mathrm{A}_2^{\mathrm{F}} \mathrm{var}_\eta(\Gamma) > \frac{1}{\delta} \left(\ell(\mathrm{p}) + \mathrm{A}_2^{\mathrm{F}}\right) \mathrm{cov}_\eta(\Gamma, \Delta)$$
$$\frac{1}{\delta^2} \mathrm{var}_\eta(\Delta) + \mathrm{A}_2^{\mathrm{F}} \mathrm{A}_2^{\mathrm{F}} \mathrm{var}_\eta(\Gamma) > \frac{1}{\delta} (\mathrm{A}_2^{\mathrm{F}} + \mathrm{A}_2^{\mathrm{F}}) \mathrm{cov}_\eta(\Gamma, \Delta)$$

one immediately sees that the phenomenon lies in the precise algebraic structure of the coefficients. Namely, $\ell(\mathrm{p}) > \mathrm{A}_2^{\mathrm{F}}$. *See,* (9.1).

REMARK 11.2 *By Proposition* 9.1, the criterion (10.3) for the strict positivity of the term premium (see, (F3)) is incompatible with both (10.4), (10.5). In other words, it is impossible to simultaneously ensure that the term premium is positive and that any of the equity premia increase. In particular, this result contradicts the "conventional wisdom" that in standard equilibrium models the term premium and the equity premium relative to short term bonds rise and fall together.

REMARK 11.3 *Let X_t, $t \geqslant 1$ be a sequence of nonnegative, independent, identically distributed random variables with compactly supported distribution function $F(x)$ and let*

$$W_t = \prod_{k=1}^{t} X_k, \qquad t \geqslant 1$$

be the corresponding geometric random walk. The equity premium relative to short term bonds for a homogeneous economy $\mathrm{h} = \mathrm{h}(\gamma, \delta)$ *with aggregate endowment process* W_t *is*

$$\tau^{-1} \log \frac{\mathrm{R}^{\mathrm{E}}_{\mathrm{h}(\gamma,\delta)}(t_1, t_2)}{\mathrm{R}^{\mathrm{F}}_{\mathrm{h}(\gamma,\delta)}(t_1, t_2)} = \log \frac{E[X_1]\, E[X_1^{-\gamma}]}{E[X_1^{1-\gamma}]}$$

It is a nonnegative, strictly increasing function of γ. *Therefore, by Remark* 6.3,

$$\tau^{-1} \log \frac{\mathrm{R}^{\mathrm{E}}_{\mathrm{h}(\gamma,\delta)}(t_1, t_2)}{\mathrm{R}^{\mathrm{F}}_{\mathrm{h}(\gamma,\delta)}(t_1, t_2)}(\gamma) < \tau^{-1} \log \frac{\mathrm{R}^{\mathrm{E}}_{\mathrm{h}(\gamma^*,\delta)}(t_1, t_2)}{\mathrm{R}^{\mathrm{F}}_{\mathrm{h}(\gamma^*,\delta)}(t_1, t_2)} < \log \frac{E[X_1]}{\inf \operatorname{supp} F}$$

for all $\gamma < \gamma^*$. Here, $\inf \operatorname{supp} F$ is the infimum of the support of $F(x)$. Clearly, the size of the equity premium is controlled by $\inf \operatorname{supp} F$. Note, that the equity premium does not depend on δ, but the upper bound depends on δ through γ^*. The **Calibration Hypothesis** implies

$$\gamma_*(0.9) = 0 \qquad \text{and} \qquad \gamma^*(0.9) \approx 42$$

and

$$\tau^{-1} \log \frac{\mathrm{R}^{\mathrm{E}}_{\mathrm{h}}(t_1, t_2)}{\mathrm{R}^{\mathrm{F}}_{\mathrm{h}}(t_1, t_2)}(\gamma^*) \approx \log 1.033$$

and

$$\log \frac{E[X_1]}{\inf \operatorname{supp} F} = \frac{\mathrm{u} + \mathrm{d}}{2d} \approx 1.037$$

This is much less than the "observed" $\log 1.07$ (see, Campbell (2003), Mehra and Prescott (1985)).

It has become a generally accepted convention (Campbell (2003)) that the random variables X_t are independent and have identical log normal distributions $N(x)$. The material point is that $\inf \operatorname{supp} N(x) = 0$ and consequently there is no a-priori upper bound on the homogeneous equity premium.

REMARK 11.4 **(Equity premium and positive covariances)** *By Definition* 7.6,

$$\mathrm{A}^{\mathrm{F}}_2(\gamma) > 0$$

when $\gamma < \mathfrak{G}$. Consequently, by (9.1) the quotients

$$\frac{\ell(\mathrm{p}')\mathrm{A}^{\mathrm{E}}_2 - \ell(\mathrm{p})\mathrm{A}^{\mathrm{F}}_2}{\ell(\mathrm{p}') + \mathrm{A}^{\mathrm{E}}_2 - \mathrm{A}^{\mathrm{F}}_2 - \ell(\mathrm{p})} \qquad \text{and} \qquad \frac{\ell(\mathrm{p}')\mathrm{A}^{\mathrm{E}}_2 - \left(\mathrm{A}^{\mathrm{F}}_2\right)^2}{\ell(\mathrm{p}') + \mathrm{A}^{\mathrm{E}}_2 - 2\mathrm{A}^{\mathrm{F}}_2}$$

appearing in the constraints (10.4), (10.5) are strictly positive when the risk aversion $\gamma < \mathfrak{G}$. If the quotients are positive, then (10.4) and (10.5) imply that a positive covariance between Γ and Δ is necessary to make the equity premium increase. The constraint (10.4), forcing the equity premium relative to short term bonds to increase, is stronger than the constraint (10.5) that forces the equity premium relative to long term bonds to increase, since, by Proposition 9.1,

$$\frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} > \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \left(\mathrm{A}_2^{\mathrm{F}}\right)^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}}$$

In other words, the set of directions in which the equity premium relative to long term bonds increases, strictly contains the set of directions in which the equity premium relative to short term bonds increases.

There is very little reliable data about standard statistical means and variances of risk aversion and discount factors in which all agents are weighted equally. Not surprisingly, there is no empirical data at all on wealth weighted averages of risk aversion and discount factors and on their corresponding wealth weighted variances and covariance. Still, the model generates concrete predictions that we now derive.

For this purpose, recall that

$$\gamma_i = \gamma + \varepsilon \Gamma_i \qquad , \qquad \delta_i = \delta + \varepsilon \Delta_i$$

for all $i = 1, \cdots, n$. Substituting,

$$\frac{\mathrm{cov}_\eta\big((\gamma_1, \cdots, \gamma_n), (\delta_1, \cdots, \delta_n)\big)}{\mathrm{var}_\eta(\gamma_1, \cdots, \gamma_n)} = \frac{\mathrm{cov}_\eta(\Gamma, \Delta)}{\mathrm{var}_\eta(\Gamma)}$$

and therefore (10.4) is equivalent to

$$\frac{\mathrm{cov}_\eta\big((\gamma_1, \cdots, \gamma_n), (\delta_1, \cdots, \delta_n)\big)}{\mathrm{var}_\eta(\gamma_1, \cdots, \gamma_n)} > \delta \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} \tag{11.1}$$

The **Calibration Hypothesis** implies that the right hand side of (11.1) decreases to $0.01\,\delta$ when γ approaches 10. These observations and the assumption that the average δ is bigger than 0.9 (see, Remark 6.3) imply that the sloppy inequality

$$\mathrm{var}_\eta(\gamma_1, \cdots, \gamma_n) < 112\, \mathrm{cov}_\eta\big((\gamma_1, \cdots, \gamma_n), (\delta_1, \cdots, \delta_n)\big) \tag{11.2}$$

follows from (11.1). We now see that the "equity premium puzzle" (see, stylized fact (F3) above) implies that the wealth weighted covariance between risk aversion and patience is positive and bounded below by $1/112$ of the wealth weighted variance of risk aversion. One is therefore tempted to make the social prediction that in an economy with a large equity premium the wealth weighted covariance between risk aversion and patience is positive. Amusingly, the inequality above is consistent with the experimental work of Van Praag and Booji (2003) in which it is "observed" that the average risk aversion is 1.5 and

$$\mathrm{var}(\gamma_1, \cdots, \gamma_n) \approx 14.3$$
$$\mathrm{cov}((\gamma_1, \cdots, \gamma_n), (\delta_1, \cdots, \delta_n)) \approx 0.33$$

Here, each agent is assigned the weight n^{-1} when evaluating expectations.

REMARK 11.5 **(Equity premium and the quotient of variances)** *By an application of Schwartz's inequality,* (10.4) *implies*

$$\left(\frac{\mathrm{var}_\eta(\Delta)}{\mathrm{var}_\eta(\Gamma)}\right)^{1/2} > \delta \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} \tag{11.3}$$

Similarly, (10.5) *implies*

$$\left(\frac{\mathrm{var}_\eta(\Delta)}{\mathrm{var}_\eta(\Gamma)}\right)^{1/2} > \delta \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \left(\mathrm{A}_2^{\mathrm{F}}\right)^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}} \tag{11.4}$$

If $\gamma < \mathfrak{G}$, *the quotients are positive and impose concrete restrictions on the variances of* Γ *and* Δ. *If* $1 \leqslant \gamma < \mathfrak{G}$, *then, by Proposition* 9.1,

$$\delta\,\ell(\mathrm{p}) > \delta \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} > \delta \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \left(\mathrm{A}_2^{\mathrm{F}}\right)^2}{\ell(\mathrm{p}') + \mathrm{A}_2^{\mathrm{E}} - 2\mathrm{A}_2^{\mathrm{F}}}$$

In other words, the right hand sides of (11.3) and (11.4) are simultaneously dominated by the γ independent, expected growth rate of the economy multiplied by the average discount factor. Invoking the **Calibration Hypothesis**, the expected growth rate $\ell(\mathrm{p}) = 0.0172$. Therefore, (11.3) and (11.4) permit a "realistic" $\mathrm{var}_\eta(\Gamma)$ to be larger than $\mathrm{var}_\eta(\Delta)$.

It follows from (11.2) and Schwartz's inequality that

$$\mathrm{var}_\eta(\Gamma) \leqslant 12{,}600\ \mathrm{var}_\eta(\Delta) \tag{11.5}$$

At first sight, the factor of 12600 appearing on the right hand side of the last inequality looks huge. However, recalling that

$$\gamma_i = \gamma + \varepsilon\Gamma_i \qquad , \qquad \delta_i = \delta + \varepsilon\Delta_i$$

for all $i = 1, \cdots, n$,

$$\frac{\mathrm{var}_\eta(\gamma_1, \cdots, \gamma_n)}{\mathrm{var}_\eta(\delta_1, \cdots, \delta_n)} = \frac{\mathrm{var}_\eta(\Gamma)}{\mathrm{var}_\eta(\Delta)}$$

If we adopt the standard convention (see, e.g. Weitzman (2001)) that the patience

$$\delta_i \in [0.8, 1),\ i = 1, \cdots, n,$$

we obtain the sloppy upper bound

$$\mathrm{var}_\eta(\delta_1, \cdots, \delta_n) \leqslant 0.01$$

It now follows directly from (11.5) that

$$\mathrm{var}_\eta(\gamma_1, \cdots, \gamma_n) \leqslant 126$$

One should observe that this sloppy bound on the variance $\mathrm{var}_\eta(\gamma_1, \cdots, \gamma_n)$ depends sensitively on the **Calibration Hypothesis** and the standard convention invoked above. For example, if

$$\delta_i \in [0.9, 1), \; i = 1, \cdots, n,$$

then

$$\mathrm{var}_\eta(\gamma_1, \cdots, \gamma_n) \leqslant 31.5$$

Nevertheless, it is amusing that this primitive model with heterogeneous agents generates quasi-realistic estimates for socially relevant quantities.

REMARK 11.6 **(Equity premium / risk free rate puzzle)** *It follows from Remark* 7.5, Remark 7.8 and direct calculation that the calibrated, homogeneous model has the following property: either the equity premium is too small when $\gamma < \mathfrak{G}$ and $\delta < 1$ are chosen so that the risk free rate is small, or the risk free rate is too large when $\gamma < \mathfrak{G}$ is chosen so that the equity premium is large. That is, varying a homogeneous economy in a homogeneous direction gives no insight into the equity premium/risk free rate puzzle.

One of the main points of this paper is that one can obtain a great deal of insight into this "puzzle" and other stylized facts by varying a homogeneous economy in a heterogeneous direction.

To ensure that the equity premium goes up and the risk free rate goes down, the direction (Γ, Δ) should lie in the intersection of the regions defined by the inequalities (10.4) and (10.1), that is

$$\frac{1}{\delta\left(\ell(\mathrm{p})+\mathrm{A}_2^{\mathrm{F}}\right)} \frac{\mathrm{var}_\eta(\Delta)}{\mathrm{var}_\eta(\Gamma)} + \delta \frac{\ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\left(\ell(\mathrm{p})+\mathrm{A}_2^{\mathrm{F}}\right)} > \frac{\mathrm{cov}_\eta(\Gamma,\Delta)}{\mathrm{var}_\eta(\Gamma)} > \delta \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}')+\mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} \tag{11.6}$$

By Assumption 2 and Remark 7.8, the denominator $\ell(\mathrm{p}) + \mathrm{A}_2^{\mathrm{F}}$ is strictly positive when $\gamma < \mathfrak{G}$. Our assumptions also imply

$$\frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}')+\mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} > \frac{\ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\left(\ell(\mathrm{p})+\mathrm{A}_2^{\mathrm{F}}\right)} \tag{11.7}$$

It follows from these observations that a pre-requisite for forcing the equity premium to increase and the risk free rate to decrease is the constraint on the ratio of variances

$$\frac{\mathrm{var}_\eta(\Delta)}{\mathrm{var}_\eta(\Gamma)} > \delta^2\left(\ell(\mathrm{p})+\mathrm{A}_2^{\mathrm{F}}\right)\left(\frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}')+\mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})} - \frac{\ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\left(\ell(\mathrm{p})+\mathrm{A}_2^{\mathrm{F}}\right)}\right) \tag{11.8}$$

It is interesting that the constraint (11.8) follows from the condition (10.4),

$$\frac{\mathrm{cov}_\eta(\Gamma,\Delta)}{\mathrm{var}_\eta(\Gamma)} > \delta \frac{\ell(\mathrm{p}')\mathrm{A}_2^{\mathrm{E}} - \ell(\mathrm{p})\mathrm{A}_2^{\mathrm{F}}}{\ell(\mathrm{p}')+\mathrm{A}_2^{\mathrm{E}} - \mathrm{A}_2^{\mathrm{F}} - \ell(\mathrm{p})}$$

that guarantees an increase in the equity premium and the additional assumption $\ell(\mathrm{p}) < 1/2$. This is an immediate consequence of (11.3), (11.7) and the inequality $\ell(\mathrm{p}) > \mathrm{A}_2^{\mathrm{F}}$ (see, (9.1)). In the calibrated model, $\ell(\mathrm{p}) = 0.0172$.

The preceding discussion is roughly equivalent to the intuition that one can break the "connection" in a homogeneous economy between the risk free rate and the equity premium with a strong, but not too strong, (see, (11.6)) positive covariance $\mathrm{cov}_\eta(\Gamma, \Delta)$.

REMARK 11.7 **(Equity premium / term premium)** *The yield curve of a generic heterogeneous economy decreases. That is, for a generic direction* (Γ, Δ), *the long term risk free rate decreases with* τ *(see, Remark* 11.1). Therefore, the equity premium relative to long term bonds / long term risk free rate "puzzle" evaporates.

REMARK 11.8 **(Counter Cyclical variation of the equity premium)** *An interesting consequence of heterogeneous risk aversion is that the risk aversion of the canonical representative agent becomes a decreasing function of his consumption (see, Benninga and Mayshar (2000)). Intuitively, this property of the representative agent should force the equity premium to vary counter cyclically with the aggregate endowment. Theorem* 10.4 and Theorem 10.5 confirm this intuition.

REMARK 11.9 **(Pro cyclical price dividend ratios, counter cyclical conditional variance of returns and negative autocorrelation of returns)** [(F5), (F8), (F9)] *It is surprising that the regions in which these three stylized facts hold are identical and determined by the simple inequality* (10.6),

$$\frac{\mathrm{cov}_\eta(\Gamma, \Delta)}{\mathrm{var}_\eta(\Gamma)} < \delta \mathrm{A}_2^{\mathrm{E}}$$

This suggests, that there is a "hidden connection" between the pro cyclical variation of price dividend ratios, the negative autocorrelation of asset returns and the counter cyclical variation of the return volatility.

Observe that, by Proposition 9.1, condition (10.6) is compatible with (10.4) if and only if $\gamma \leqslant 1$. Recall, that (10.4) forces the equity premium relative to short term bonds up. On the other hand, (10.6) is always compatible with the weaker condition (10.5) that forces the equity premium relative to long term bonds up.

It follows from the preceding discussion that for $\gamma < 1$, there is an open region of directions (Γ, Δ) in which (F5), (F8) and (F9) are valid and simultaneously both equity premia go up. For $\gamma > 1$, there is no region of this kind, some economically desirable feature is lost. We do not know what to make of this dichotomy between economies with average risk aversion less then one and those with average risk aversion larger than one.

REMARK 11.10 **(Positive autocorrelation of equity prices and negative correlations of prices with returns)** *The moral of Theorem* 10.4, parts (F6) and (F7), is that heterogeneous risk aversion all by itself generates (1) a positive autocorrelation of equity price dividend ratios and (2) a negative correlation of price dividend ratios with equity returns. Here, it is important that the aggregate endowment process is stationary and positively autocorrelated.

REMARK 11.11 **(Conditional correlation of returns with consumption growth and changes in price-dividend ratios)** [(F10), (F11)] *The really important point here is that weak heterogeneity alone does not generate non trivial dynamics.*

REMARK 11.12 *It is also possible to determine the behavior of the normalized, log risk free rates and the normalized, log equity premia as* t_1 *becomes large. This information allows us to analyze the quantities*

$$E\left[\frac{\sum_{t=1}^{T}\tau^{-1}\log \mathrm{R}^{\mathrm{F}}(t,t+\tau)}{T}\right]$$

and

$$E\left[\frac{\sum_{t=1}^{T}\tau^{-1}\left(\log \mathrm{R}^{\mathrm{E}}(t,t+\tau)-\log \mathrm{R}^{\mathrm{F}}(t,t+\tau)\right)}{T}\right]$$

as T *becomes large. Mehra and Prescott (1985) discuss the special case when* $\tau = 1$. *We have,*

(F1) *For all sufficiently large* t, *the expected normalized, log risk free rate*

$$E\left[\tau^{-1}\log \mathrm{R}^{\mathrm{F}}(t,t+\tau)\right]$$

is smaller than that in the homogeneous economy in the direction (Γ,Δ) *if and only if* (10.1) *is fulfilled.*

(F3) *For all sufficiently large* t, *the expected normalized, log equity premium*

$$E\left[\tau^{-1}\log\frac{\mathrm{R}^{\mathrm{E}}(t,t+\tau)}{\mathrm{R}^{\mathrm{F}}(t,t+\tau)}\right]$$

is larger than that in the homogeneous economy in the direction (Γ,Δ) *if and only if*

$$\frac{\mathrm{cov}_\eta(\Gamma,\Delta)}{\mathrm{var}_\eta(\Gamma)} > \delta\,\ell(\mathrm{p})$$

Observe that γ *does not appear on either side of this inequality.*

An application of the Cauchy-Schwartz inequality to the covariance $\mathrm{cov}_\eta(\Gamma, \Delta)$, *yields the inequality*

$$\frac{\mathrm{var}_\eta(\Gamma)}{\mathrm{var}_\eta(\Delta)} < \frac{1}{(\delta\,\ell(\mathrm{p}))^2} \tag{11.9}$$

Observe that, by (10.2), *the risk free rate automatically decreases when the inequality* (11.9) *holds! Thus, the region in which the risk free rate decreases automatically contains the region in which the equity premium increases.*

We summarize Remarks 11.1, 11.4, 11.6, 11.8, 11.9, 11.10 in

THEOREM 11.13 *Suppose that the risk aversion* γ *of the best homogeneous approximation is less then one,* $\gamma < 1$. *Let also* $\mathscr{H}$ *be the set of weakly heterogeneous economies* $((\gamma_1, \delta_1), \cdots, (\gamma_n, \delta_n))$ *satisfying inequalities* (10.4), (10.6) *and* (10.1). *Then, this set is open and nonempty and the stylized facts (F4), (F5), (F6), (F7), (F8), (F9) are simultaneously valid for sufficiently long horizon. Moreover, for the economies in this open set, equity premia are larger and risk free rates are smaller than those of the best homogeneous approximation.*

Our discussion of equity and bonds in a weakly heterogeneous economy is now finished.

12 A QUICK LOOK AT EUROPEAN CALL OPTIONS

So far, we have purposely ignored all assets apart from equity, modelled by the Lucas tree asset, and risk free bonds. Now, we briefly discuss the influence of heterogeneity on European call options. See, Bharma, Koga, and Uppal (2002), Benninga and Mayshar (2000) for a discussion of option prices and heterogeneity.

Recall that the price Q_t at time $t \geqslant 0$ of a general asset with dividend process D_t, $t \geqslant 0$, is given by

$$Q_t \;=\; E_t \left[\sum_{\tau=1}^{\infty} \frac{M_{t+\tau}}{M_t} \, D_{t+\tau} \right]$$

The dividend process for the "European call option on equity with strike K and maturity s" is by definition

$$D_t(K, s) \;=\; \delta_{ts}(P_s - K)^+$$

Here, δ_{ts} is the Kronecker symbol and

$$f^+ \;=\; \max(f, 0)$$

is the positive part of the function f. The price $\mathrm{Call}(K, t, t+\tau)$ at time $t \geqslant 0$ for the European Call option with strike K and maturity $t+\tau$ is

$$\mathrm{Call}(K, t, t+\tau) = E_t\left[\sum_{s=1}^{\infty} \frac{M_{t+s}}{M_t} D_{t+s}(K, t+\tau)\right] \tag{12.1}$$

$$= E_t\left[\frac{M_{t+\tau}}{M_t}(P_{t+\tau} - K)^+\right] \tag{12.2}$$

It is instructive to compute $\mathrm{Call}_{\mathrm{h}}(K, t, t+\tau)$ for the homogeneous economy with parameters (γ, δ) and the geometric random walk aggregate endowment process W_t, $t \geqslant 0$. See, Definition 6.1. In this case, the state price density, the price of equity and the risk free rate are given by

$$M_t = \delta^t W_t^{-\gamma}$$

$$P_{\mathrm{h}\,t} = e^{\mathscr{B}(1-\gamma,\delta)}(1 - e^{\mathscr{B}(1-\gamma,\delta)})^{-1} W_t$$

$$\mathrm{r}_{\mathrm{h}}^{\mathrm{F}}(\mathrm{t}, \mathrm{t}+\tau) = e^{-\tau\mathscr{B}(\gamma,\delta)}$$

where

$$\mathscr{B}(\gamma,\delta) = \log\delta + \log(\mathrm{p}\,\mathrm{u}^{-\gamma} + \mathrm{q}\,\mathrm{d}^{-\gamma})$$

is the log price of the risk free bond in the homogeneous economy (γ, δ). As usual, the normalization $W_0 = 1$ is imposed. See, Theorem 7.3 and Theorem 7.9.

Substituting,

$$\mathrm{Call}_{\mathrm{h}}(K, t, t+\tau) \tag{12.3}$$

$$= \delta^\tau E_t\left[(W_{t+\tau}/W_t)^{-\gamma}(P_{\mathrm{h}\;t+\tau} - K)^+\right] \tag{12.4}$$

$$= \delta^\tau \sum_{0 \leqslant l < \mathrm{L}} \binom{\tau}{l} \mathrm{p}^{\tau-l}\,\mathrm{q}^l\,\mathrm{u}^{-\gamma(\tau-l)}\,\mathrm{d}^{-\gamma l}\left(P_{\mathrm{h}\,t}\,\mathrm{u}^{\tau-l}\,\mathrm{d}^l - K\right) \tag{12.5}$$

where

$$\mathrm{L} = \frac{\log\left(\frac{P_{\mathrm{h}\,t}}{K}\right) + \tau\log\mathrm{u}}{\log(\mathrm{u}\,\mathrm{d}^{-1})} \tag{12.6}$$

It is convenient to make two definitions.

DEFINITION 12.1 *As usual,* W_t, $t \geqslant 0$ *is the geometric random walk*

$$W_t = X_1 \cdots X_t$$

Again, X_s, $s \geqslant 1$, *are independent identically distributed random variables that take the value* u *with probability* r *and the value* d *with probability* $1-\mathrm{r}$. *By definition,* $\mathrm{Prob}^{(\mathrm{r})}$ *is the unique measure on* Ω *satisfying*

$$\mathrm{Prob}^{(\mathrm{r})}\left[\left\{\begin{smallmatrix}\text{paths that step up at } s_1, \cdots, s_k \leqslant t \\ \text{and at no other times before } t\end{smallmatrix}\right\}\right] = \mathrm{r}^k(1-\mathrm{r})^{t-k}$$

for any $t \geqslant 1$ *and all* $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant t$.

We denote by $E^{(\mathrm{r})}[\cdot]$ *the expectation with respect to* $\mathrm{Prob}^{(\mathrm{r})}$. *Similarly,* $\mathrm{cov}^{(\mathrm{r})}$ *and* $\mathrm{var}^{(\mathrm{r})}$ *stand for the covariance and variance with respect to* $\mathrm{Prob}^{(\mathrm{r})}$.

DEFINITION 12.2 *Let* $\mathrm{p} \in (0,1)$ *and* $\mathrm{u} > \mathrm{d} > 0$. *For all real* γ,

$$\mathrm{p}_\gamma = \frac{\mathrm{p}\,\mathrm{u}^{-\gamma}}{\mathrm{p}\,\mathrm{u}^{-\gamma} + \mathrm{q}\,\mathrm{d}^{-\gamma}}$$

$$\mathrm{q}_\gamma = 1 - \mathrm{p}_\gamma$$

and

$$\pi(\tau,\gamma) = \mathrm{Prob}^{(\mathrm{p}_\gamma)}\left[\, P_{\mathrm{h}\,t+\tau} > K \,|\, P_{\mathrm{h}\,t}\,\right] = \sum_{0 \leqslant l \leqslant L} \binom{\tau}{l} \mathrm{p}_\gamma^{\tau-l}\, {q_\gamma}^{l}$$

We have consciously suppressed the dependence on K *and* $P_{\mathrm{h}\,t}$. *Here, the quantity* $\mathrm{Prob}^{(\mathrm{p}_\gamma)}[\,\cdot \mid P_{\mathrm{h}\,t}]$ *is the conditional probability given the price of equity at the moment* t.

THEOREM 12.3 *Let* W_t *be the geometric random walk, introduced above, with probability* p *of stepping up and let* $\rho_t = t^{-1} \log W_t$ *be corresponding growth rate. Let* (γ, δ) *be a fixed homogeneous economy. Set*

$$\begin{aligned} \mathrm{Call}_{\mathrm{h}}(K, t, t+\tau) &= P_{\mathrm{h}\,t}\, e^{\tau \mathscr{B}(\gamma-1,\delta)}\, \pi(\tau,\gamma-1) - K\, e^{\tau \mathscr{B}(\gamma,\delta)}\, \pi(\tau,\gamma) \\ &= \mathrm{Call}_{\mathrm{h}1} - \mathrm{Call}_{\mathrm{h}2} \end{aligned} \tag{12.7}$$

and introduce

$$\mathrm{A}_2^{\mathrm{opt}} = \tau^{-1} \frac{\partial}{\partial\gamma} \mathrm{Call}_{\mathrm{h}}(K, t, t+\tau) - \tau^{-1} e^{\tau \mathscr{B}(\gamma-1,\delta)}\, \pi(\tau,\gamma-1) \frac{\partial}{\partial\gamma} P_{\mathrm{h}\,t}$$

$$\mathrm{Call}_{\mathrm{h}3} = \tau^{-1} P_{\mathrm{h}\,t} \frac{\partial}{\partial\gamma}\left(e^{\tau \mathscr{B}(\gamma-1,\delta)}\, \pi(\tau,\gamma-1)\right)$$

Here, as in Theorem 7.9, $P_{\mathrm{h}\,t}$ is the price of equity in the homogeneous economy (γ,δ).

Suppose, (γ,δ) is the best homogeneous approximation to

$$(\gamma,\delta) + \varepsilon(\Gamma_i, \Delta_i), \qquad i = 1, \cdots, n$$

and that $K \neq P_{\mathrm{h\,t}}\mathrm{u}^{\tau-l}\mathrm{d}^l$ for all $l = 0, \cdots, \tau$. Fix t and τ. For all sufficiently small ε, the option price is given by

$$\mathrm{Call}(K, t, t+\tau) = \mathrm{Call}_{\mathrm{h}}(K, t, t+\tau) + \varepsilon^2 \tau \left(t\mathrm{B}_1^{\mathrm{opt}} + \tau \mathrm{B}_2^{\mathrm{opt}} + \mathrm{B}_3^{\mathrm{opt}}\right) + O(\varepsilon^3)$$

where

$$\begin{aligned} \gamma \mathrm{B}_1^{\mathrm{opt}} = &- \mathrm{A}_2^{\mathrm{opt}} \mathrm{var}_\eta(\Gamma)\, \rho_t + \delta^{-1}\left(\mathrm{A}_2^{\mathrm{opt}} - \mathrm{Call}_{\mathrm{h}}(K, t, t+\tau)\,\rho_t\right) \mathrm{cov}_\eta(\Gamma,\Delta) \\ &+ \delta^{-2}\, \mathrm{var}_\eta(\Delta)\, \mathrm{Call}_{\mathrm{h}}(K, t, t+\tau) \\ &- \tau^{-1}\, \mathrm{B}_1^{\mathrm{P}}\, \mathrm{Call}_{\mathrm{h}1} \end{aligned}$$

and

$$\gamma\, \mathrm{B}_2^{\mathrm{opt}} \;=\; -\; \mathrm{A}_2^{\mathrm{opt}} \left(\mathrm{var}_\eta(\Gamma)\, \rho_t \;-\; \delta^{-1}\, \mathrm{cov}_\eta(\Gamma\,,\Delta)\right) \\ +\; \delta^{-2}\, \mathrm{var}_\eta(\Delta)\, \mathrm{Call}_{\mathrm{h}}\, (K\,,\, t\,,\, t+\tau) \;-\; \tau^{-1}\, \mathrm{C}_2^{\mathrm{opt}}$$

with

$$\mathrm{C}_2^{\mathrm{opt}} \;=\; \mathrm{C}_{21}^{\mathrm{opt}}\, \mathrm{var}_\eta(\Delta) \;+\; \mathrm{C}_{22}^{\mathrm{opt}}\, \mathrm{cov}_\eta(\Gamma\,,\Delta) \;+\; \mathrm{C}_{23}^{\mathrm{opt}}\, \mathrm{var}_\eta(\Gamma)$$

and

$$\mathrm{C}_{21}^{\mathrm{opt}} = -\frac{\mathrm{A}_1^{\mathrm{P}}}{\delta}\, \mathrm{Call}_{\mathrm{h}\,1} \\ +\; \left(2^{-1}\,\gamma\,\delta^{-2}\; (1 \;+\; 2\,\gamma^{-1}\; \mathscr{E}(\xi))\right)\; \mathrm{Call}_{\mathrm{h}}\, (K\,,\, t\,,\, t+\tau)$$

$$\mathrm{C}_{22}^{\mathrm{opt}} = -\frac{\mathrm{A}_2^{\mathrm{P}}}{\delta}\, \mathrm{Call}_{\mathrm{h}\,1} \;+\; \mathrm{A}_1^{\mathrm{P}}\, \mathrm{Call}_{\mathrm{h}\,3} \\ -\; \mathrm{A}_2^{\mathrm{opt}}\, \delta^{-1}\; \mathscr{E}(\xi) \;+\; \delta^{-1}\; (\mathscr{E}(\zeta) - 1)\; \mathrm{Call}_{\mathrm{h}}\; (K\,,\, t\,,\, t+\tau)$$

$$\mathrm{C}_{23}^{\mathrm{opt}} = \mathrm{A}_2^{\mathrm{P}}\, \mathrm{Call}_{\mathrm{h}\,3} \;-\; \mathrm{A}_2^{\mathrm{opt}}\; (1 - \mathscr{E}(\zeta))$$

Finally,

$$\mathrm{B}_3^{\mathrm{opt}} = \tau^{-1}\, \mathrm{Call}_{\mathrm{h}\,1}\, \mathrm{B}_3^{\mathrm{P}}$$

See, Theorem 7.9 for the definitions of $\mathrm{B}_1^{\mathrm{P}}$, $\mathrm{B}_3^{\mathrm{P}}$.

REMARK 12.4 *There are many stylized facts about options. In principle, Theorem* 12.3 puts us in position to determine the status of such stylized facts. But, this requires more than the "quick look" we have permitted ourselves.

REMARK 12.5 *Let W_t be a geometric random walk with log normal steps*

$$W_t \;=\; X_1 \cdots X_t$$

Here, $\log X_t\,,\; t \;\geqslant\; 1$, *are identically distributed normal random variables with mean μ and volatility (variance) σ.*

In the homogeneous economy $(\gamma\,,\delta)$, the state price density, the price of equity and the risk free rate now become

$$M_t \;=\; \delta^t\, W_t^{-\gamma}$$

$$P_{\mathrm{h}\,t} \;=\; \delta\, e^{\mu\,(\gamma-1)\,-\,0.5\,(\gamma-1)^2\,\sigma^2} \left(1 - \delta\, e^{\mu\,(\gamma-1)\,-\,0.5\,(\gamma-1)^2\,\sigma^2}\right)^{-1}\, W_t$$

$$\mathrm{r}_{\mathrm{h}}^{\mathrm{F}}(\mathrm{t}\,,\mathrm{t}+\tau) \;=\; e^{r\,\tau}$$

where, we introduce the local notation

$$r = -\log \delta - \mu \gamma + 0.5 \gamma^2 \sigma^2$$

Substituting,

$$\begin{aligned}
&\mathrm{Call}_{\mathrm{h}}(K, t, t+\tau) \\
&= \delta^{\tau} E_t \left[(W_{t+\tau}/W_t)^{-\gamma} (P_{\mathrm{h}\ t+\tau} - K)^+ \right] \\
&= \frac{1}{\sqrt{2\pi\sigma^2 \tau}} \int_{\log(K/P_{\mathrm{h}\,t})}^{+\infty} e^{-\gamma x} (P_{\mathrm{h}\,t} e^x - K)\, e^{-\frac{(x-\tau\mu)^2}{2\sigma^2 \tau}}\, dx
\end{aligned}$$

We now make a standard calculation. For this purpose, set

$$x = \tau \mu - \sigma \sqrt{\tau}\, y$$

Changing variables in the last integral, one obtains

$$\begin{aligned}
&\mathrm{Call}_{\mathrm{h}}(K, t, t+\tau) \\
&= P_{\mathrm{h}\,t} \frac{e^{-\gamma\tau\mu}}{\sqrt{2\pi}} \int_{-\infty}^{(\sigma\sqrt{\tau})^{-1}(\log(P_{\mathrm{h}\,t}/K)+\tau\mu)} e^{\gamma\sigma\sqrt{\tau} y}\, e^{\tau\mu - \sigma\sqrt{\tau}\, y}\, e^{-y^2/2}\, dy \\
&- K \frac{e^{-\gamma\tau\mu}}{\sqrt{2\pi}} \int_{-\infty}^{(\sigma\sqrt{\tau})^{-1}(\log(P_{\mathrm{h}\,t}/K)+\tau\mu)} e^{\gamma\sigma\sqrt{\tau} y}\, e^{-y^2/2}\, dy \\
&= P_{\mathrm{h}\,t}\, e^{(1-\gamma)\tau\mu + 0.5(1-\gamma)^2\sigma^2\tau} N\left(\frac{\log(P_{\mathrm{h}\,t}/K) + \tau\mu + (1-\gamma)\sigma^2\tau}{\sigma\sqrt{\tau}} \right) \\
&- K\, e^{-\gamma\tau\mu + 0.5\gamma^2\sigma^2\tau} N\left(\frac{\log(P_{\mathrm{h}\,t}/K) + \tau\mu - \gamma\sigma^2\tau}{\sigma\sqrt{\tau}} \right)
\end{aligned}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx$$

The final difference is referred to as the Black - Schoules formula for the European Call Option. The "Black - Schoules formula" can also be obtained indirectly by approximating the continuous log normal distribution by the discrete binomial distribution and passing to the limit in expression (12.7)*. See, Cox, Ross, and Rubinstein (1979).*

References

Aase, K. K. (1993): "Equilibrium in a Reinsurance Syndicate; Existence, Uniqueness and Characterization," *ASTIN Bulletin*, 23(2), 185–211.

Benninga, S., and J. Mayshar (2000): "Heterogeneity and Option Pricing," *Review of Derivatives Research*, 4(1), 7–27.

Bharma, H., L. Koga, and R. Uppal (2002): "Non-redudndant derivatives in a dynamic general equilibrium economy," working paper.

Brav, A., G. M. Constantinides, and C. C. Geczy (2002): "Asset Pricing with Heterogenous Consumers and Limited Participation: Empirical Evidence," *Journal of Political Economy*, 110(4), 793–824.

Campbell, J. Y. (2003): "Consumption-Based Asset Pricing," in *Handbook of the Economics of Finance*, ed. by G. Constantinides, M. Harris, and R. Stulz, vol. 1, pp. 803–887. North-Holland, Amsterdam.

Campbell, J. Y., and J. H. Cochrane (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107(2), 205–251.

Chan, Y. L., and L. Kogan (2002): "Catching Up with the Joneses: Heterogenous Preferences and the Dynamics of Asset Prices," *Journal of Political Economy*, 110(6), 1255–1285.

Constantinides, G. M. (1982): "Intertemporal asset pricing with heterogeneous consumers and without demand aggregation," *Journal of Business*, 55(2), 253–267.

Constantinides, G. M., and D. Duffie (1996): "Asset Pricing with Heterogenous Consumers," *Journal of Political Economy*, 104(2), 219–240.

Cox, J., S. Ross, and M. Rubinstein (1979): "Option Pricing: A Simplified Approach," *Journal of Financial Econometrics*, 7, 229–263.

Dana, R.-A. (1993a): "Existence and Uniqueness of Equilibria when Preferences are Additively Separable," *Econometrica*, 61(4), 953–957.

——— (1993b): "Existence, Uniqueness and Determinacy of Arrow-Debreu Equilibria in Finance Models," *Journal of Mathematical Economics*, 22.

Duffee, G. (forth.): "Time-variation in the covariance between stock returns and consumption growth," *Journal of Finance*.

Duffie, D., and C.-F. Huang (1985): "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities," *Econometrica*, 53(6), 1337–1356.

DUMAS, B. (1989): "Two-Person Dynamic Equilibrium in the Capital Market," *Review of Financial Studies*, 2(2), 157–188.

GOLLIER, C., AND R. J. ZECKHAUSER (2005): "Aggregation of Heterogeneous Time Preferences," *Journal of Political Economy*, 113(4), 878–896.

GUESNERIE, R., AND J.-Y. JAFFRAY (1974): "Optimality of Equilibrium of Plans, Prices, and Price Expectations," in *Allocation under Uncertainty: Equilibrium and Optimality*, ed. by J. Drèze, pp. 71–86. MacMillan, London.

KARATZAS, I., J. P. LEHOCZKY, AND S. E. SHREVE (1990): "Existence and Uniqueness of Multi-Agent Equilibria in a Stochastic, Dynamic Consumption/Investement Model," *Mathematics of Operations Research*, 15, 80–128.

KOGAN, L., I. MAKAROV, AND R. UPPAL (2003): "The Equity Risk Premium and the Riskfree Rate in an Economy with Borrowing Constraints," working paper.

KOGAN, L., AND R. UPPAL (2001): "Risk aversion and optimal portfolio policies in partial and general equilibrium economies," working paper, NBER.

KREPS, D. M. (1982): "Multiperiod Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model," in *The Economics of Uncertainty and Information*, ed. by J. McCall. University of Chicago Press, Chicago.

LENGWILER, Y. (2005): "Heterogenous Patience and the Term Structure of Real Interest Rates," *American Economic Review*, 95(3), 890–896.

LENGWILER, Y., S. MALAMUD, AND E. TRUBOWITZ (2005): "Asset Pricing in Heterogeneous Economies," Working paper http://ssrn.com/abstract=755984, SSRN.

LUCAS, R. E. J. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46(6), 1429–1445.

MALAMUD, S., W. PETERSEN, AND E. TRUBOWITZ (2006): "Asset Pricing in Strongly Heterogeneous Economies," Working paper, ETH-Zürich, Department of Mathematics.

MALAMUD, S., AND E. TRUBOWITZ (2006): "Asset Pricing in Heterogeneous Economies II. The Long Horizon Behavior," *submitted.*

MAS-COLELL, A., AND W. ZAME (1991): "Equilibrium Theory in Infinite Dimensional Spaces," in *Handbook on Mathematical Economics*, ed. by W. Hildenbrand, and H. Sonnenschein, vol. IV, chap. 34, pp. 1835–1898. North-Holland, Amsterdam.

MEHRA, R., AND E. C. PRESCOTT (1980): "Recursive Competitive Equilibrium: The Case of Homogeneous Households," 48, 1365–1379.

——— (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15(2), 145–161.

ROSS, S. (1989): "Intertemporal asset pricing," in *Theory of valuation, Frontiers of modern financial theory*, ed. by S. Bhattachariya, and G. M. Constantinides, vol. 1, pp. 85–96.

RUBINSTEIN, M. (1974): "An Aggregation Theorem for Securities Markets," *Journal of Financial Economics*, 1(3), 225–244.

VAN PRAAG, B. M. S., AND A. S. BOOJI (2003): "Risk Aversion and the Subjective Time Discount Rate: A Joint Approach," Discussion Paper TI 2003–018/3, Tinbergen Institute.

WANG, J. (1993): "A model of intertemporal asset pricing under asymmetric information," *The Review of financial studies*, 60, 249–282.

——— (1994): "A model of competitive stock trading volume," *Journal of Political Economy*, 102, 127–168.

——— (1996): "The Term Structure of Interest Rates in a Pure Exchange Economy with Heterogenous Investors," *Journal of Financial Economics*, 41(1), 75–110.

WEITZMAN, M. L. (2001): "Gamma Discounting," *American Economic Review*, 91(1), 260–271.

APPENDICES

A THE MODEL

A.1 *Assets and state price densities*

Time is discrete and the horizon is infinite $t \in \{0, 1, 2, \cdots\} =: \mathbf{T}$. Uncertainty is captured by a probability space $s \in \Omega$ and an infinite filtration (information structure) $(\mathscr{F}_t)_{t=0}^{\infty}$ of σ-algebras of subsets of Ω.

There is a single commodity ("consumption"). There is a (possibly infinite) collection of assets $j \in \mathbb{N}$ who pay divedend in this single commodity. An asset is characterized by his dividend process D_{jt} and price process Q_{jt}. It is known that under some mild technical conditions (see, e.g. Duffie and Huang (1985)) the absence of arbitrage opportunities is equivalent to the existence of positive state price densities M_t, $t \geqslant 0$, such that

$$Q_{jt} = E_t\left[(Q_{j\,t+1} + D_{j\,t+1})\frac{M_{t+1}}{M_t}\right] \tag{A.1}$$

Here, E_t the conditional expectation with respect to the σ-algebra $\mathscr{F}_t$.

Moreover, the market is completeness is "almost" equivalent to the uniqueness of state price densities M_t, $t = 1, \cdots, T$. From now on we assume that the market is complete and that the state price densities are unique[7].

For the filtration, generated by the geometric random walk the situation is even simpler and the standard arguments (see, e.g. (Guesnerie and Jaffray, 1974; Kreps, 1982; Duffie and Huang, 1985)) immediately imply

PROPOSITION A.1 *Suppose that sigma-algebra $\mathscr{F}_t$ contains only a finite number of subsets for each finite t. Then there is no arbitrage if and only if there exist positive state price densities M_t, $t \geqslant 0$ and the market is dynamically complete if and only if the state price densities are unique up to constant factor.*

ASSUMPTION 3 *We assume that there are no bubbles in the asset prices. That is,*

$$\lim_{t\to\infty} E[Q_{j,t}M_t] = 0$$

The following result is an immediate consequence of (A.1).

PROPOSITION A.2 *Under Assumption 3, the price of any asset j is given by*

$$Q_{jt} = E_t\left[\sum_{\tau=1}^{\infty} \frac{M_{t+\tau}}{M_t} D_{t+\tau}\right]$$

[7] If T is finite and Ω is infinite, a complete market would require an infinite number of assets. Thus, it makes sense to work with an infinite Ω only if T is infinite. In that case, it may be possible to create a dynamically complete market by trading a finite set of assets infinitely many times (Guesnerie and Jaffray, 1974; Kreps, 1982; Duffie and Huang, 1985). Yet, there are contributions that work with an infinite Ω even with a finite T and still assume complete markets (e.g. Aase, 1993), and our model allows for this as well.

A.2 *Agents*

There is a finite collection of agents, $i \in \{1,\dots,n\} =: N$. Agent i has constant relative risk aversion (CRRA) utility function. He chooses his random consumption $x_{i\,t}$ at each time $t \geqslant 0$ and each possible state of the world to maximize expected discounted intertemporal utility function[8]

$$E\left[\sum_{t=0}^{T} \delta_i^t\, u_i(x_{i\,t})\right]$$

where

$$u_i(z) = \frac{z^{1-\gamma_i}-1}{1-\gamma_i}$$

Here, γ_i is the relative risk aversion of agent i and δ_i is his discount factor (patience). The budget constraint of agent i is determined by his individual endowment process $w_{i\,t},\ t \geqslant 0$. Namely, since, by assumption, the market is complete and the state price densities are unique, the set of admissible consumption streams $\mathbf{x}_i = (x_{i\,t})_{t\geqslant 0}$ is

$$\left\{x_{i\,t} > 0 : E\left[\sum_{t=0}^{\infty} x_{i\,t} M_t\right] = E\left[\sum_{t=0}^{\infty} w_{i\,t} M_t\right]\right\}$$

Agent i is thus fully described by the pair (γ_i, δ_i) and the endowment process $w_{i\,t},\ t \geqslant 0$.

LEMMA A.3 *The utility maximization problem for of agent i*

$$\max\left\{E\left[\sum_{t=0}^{\infty} \delta_i^t\, u_i(x_{i\,t})\right] \mid E\left[\sum_{t=0}^{\infty} x_{i\,t} M_t\right] = E\left[\sum_{t=0}^{\infty} w_{i\,t} M_t\right]\right\} \tag{A.2}$$

is solvable if and only if the state price densities satisfy

$$\sum_t E[\,w_{i\,t} M_t] < \infty \qquad \textit{and} \qquad \sum_t \delta^{t\,b_i} E[M_t^{1-b_i}] < \infty \tag{A.3}$$

In this case the optimal consumption stream $\mathbf{x}_i$ is given by

$$x_{i\,t} = M_t^{-b_i}\, \delta_i^{t\,b_i}\, x_{i\,0} \tag{A.4}$$

and

$$x_{i\,0} = \frac{\sum_{i\in N} E[\,w_{i\,t} M_t]}{\sum_t \delta_i^{t\,b_i} E[M_t^{1-b_i}]}. \tag{A.5}$$

Recall, that

$$W_t(s) := \sum_{i\in N} w_{i\,t}(s)$$

and $g_t(s) := (W_t(s)/W_0)^{1/t}$. Finally,

$$\rho_t = \log g_t = t^{-1} \log W_t$$

is the growth rate of the aggregate endowment

[8] The reciprocal of relative risk aversion, $b_i = 1/\gamma_i$ is called cautiousness

DEFINITION A.4 *State price densities* $\mathbf{M} := (M_t(s))_{s\in\Omega,\, t\in\mathbf{T}}$ *are an equilibrium if*

$$\sum_{i\in N} x_{i\,t} \;=\; W_t$$

where the optimal consumption $x_{i\,t}$ *is defined in* (A.4) *and* (A.5)

From now on we use the
Normalization. We normalize $W_0 = 1$ and $M_0 = 1$.

Thus, we have

PROPOSITION A.5 *State price densities* M_t, $t \geqslant 1$ *are an equilibrium if and only if*

$$\sum_{i\in N} M_t(s)^{-b_i}\delta_i^{t\,b_i} x_{i\,0} \;=\; W_t(s), \qquad s\in\Omega, t\in\mathbf{T}, \tag{A.6}$$

$$with \quad x_{i\,0} = \frac{\sum_{t\in\mathbf{T}} E[w_{i\,t} M_t]}{\sum_{t\in\mathbf{T}} \delta_i^{tb_i} E[M_t^{1-b_i}]}, \qquad i\in N. \tag{A.7}$$

REMARK A.6 *Observe that there are no conditional expectations in the utility maximization problem. The reason for this is that it is assumed that all decisions are made "at the beginning of time." All decisions are made based on the same amount of information. To describe the time series properties of assets, however, we make use of conditional expectations, because the price of an asset in period* t *is the value of its future dividends, conditional on the information that is available in period* t.

γ_i, δ_i	risk aversion and disc. factor of agent i. Theorem 2.1
$w_{i\,t},\ t \geqslant 0$	individual endowment process of agent i Theorem 2.1
W_t	Theorem 2.1 and Definition 6.1
(Γ, Δ)	Direction of perturbation. (3.1)
η_i	(3.2)
$\mathscr{E}$	(3.3)
ξ and ζ	Lengwiler, Malamud, and Trubowitz (2005), (E.3)
ρ_t	(3.4)
M_t	(3.5) and (A.1)
P_t	(4.1)
$\beta^{\mathrm{F}}(t_1, t_2)$	(4.2)
$\mathrm{r}_t^{\mathrm{E}}$	(4.3)
$\mathrm{r}^{\mathrm{F}}(t_1, t_2)$	(4.4)
$\mathrm{r}^{\mathrm{E}}(t_1, t_2)$	(4.5)
$\mathrm{R}^{\mathrm{E}}(t_1, t_2)$	(4.7)
$\mathrm{R}^{\mathrm{F}}(t_1, t_2)$	(4.8)
$\mathrm{u}, \mathrm{d}, \mathrm{p}, \mathrm{q}$	Definition 6.1
$\ell(\mathrm{p})$	Definition 7.1
p'	Definition 7.1
γ_* and γ^*	Remark 6.3
Calibration Hypothesis	the end of Section 6
$\tau^{-1} \log \mathrm{R}_{\mathrm{h}}^{\mathrm{E}}$	Theorem 7.2
$\mathrm{A}_1^{\mathrm{E}}, \mathrm{A}_2^{\mathrm{E}}$	(7.2)
$\tau^{-1} \log \mathrm{R}_{\mathrm{h}}^{\mathrm{F}} = \tau^{-1} \log \mathrm{r}_{\mathrm{h}}^{\mathrm{F}}$	Theorem 7.3
$\mathrm{A}_1^{\mathrm{F}}, \mathrm{A}_2^{\mathrm{F}}$	(7.6)
$P_{\mathrm{h}\,t} W_t^{-1} = P_{\mathrm{h}\,0}$	Theorem 7.9
$\mathrm{A}_1^{\mathrm{P}}, \mathrm{A}_2^{\mathrm{P}}$	(7.12)
$\mathfrak{G}$	Definition 7.6
$\mathrm{h}(\gamma, \delta)$	Homogeneous economy (γ, δ)
$\mathrm{Call}(K, t, t+\tau)$	(12.1)
$\mathrm{Call}_{\mathrm{h}}(K, t, t+\tau)$	(12.3)
$\mathrm{Prob}^{(\mathrm{r})}, E^{(\mathrm{r})}[\cdot], \mathrm{cov}^{(\mathrm{r})}, \mathrm{var}^{(\mathrm{r})}$	(12.1)
$\pi(\tau, \gamma)$	Definition 12.2