

National Centre of Competence in Research
Financial Valuation and Risk Management

Working Paper No. 532

An Intergenerational Cross-Country Swap

Miret Padovani

Paolo Vanini

First version: April 2009
Current version: June 2009

This research has been carried out within the NCCR FINRISK project on
“Credit Risk and Non-Standard Sources of Risk in Finance”

AN INTERGENERATIONAL CROSS-COUNTRY SWAP

MIRET PADOVANI AND PAOLO VANINI

ABSTRACT. This paper addresses the issue of intergenerational and international sharing of longevity and growth risks. Current research on worldwide demographic changes highlights the importance of longevity risk on financial markets and the need to devise optimal hedging vehicles. We present a potential financial innovation between two countries at different stages of economic development and with different long-term challenges. This 30-year-long swap is structured in such a way to capture the different timing of needed funds of the two countries and the funding capabilities of each generation: the more developed economy requires funds in the future to cover expenses for its ageing population, while the developing economy needs funds today to pay for educational, technological, and other infrastructural services. To price the swap, we apply an exponential-utility-based pricing method and define an interval of prices allowing a contract to be agreed upon. We show how the bid-ask spread varies with respect to the governments' risk and time preferences.

Date: First version: April 2009. This version: June 7, 2009.

Key words and phrases. Financial innovation; longevity risk; old-age dependency ratio; growth risk; exponential-utility-based pricing; international finance; intergenerational risk-sharing; international risk-sharing.

We are thankful to Michel Habib, Hyun Song Shin, Alexander Wagner, and participants at the seventh Swiss doctoral workshop in finance (Gerzensee, June 2008) for helpful comments on a preliminary draft of the paper. Special thanks also go to our colleagues at the Zürcher Kantonalbank for interesting discussions and feedback. Part of this research has been carried out within the research project on “Credit risk and non-standard sources of risk in finance” led by Rajna Gibson at the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK). The NCCR FINRISK is a research instrument of the Swiss National Science Foundation.

1. INTRODUCTION

Countries at different stages of economic development face different long-term challenges. On the one hand, take a country such as Switzerland, which faces the challenge of being able to provide for the pensions and long-term care of its ageing population without jeopardizing its economic competitiveness.¹ On the other hand, take Egypt, which faces the challenge of being able to attract sufficient funds and encourage new businesses to foster its economic development; Egypt is not constrained by longevity-related expenses, and higher longevity due to better health and living conditions may actually signify higher work productivity and faster economic growth. So Switzerland faces longevity risk, while Egypt faces growth risk. Rather than bear those risks, stakeholders in the two countries may benefit from transferring them to the financial markets. Yet the question is how.

We may very naively divide the world economy into two groups: developed and developing economies; or - political correctness aside - rich and poor. The former countries are short longevity, the latter are long longevity.

The contribution of this paper is threefold. We delineate the structure of a potential future financial innovation enabling one party - the rich country - to mitigate its longevity risk and the other - the poor country - to widen its pool of funding options. This intergenerational cross-country swap would mirror the different timing of needed funds of the two countries and the funding capacities of each generation. Secondly, we price the swap using an exponential-utility-based pricing method and determine the highest possible bid and lowest possible ask price. Finally, we show how the bid-ask spread varies with governments' risk and time preferences.

This swap permits risk management to be extended far beyond its former realm, covering a new class of risk: in this case, population ageing. It also changes the assumptions about what can be insured and hedged - take the old-age dependency ratio - and has a potential major impact on human welfare.²

¹See the 2008 Swiss Health Observatory study [[SBJR+08](#)].

²See the discussion on "radical financial innovations" by Shiller [[Shi04b](#)].

	2005	2020	2030	2040
Australia	AAA	AA	BBB	Non-IG
Canada	AAA	AAA	AAA	AA
France	AAA	A	Non-IG	Non-IG
Italy	AA	A	Non-IG	Non-IG
Japan	AA	Non-IG	Non-IG	Non-IG
Spain	AAA	AAA	BBB	Non-IG
Sweden	AAA	AAA	A	Non-IG
United Kingdom	AAA	AAA	A	Non-IG
USA	AAA	BBB	Non-IG	Non-IG

TABLE 1.1. Hypothetical projected long-term sovereign ratings according to baseline scenario if no adjustment in government budget occurs. *Source:* [Kra02].

A crucial aspect in this framework is the interrelationship between longevity and growth risks, since inadequate management of the high costs associated with an ageing society may lead a country to economic deterioration. The Standard & Poor’s study [Kra02] documents how ageing-related government liabilities may result in downgrades of sovereign ratings if no adjustment in government budget occurs (see table 1.1). This highlights the important role governments should take as managers of key long-term risks related to population ageing, as discussed by Groome *et al.* [GBH⁺06]. The literature investigating the economic effects of population ageing on financial markets and possible policy reforms is quite vast;³ we refer to Groome *et al.* [GBR06] for a review of the most recent research to date.

Our interest here lies in international financial innovations motivated by worldwide asymmetric demographic trends. Bryant [Bry06] and Batini *et al.* [BCM06] independently study the interactions between developed and developing countries as their populations follow different evolution paths. Their analyses show that population ageing in industrialised countries will reduce growth and negatively affect savings and investments; on the other hand, developing countries will enjoy a “demographic dividend” that should result in stronger growth over the next couple of decades, before ageing sets in. Takling pretty much the same issue, Alho and Borgy [AB07] employ a multi-regional model to analyze the uncertainty induced into key

³See, among others, Abel [Abe03] and Brooks [Bro02].

macroeconomic variables by uncertain future demographics. The authors observe that the macroeconomic adjustments can differ substantially if they consider independence or high correlation across the regions.

In a series of thought-provoking studies, Robert Shiller and his co-authors stress the desirability of an international risk transfer of economic growth risks.⁴ Consider the exemplifying “radical financial innovation” suggested by Shiller [Shi04a]:

“Suppose that a ten-year contract were made between a poor country on one side - in this example, India - and such wealthier countries as Canada, Mexico, the United States, Brazil, Japan, France, Germany, Italy, and the United Kingdom on the other, to swap unexpected future changes in Indian GDP for unexpected future changes in the combined GDPs of the other countries. (Shiller [Shi04a])”

The swap we suggest differs from the one above in that it combines growth with longevity risk. Eijffinger and Wagner [EW03] show that international risk-sharing gains through existing financial assets actually result from intertemporal rather than cross-sectional gains, particularly because of less severe incentive problems. Such evidence provides strong support for the idea that a country which is short longevity would gain if it were to transfer its longevity risk through an intergenerational agreement with another country which is long longevity. The open question this paper aims to answer is: how can one concretely and successfully structure derivatives for a simultaneous intergenerational and international risk transfer?

⁴See Shiller [Shi04a] and Athanasoulis and Shiller [AS01].

Our derivative idea is also related to several recent studies in financial engineering - namely, on longevity-based securities,⁵ economic derivatives,⁶ as well as commercial microfinance.⁷ We do not discuss this literature here, but do highlight their importance, since they give an idea of the challenges encountered when engineering, pricing, and hedging innovative financial products.

The remainder of the paper is organized as follows. The next section illustrates the structuring of the swap and justifies the choice of the old-age dependency ratio as its underlying variable. The following section applies an exponential-utility-based method to price this innovative payoff in an incomplete market. Section 4 discusses how the arbitrage-free interval of possible swap prices varies with respect to the model parameters. Section 5 concludes and outlines future research work.

2. STRUCTURING THE SWAP

The financial innovation we present is an agreement between a rich country's government and a poor country's government to exchange a sequence of cash flows at specified settlement dates. Its structure is designed in such a way to capture the temporal asymmetry between the two countries: the richer country needs funds

⁵The idea behind derivatives such as longevity bonds (see Blake and Burrows [BB01], Blake *et al.* [BCDM06]) and longevity swaps (see Dowd *et al.* [DBCD06]) is a risk-exchange contract between two parties, one being short longevity and the other being long longevity. Typical counterparties for such a contract may be the sponsor of a pension plan and a life insurer. The payoffs of these derivatives depend upon the realized value of an index of longevity rates. Several investment banks and option exchanges - namely, JPMorgan, Goldman Sachs, Credit Suisse, and Deutsche Börse - have recently developed longevity indices with the aim to contribute to the take-off of the longevity derivatives market. Mitchell *et al.* [MPSY06] provide an interesting overview of financial product innovations in this field.

⁶Economic derivatives allow investors to take direct positions on the outcomes of macroeconomic data releases. The first economic derivatives - futures and options on non-farm payroll figures - started trading in 2002 on the Chicago Mercantile Exchange but were subsequently discontinued in 2007. Among the reasons given were lack of interest and illiquidity; indeed, little disagreement on the outcome of data makes it difficult for traders to find a counterparty willing to take the other side of the trade. There is ongoing work on how to best structure financial derivatives to mitigate event risks, such as adverse fluctuations in unemployment or national income figures (see Gadanecz *et al.* [GMU07]). Similar to the longevity derivatives market, the economic derivatives market is currently struggling to take off. Economic derivatives do entail, however, lower basis risk than more conventional instruments for taking positions on macroeconomic data.

⁷The objective of commercial microfinance is to increase the set of funding alternatives available to poor countries and microfinance institutions beyond the plain intergovernmental loans they more typically receive. As an example, Bystrom [Bys08] studies microfinance collateralized loan obligations as a tool for economic development. Another type of product are microfinance investment funds.

in the future when it will have to cover expenses for the elderly, while the poorer country needs funds today to pay for educational, technological, and other infrastructural services.

The swap has a long-term duration, say 30 years as for available government debt. The rich country represents the fixed leg, the poor country the floating leg. The net cash flows from rich to poor are much higher in the first years of the contract's life, but this asymmetry is reversed over time, mirroring the different timing of needed and available funds stated above.

The swap's structure begs the question of how far ahead one can forecast growth and whether 30-year-ahead growth and longevity forecasts can be sufficiently reliable. Given the actual difficulty in forecasting growth and longevity over a period exceeding 18 months,⁸ the swap is rolled over every 10 years. This also allows the parties to take into account sovereign default risk: If country creditworthiness declines, then the swap spread can be raised at the time the contract is rolled over. But as with any rolling strategy, it is important to account for possible rolling or tracking errors and their severity.

The main advantage for the rich country in this fixed-for-floating swap is to transform future cash-outs at a floating rate - its elderly-related expenses which depend on a stochastic population ageing rate - into payments at a fixed rate, thus locking in a "sustainable" population ageing rate. As for the poor country, its main advantage is to hedge against adverse changes in the development aid it receives due to population ageing in the donor country. A long-term contract as this 30-year swap entails, though, the risk of significant fluctuations in exchange rates and interest rates. These risks may be hedged together by adding a (fixed-for-floating) cross-currency interest-rate swap, to be rolled over during the 30-year lifetime of the swap.

A couple of benefits of this intergenerational cross-country swap over a simple developmental loan from rich to poor are particularly relevant. One first benefit is

⁸See Isiklar and Lahiri [IL07].

the possibility for both parties to spread the cash out- and inflows over time. Secondly, the swap provides the possibility of including a rebate in case the contract is interrupted because of sovereign default or economic recession in either country. But this swap also offers advantages over a longevity derivative between two counterparties based in the rich country. The problem of finding a counterparty willing to take on the long-longevity side of the transaction is often cited by practitioners as the main problem currently hindering the take-off of longevity derivatives. We will show in what follows that a poor country may be willing to assume the longevity risk of a richer country if this risk is forecast to adversely affect the developmental support it receives from the rich country.

The issue of identifying the variable on which to base the swap's cash flows is not trivial, given that no existing measure to date serves our purpose fully and satisfactorily. We need a rate that reveals the strength with which longevity trends impact each country's wealth endowments.

Gross domestic product (GDP) is by far the most widely-used indicator of economic growth, but it does have its shortcomings, too. A crucial shortcoming of GDP measures is that they do not take into account life expectancy or other demographic variables such as fertility, which are important indicators of a country's well-being. What is important for the purpose of this study, however, is to quantify the burden of elderly-related expenses on an economy. These expenses tend to actually raise GDP figures through an increase in government expenditures; but, as discussed above, longevity is forecast to negatively affect the economy of the more advanced economies.

Several studies in growth theory attempt to come up with a growth measure reflecting the welfare gains from both quality and quantity of life.⁹ The aim of longevity-adjusted growth rates is to quantify the extent to which national economic growth is affected by population age structure. The literature on the relationship between income growth and life expectancy has taken off in the seventies

⁹See Becker *et al.* [BPS06].

following studies by Usher [Ush73] and Preston [Pre75]. There have been several country-specific applications since then; e.g. Ponthière [Pon08] applies the Usher-Williamson-Miller framework to include longevity data into Belgian national income accounting. In a major study in this field, Becker *et al.* [BPS06] develop a statistical model that accounts for the impact of longevity on the evolution of welfare across almost 100 countries from 1960 to 2000. Their model measures the growth of individual income plus the value placed on the growth of an individual's life expectancy. A common result of all these studies is that, by not taking into account increases in longevity, GDP underestimates the extent to which developing countries are gaining relative to developed countries.

One indicator of the economic burden of rising population longevity is the old-age dependency ratio. This gives the number of retirees in percentage of the total population in working age, i.e.

$$\text{old-age dependency ratio} = \frac{\text{population aged 65+}}{\text{population aged 15-64}}.$$

The evolution of this ratio depends on future trends in fertility, mortality and international migration. A higher forecast dependency ratio is a clear signal of an increasing burden of ageing-related expenses and of the need for governments to put in place the necessary budget reforms. Given the availability of historical and forecast data on the old-age dependency ratio for several countries and their widespread use in policy reform discussions, we choose this variable as the underlying of our swap.¹⁰

Having identified the underlying and the main features of the swap, we need to guarantee that such a long-term contract does not neglect the risk of sovereign default. A knock-out feature or credit trigger acts as an insurance against sovereign

¹⁰The United Nations Population Division (UNPD) publishes forecasts of the old-age dependency ratio for most countries in the world every year. The most recent population projects are reported in [UNP07]. To project the population until 2050, the UNPD relies on a set of assumptions regarding future trends in fertility, mortality, and international migration. Because future trends cannot be known with certainty, several projection variants are produced. We refer to the UN study [UNP07] for data on historical and forecast old-age dependency ratios around the world according to the medium, high, and low projection variants.

default and causes the payments from rich to poor to be interrupted as soon as the creditworthiness of the poor country falls below a certain level. In a worst-case scenario, the payments from rich to poor cease within a few years and can be taken as a charitable payment.

3. PRICING THE SWAP

Given the innovative nature of our swap's structure, we need to price in an incomplete market. The literature on pricing derivatives when risk-neutral valuation is not sufficient is vast. We here follow a utility-based valuation approach, which relies on the individual rationality requirement of the agent's expected utility from participation in the contract to (weakly) exceed his reservation utility. We take the exponential utility function, which implies constant relative risk aversion and linear risk tolerance. This utility function is widely employed in the literature because of its mathematical tractability. For the mathematical details of exponential utility indifference valuation we refer to Mania and Schweizer [MS05] and Frei and Schweizer [FS08].

In a complete financial market every contingent claim can be perfectly replicated by a portfolio of traded securities and therefore admits a unique arbitrage-free price. In an incomplete market, to every contingent claim is associated an interval of arbitrage-free prices and arbitrage arguments alone are not sufficient to lead to a unique price, i.e. to a replication strategy. As the lower and upper endpoints of this interval coincide with the sub- and super-replication costs of the contingent claim, respectively, any price in the middle will lead to a possible profit & loss at maturity. Hence, the choice of an arbitrage-free price must be made with respect to another criterion.

The utility indifference pricing method has been proved to provide a narrower range of price bounds than the arbitrage-free pricing method.¹¹ The pricing of the swap is based on the condition of individual rationality for both parties to enter

¹¹See Mania and Schweizer [MS05].

the deal. The participation constraints allow us then to determine an interval of prices acceptable to both seller and buyer.

We make use of the following notation:

- W_t denotes the wealth endowment of the rich government at time t - national budget set aside for pensions and long term health care - adjusted for reference population size and inflation;
- \tilde{W}_t denotes the wealth endowment of the poor government at time t - national budget set aside for infrastructure projects - adjusted for reference population size and inflation;
- s_{ask} denotes the fixed ask swap rate;
- s_{bid} denotes the fixed bid swap rate;
- $t = 0, 1, \dots, T$ denotes the time frame, with T the contract maturity;
- $U(W)$ denotes the utility function;
- $D_t(W_t)$ denotes the discount factor;
- \mathbb{P}^W denotes the probability law of the wealth dynamics;
- \mathbb{P}^a denotes the probability law of the old-age dependency ratio dynamics.

Note that we use the tilde notation $\tilde{\cdot}$ for parameters related to the poor country.

The indifference valuation criterium demands that the investor valuing a contingent claim should achieve the same expected utility both in case he does not possess the claim and in case he does possess the claim but his initial capital is reduced by the amount of indifference value of the claim. In this framework, rich and poor country must be made indifferent between bearing longevity and income growth risks without entering a swap agreement under a medium population projection variant and no longer bearing these risks after entering the swap agreement. Thus, the indifference values of the swap to the rich and the poor country are defined through

$$V(W_0, s_{bid}, a_0) \geq V(W_0, 0, 0)$$

and

$$\tilde{V}(\tilde{W}_0, s_{ask}, a_0) \geq \tilde{V}(\tilde{W}_0, 0, 0),$$

respectively.

The developed economy (or fixed leg or buyer of the swap) pays fixed value and receives realized value at each settlement date, so that its participation constraint is defined by

$$(3.1) \quad \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[\sum_{t=0}^T D_t(W_t) * U(W_t, s_{bid}, a_t) \right] \geq \mathbb{E}_0^{\mathbb{P}^W} \left[\sum_{t=0}^T D_t(W_t) * U(W_t, 0, 0) \right].$$

This gives the maximum bid price. We simplify the pricing framework by assuming that the government spends its entire budget; hence, we do not include a savings rate.

The developing economy (or floating leg or seller of the swap) pays realized value and receives fixed value at each settlement date, so that its participation constraint is defined by

$$(3.2) \quad \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[\sum_{t=0}^T D_t(\tilde{W}_t) * U(\tilde{W}_t, s_{ask}, a_t) \right] \geq \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[\sum_{t=0}^T D_t(\tilde{W}_t) * U(\tilde{W}_t, 0, 0) \right].$$

This gives the minimum ask price. These two participation constraints allow us to derive a range of prices within which the swap price (i.e. the fixed swap rate) needs to lie.

We describe social welfare by the negative exponential utility function:

$$(3.3) \quad U(x_t) = -e^{-\lambda x_t}, \quad \lambda \in (0, \infty).$$

Each country discounts its future utility according to its specific social discount rate. The discounted marginal utility of government income is specified by

$$u = U'(W_t) e^{-\rho t} = \lambda e^{-\lambda W_t} e^{-\rho t}$$

and the social discount rate by

$$(3.4) \quad d_t = -\frac{\dot{u}}{u} = \rho + \lambda \dot{W}_t \approx \rho + \lambda * (W_t - W_{t-1}),$$

where ρ denotes the social rate of time preference, representing the value society attaches to present consumption relative to future consumption. Hence, the social discount factor is given by

$$(3.5) \quad D_t(W_t) = e^{-d_t t} = e^{-\rho t - \lambda^*(W_t - W_{t-1})t}.$$

The environmental finance literature has highlighted several reasons why it is inappropriate to simply use market risk-free long-term interest rates, such as the rates on government bonds with equivalent maturity.¹² The primary reason is intergenerational concerns, which a government is supposed to take into account when initiating projects that are deemed to affect future generations. The importance given to future generations depends on current wealth and expected future wealth levels. If age structure changes are expected to slow down future economic growth rates, the social discount rate should decline accordingly. This translates into greater sacrifices on behalf of the current generation to take into account future generations' well-being. Pearce *et al.* [PGHK03] argue that in the extreme case of projected long-term recession, the social discount rate should be negative.¹³

Proposition 3.1. *Consider the two participation constraints (3.1) and (3.2), the utility function (3.3), and the discount factor (3.5). Then the fixed swap rate, s , must lie within a range with upper bound*

$$(3.6) \quad \bar{s}_{bid} = \frac{1}{\lambda} \left(\ln \sum_{t=0}^T e^{-\rho t} \mathbb{E}_0^{\mathbb{P}_W} \left[e^{-\lambda W_t(1+t) + \lambda W_{t-1}t} \right] - \ln \sum_{t=0}^T e^{-\rho t} \mathbb{E}_0^{\mathbb{P}_W \times \mathbb{P}_a} \left[e^{-\lambda W_t(1+t) + \lambda W_{t-1}t - \lambda a_t} \right] \right)$$

¹²See Pearce *et al.* [PGHK03].

¹³The question whether society should place a lower value on a future gain or loss than on the same gain or loss occurring now is highly controversial. A large body of literature analyzes social discounting for environmental policies and how governments should value climate change damages; see, among others, Weitzman [Wei01], Weitzman [Wei07], Pearce *et al.* [PGHK03], and Hepburn [Hep06]. Application examples include the *Stern review on the economics of climate change* and the cost-benefit analyses employed by the Copenhagen Consensus to examine solutions to ten of the world's biggest challenges.

and lower bound

$$(3.7) \quad \begin{aligned} s_{ask} = & \frac{1}{\tilde{\lambda}} \left(\ln \sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[e^{-\tilde{\lambda}\tilde{W}_t(1+t) + \tilde{\lambda}\tilde{W}_{t-1}t + \tilde{\lambda}a_t} \right] \right. \\ & \left. - \ln \sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda}\tilde{W}_t(1+t) + \tilde{\lambda}\tilde{W}_{t-1}t} \right] \right). \end{aligned}$$

Proof. See appendix section A.1. \square

Equations (3.6) and (3.7) show that the price bounds are a function of λ , $\mathbb{E}_0^{\mathbb{P}^W} [W_t]$, and $\mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} [W_t, a_t]$ for the bid price; $\tilde{\lambda}$, $\mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} [\tilde{W}_t]$, and $\mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} [\tilde{W}_t, a_t]$ for the ask price. Furthermore, these price bounds need to satisfy a couple of conditions. First of all, both the minimum price the seller is willing to receive for the swap and the maximum price the buyer is willing to pay for the swap must be positive. Secondly, the maximum bid price should not be lower than the minimum ask price; otherwise, no agreement between the two parties is possible.

The next step in pricing the swap is to define the stochastic paths of the three sources of uncertainty, a_t , W_t , and \tilde{W}_t . Forecasts for years 2010 up to 2030 show a clear upward trend in population ageing in the industrial countries, though this trend may experience ‘up’ or ‘down’ episodes, for example, due to extreme heat waves. We do not get into the details of longevity forecasting, but nevertheless highlight a few contentious points, such as whether and how medical advances should be taken into account, whether or not there exists a biological limit to life, and whether lifespans could eventually be extended through genetic changes or non-genetic interactions. Summing up, we may represent this series by the following upward-trending process:

$$(3.8) \quad a_t = a_0 + \delta t + \epsilon_t^a,$$

where the error term, ϵ_t^a , follows an autocorrelation process with a lag of order k :

$$\epsilon_t^a = \sum_{i=1}^k \varsigma_i \epsilon_{t-i}^a + \sigma^a \xi_t^a, \quad \xi^a \sim \mathcal{N}(0, 1).$$

Fitting this process to historical and forecast data for Switzerland for years 1991 to 2050, we have derived parameter values $a_0 = 23.4$ and $\delta = 0.45$.¹⁴ In a future companion paper, we will estimate the parameters of the autocorrelation process and the lag order for a number of selected countries.

Evolution of the wealth endowment over time in the developed country can be thought to follow a mean-reverting process, converging to some long-term value. The wealth level is adjusted for inflation and for population growth, and so is expressed in real per-capita terms. This value of mean reversion can be thought of as a proportion of national income the government desires to devote to elderly-related expenses, e.g. long-term health care. Hence, this process needs to mirror the evolution of per-capita national income in real terms.

We may take the following AR(1) process:

$$(3.9) \quad W_t = \alpha + \beta W_{t-1} + \epsilon_t, \quad 0 \leq \beta < 1,$$

where $1 - \beta$ measures the speed of mean reversion in the rate of wealth changes. The error term is given by

$$(3.10) \quad \epsilon_t = \sigma \xi_t + \sigma \sum_{i=0}^m \eta_i \xi_{t-i}^a, \quad \xi, \xi^a \sim \mathcal{N}(0, 1).$$

with $0 \leq m \leq +\infty$ and where ξ_t captures the randomness due to all factors other than the uncertainty related to the population age structure; $\xi_t^a, \dots, \xi_{t-m}^a$ are the current and lagged innovations of the old-age dependency ratio; η_t are the sensitivity parameters between the wealth process and population ageing. In the extreme case of W and a being independent, i.e. when population ageing has no effect on the wealth endowment, the sensitivity parameters η_t are all equal to zero, so that the error term is simply an i.i.d. standard normal random variable, i.e. $\epsilon_t = \sigma \xi_t$. The

¹⁴We have used publicly-available data from the Bundesamt für Statistik and the State Secretariat for Economic Affairs.

solution to (3.9) is

$$(3.11) \quad W_t = \alpha \sum_{i=0}^{t-1} \beta^i + \beta^t W_0 + \sum_{i=0}^{t-1} \beta^i \epsilon_{t-i}.$$

For the developing economy we take the following AR(1) process:

$$(3.12) \quad \tilde{W}_t = \tilde{\alpha} + \tilde{\beta} \tilde{W}_{t-1} + \tilde{\gamma} t + \tilde{\epsilon}_t, \quad 0 \leq \tilde{\beta} < 1, \gamma > 0,$$

where, once again, $1 - \tilde{\beta}$ measures the speed of mean reversion and the error term is given by

$$(3.13) \quad \tilde{\epsilon}_t = \tilde{\sigma} \tilde{\xi}_t + \tilde{\sigma} \sum_{i=0}^m \tilde{\eta}_i \xi_{t-i}^a, \quad \tilde{\xi}, \xi^a \sim \mathcal{N}(0, 1).$$

The term $\tilde{\gamma} t$ allows for a small upward trend which captures economic growth of the developing economy. The solution to (3.12) is

$$(3.14) \quad \tilde{W}_t = \tilde{\alpha} \sum_{i=0}^{t-1} \tilde{\beta}^i + \tilde{\beta}^t \tilde{W}_0 + \tilde{\gamma} \sum_{i=0}^{t-1} t \tilde{\beta}^i + \sum_{i=0}^{t-1} \tilde{\beta}^i \tilde{\epsilon}_{t-i}.$$

We may solve the geometric sum in (3.11) and rewrite the equation in a more compact and tractable form as

$$(3.15) \quad W_t = \varphi_0 + \varphi_1 \beta^t + \sum_{i=0}^{t-1} \beta^i \epsilon_{t-i},$$

with

$$\varphi_0 = \frac{\alpha}{1 - \beta}, \quad \varphi_1 = -\frac{\alpha}{1 - \beta} + W_0.$$

Similarly for (3.14) we obtain

$$(3.16) \quad \tilde{W}_t = \tilde{\varphi}_0 + \tilde{\varphi}_1 \tilde{\beta}^t + \tilde{\varphi}_2 t + \sum_{i=0}^{t-1} \tilde{\beta}^i \tilde{\epsilon}_{t-i},$$

with

$$\tilde{\varphi}_0 = \frac{\tilde{\alpha}}{1 - \tilde{\beta}}, \quad \tilde{\varphi}_1 = -\frac{\tilde{\alpha} + \tilde{\gamma}}{1 - \tilde{\beta}} + \tilde{W}_0, \quad \tilde{\varphi}_2 = \frac{\tilde{\gamma}}{1 - \tilde{\beta}}.$$

Inserting (3.14) and (3.11) into the bounds (3.6) and (3.7) gives us explicit expressions for the minimum ask price and maximum bid price, as outlined in the following proposition.

Proposition 3.2. *Under wealth dynamics (3.9) and (3.12) and population ageing (3.8), the fixed swap rate, s , must lie within a range with upper bound*

$$\bar{s}_{bid} = \frac{1}{\lambda} \left(\ln \sum_{t=0}^T \Psi_t \mathbb{E}_0^{\mathbb{P}^W} [e^{-\Lambda_t}] - \ln \sum_{t=0}^T \Psi_t^a \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} [e^{-\Lambda_t^a}] \right)$$

and lower bound

$$\underline{s}_{ask} = \frac{1}{\tilde{\lambda}} \left(\ln \sum_{t=0}^T \tilde{\Psi}_t^a \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} [e^{-\tilde{\Lambda}_t^a}] - \ln \sum_{t=0}^T \tilde{\Psi}_t \mathbb{E}_0^{\mathbb{P}^W} [e^{-\tilde{\Lambda}_t}] \right),$$

with

$$\begin{aligned} \Psi_t &:= e^{-\rho t} e^{-\lambda[\varphi_0 + \varphi_1 \beta^{t-1}[(1+t)\beta - t]]}, & \Psi_t^a &= \Psi_t e^{-\lambda \delta t}, \\ \Lambda_t &:= \lambda \left[(1+t) \epsilon_t + t \sum_{i=0}^{t-1} \beta^i \epsilon_{t-i} \right], & \Lambda_t^a &= \Lambda_t - \lambda \epsilon_t^a, \\ \tilde{\Psi}_t &:= e^{-\tilde{\rho} t} e^{-\tilde{\lambda}[\tilde{\varphi}_0 + \tilde{\varphi}_1 \tilde{\beta}^{t-1}[(1+t)\tilde{\beta} - t] + 2\tilde{\varphi}_2 t]}, & \tilde{\Psi}_t^a &= \tilde{\Psi}_t e^{\tilde{\lambda} \delta t}, \end{aligned}$$

and

$$\tilde{\Lambda}_t := \tilde{\lambda} \left[(1+t) \tilde{\epsilon}_t + t \sum_{i=0}^{t-1} \tilde{\beta}^i \tilde{\epsilon}_{t-i} \right], \quad \tilde{\Lambda}_t^a = \tilde{\Lambda}_t + \tilde{\lambda} \tilde{\epsilon}_t^a.$$

The formulas for the price bounds will differ when we assume the two stochastic processes for W and a to be either dependent or independent of each other. The latter case of processes' dependence is obviously more realistic and is precisely what motivates our innovation idea, but it is equally important to check how the price bounds look like in the former case, since only by doing so can we deduce interesting comparative statics.

4. EXPONENTIAL-UTILITY-BASED PRICE BOUNDS

Consider first the case of the stochastic processes for W_t and a_t being independent.

Proposition 4.1. *Take W_t and a_t independent. Under wealth dynamics (3.9) and (3.12) and population ageing (3.8), the fixed swap rate, s , must lie within a range with upper bound*

$$\bar{s}_{bid} = \frac{1}{\lambda} \left(\ln \sum_{t=0}^T \Psi_t \Omega_t - \ln \sum_{t=0}^T \Psi_t^a \Omega_t^a \right)$$

and lower bound

$$\underline{s}_{ask} = \frac{1}{\lambda} \left(\ln \sum_{t=0}^T \tilde{\Psi}_t^a \tilde{\Omega}_t^a - \ln \sum_{t=0}^T \tilde{\Psi}_t \tilde{\Omega}_t \right),$$

with

$$\begin{aligned} \Psi_t &:= e^{-\rho t} e^{-\lambda[\varphi_0 + \varphi_1 \beta^{t-1}[(1+t)\beta - t]]}, & \Psi_t^a &= \Psi_t e^{-\lambda \delta t}, \\ \Omega_t &:= e^{-\frac{\lambda^2}{2} \sigma^2 [(1+t)^2 + t^2 (\sum_{i=1}^{t-1} \beta^i)^2]}, & \Omega_t^a &:= e^{-\frac{\lambda^2}{2} [\sigma^2 [(1+t)^2 + t^2 (\sum_{i=1}^{t-1} \beta^i)^2] + k \varsigma_t^2 + (\sigma^a)^2]}, \\ \tilde{\Psi}_t &:= e^{-\tilde{\rho} t} e^{-\tilde{\lambda}[\tilde{\varphi}_0 + \tilde{\varphi}_1 \tilde{\beta}^{t-1}[(1+t)\tilde{\beta} - t] + 2\tilde{\varphi}_2 t]}, & \tilde{\Psi}_t^a &= \tilde{\Psi}_t e^{\tilde{\lambda} \delta t}, \end{aligned}$$

and

$$\tilde{\Omega}_t := e^{-\frac{\tilde{\lambda}^2}{2} \tilde{\sigma}^2 [(1+t)^2 + t^2 (\sum_{i=1}^{t-1} \tilde{\beta}^i)^2]}, \quad \tilde{\Omega}_t^a := e^{-\frac{\tilde{\lambda}^2}{2} [\tilde{\sigma}^2 [(1+t)^2 + t^2 (\sum_{i=1}^{t-1} \tilde{\beta}^i)^2] + k \varsigma_t^2 + (\sigma^a)^2]}.$$

Proof. See appendix section A.2. □

So if we assume the wealth endowments in both the rich and poor countries to be totally independent of population ageing in the rich country, the maximum bid price and the minimum ask price are a function of the risk aversion parameters, social discount rates, process parameters for W and a . To satisfy the two conditions of positive price bounds and a non-empty price interval, the following inequalities need to hold:

$$\lambda \delta t \neq 0; \quad \frac{\lambda^2 k \varsigma_t^2 + (\sigma^a)^2}{2} \neq 0;$$

$$(4.1) \quad \ln \sum_{t=0}^T \Psi_t \Omega_t > \ln \sum_{t=0}^T \Psi_t^a \Omega_t^a;$$

$$(4.2) \quad \ln \sum_{t=0}^T \tilde{\Psi}_t^a \tilde{\Omega}_t^a > \ln \sum_{t=0}^T \tilde{\Psi}_t \tilde{\Omega}_t.$$

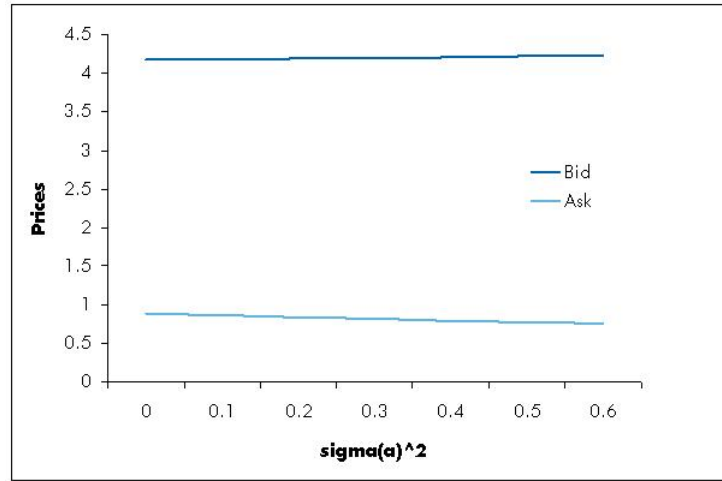


FIGURE 4.1. Changes in the bid-ask spread with respect to changes in $(\sigma^a)^2$. Prices in CHF billions.

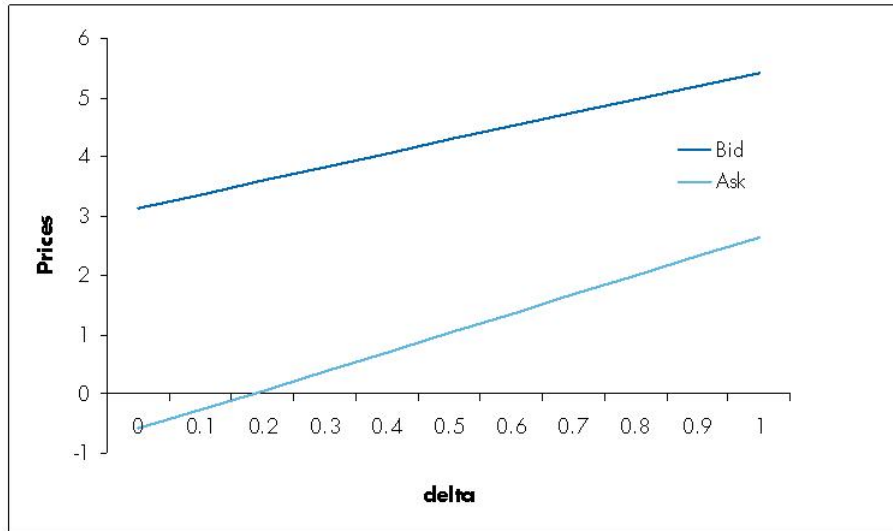


FIGURE 4.2. Changes in the bid-ask spread with respect to changes in δ . Prices in CHF billions.

Whereas condition (4.1) always holds true, condition (4.2) only holds true for T greater than some critical value T^* . Indeed, given the long-term character of the risks faced by the two counterparties, a contract with too short a maturity would not provide a useful hedge against these risks.

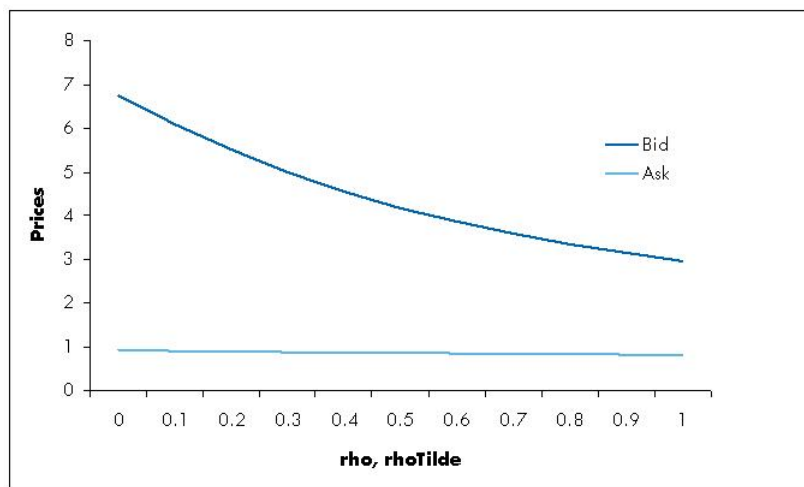


FIGURE 4.3. Changes in the bid-ask spread with respect to changes in ρ and $\tilde{\rho}$. Prices in CHF billions.

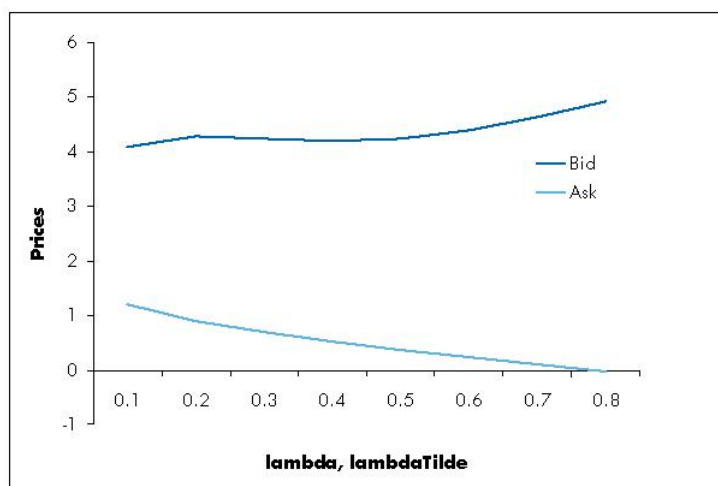


FIGURE 4.4. Changes in the bid-ask spread with respect to changes in λ and $\tilde{\lambda}$. Prices in CHF billions.

Sensitivity analysis allows us to observe whether and how the price interval widens or narrows for different parameter values. To check for the intuition behind the sensitivity analysis and to graphically illustrate the price changes, we apply the model to population and economic data on Switzerland.¹⁵

¹⁵Without loss of generality and for illustrative purposes, we assume rich and poor countries to have the same risk and time preferences. In future work, it will be interesting to check how our results vary as we take differing parameters for the two counterparties.

First of all, as contract life T increases, both \bar{s}_{bid} and \underline{s}_{ask} increase. So it will not be worthwhile for the rich and poor country to enter a contract with too short a maturity. This is a rather intuitive result, since the rich country does not forecast to experience longevity-related budget problems until some time in the further future.

For higher social rates of time preference ρ and $\tilde{\rho}$, both countries quote lower prices; however, the decrease in the quoted bid price is much stronger and the price interval narrows considerably with an increase in both countries' discount rates. This is illustrated in figure 4.3. Note, however, that the stronger decrease in the bid price is only true in absolute terms; in relative terms, the decrease in both prices is roughly equally strong.

For an increasing volatility in the innovations of the population ageing process, σ^a , both \bar{s}_{bid} and \underline{s}_{ask} increase, but rather mildly. Therefore, as figure 4.1 illustrates, the price interval does not change significantly.

If the old-age dependency ratio is expected to increase at a faster pace, i.e. for a higher value of δ , both the seller and the buyer of the swap will quote higher prices. There will also be a critical value of δ which guarantees that the seller will quote a positive price; so for too low a δ value, the buyer the seller will not be interested in entering such a contract. This result confirms precisely our intuition behind such a financial innovation, since the swap only makes economic sense if there is a severe population ageing risk to be hedged.

The more interesting result from the sensitivity analysis is with regards to the rich country's risk aversion parameter. Figure 4.4 shows indeed a kink in the shape of the bid price curve. This shape points out to the country being more interested in hedging its risk of severe population ageing either when it is relatively little risk averse or very risk averse. We believe, however, that this interesting shape of the bid price curve is worthy of further investigation.

We now take the two processes W_t and a_t to be dependent of each other and check how the price interval varies as the interdependence between the two processes comes into play.

Proposition 4.2. *Take W_t and a_t dependent as defined in (3.10). Under wealth dynamics (3.9) and (3.12) and population ageing (3.8), the fixed swap rate, s , must lie within a range with upper bound*

$$\bar{s}_{bid} = \frac{1}{\lambda} \left(\ln \sum_{t=0}^T \Psi_t \Omega_t - \ln \sum_{t=0}^T \Psi_t^a \Omega_t^a \right)$$

and lower bound

$$\underline{s}_{ask} = \frac{1}{\lambda} \left(\ln \sum_{t=0}^T \tilde{\Psi}_t^a \tilde{\Omega}_t^a - \ln \sum_{t=0}^T \tilde{\Psi}_t \tilde{\Omega}_t \right),$$

with

$$\begin{aligned} \Psi_t &:= e^{-\rho t} e^{-\lambda[\varphi_0 + \varphi_1 \beta^{t-1}[(1+t)\beta - t]]}, & \Psi_t^a &= \Psi_t e^{-\lambda \delta t}, \\ \Omega_t &:= e^{\frac{[-\lambda(1+t)\sigma - \lambda(1+t)\sigma \sum_{j=0}^m \eta_j - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j]^2}{2}}, \\ \Omega_t^a &:= e^{\frac{[-\lambda(1+t)\sigma - \lambda(1+t)\sigma \sum_{j=0}^m \eta_j - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j - \lambda \sum_{i=1}^k \varsigma_t - \lambda \sigma^a]^2}{2}}, \\ \tilde{\Psi}_t &:= e^{-\tilde{\rho} t} e^{-\tilde{\lambda}[\tilde{\varphi}_0 + \tilde{\varphi}_1 \tilde{\beta}^{t-1}[(1+t)\tilde{\beta} - t] + 2\tilde{\varphi}_2 t]}, & \tilde{\Psi}_t^a &= \tilde{\Psi}_t e^{\tilde{\lambda} \delta t}, \\ \tilde{\Omega}_t &:= e^{\frac{1}{2}[-\tilde{\lambda}(1+t)\tilde{\sigma} - \tilde{\lambda}(1+t)\tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j]^2}, \end{aligned}$$

and

$$\tilde{\Omega}_t^a := e^{\frac{1}{2}[-\tilde{\lambda}(1+t)\tilde{\sigma} - \tilde{\lambda}(1+t)\tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j + \tilde{\lambda} \sum_{i=1}^k \varsigma_t + \tilde{\lambda} \sigma^a]^2}.$$

Proof. See appendix section A.3. □

The inherent market price of risk for the old age dependency ratio results from the inability to hedge the risk of an ageing population. I.e. the interrelationship between population ageing in the developed country and the wealth endowments in both countries plays an important role in determining the risk premium embedded in the value of this contract.

If the rich country's wealth endowment is very sensitive to the ageing of its population, then it is more interested in entering the swap agreement and is willing to pay more to hedge against adverse demographic developments. This raises the maximum bid quote. Similarly, if the poor country's wealth endowment is very sensitive to the ageing of the rich country's population, then it is more interested in

entering the swap agreement and is willing to receive less to hedge against adverse demographic developments. This is especially the case if the poor country is highly dependent on financial aid from the rich country and is at risk of receiving less aid if the rich country needs to devote extra funds to its own elderly population. It may be interesting to check the extent to which developmental aid flows are or may be in the future affected by population ageing in the donor countries.

The amount of the risk premium can also be shown to depend on the financial position of both countries, too. Not only does the sensitivity of the wealth endowment matter, but also the amount of government income invested in the endowment. Higher values of W_0 and \tilde{W}_0 raise φ_1 and $\tilde{\varphi}_1$, thus shifting the bid-ask interval towards lower prices. So countries entering the swap agreement with higher wealth are likely to demand a lower risk premium.

5. CHALLENGES

The swap we have presented throughout the previous sections does not yet exist. We may think of three significant challenges arising when implementing this inter-generational cross-country swap. First of all, what percentage of its GDP would the rich country be willing to pay for longevity protection? Given our model assumptions, together with historical GDP and forecast longevity data for Switzerland, we estimate the maximum bid price to lie between 3 to 5 billion Swiss francs, which is roughly not more than 1% of current real Swiss GDP.¹⁶ We need to compare this figure with the amount of funds the Swiss government forecasts to spend on pensions and long-term health care over the next three decades. According to the base scenario outlined in the Swiss Health Observatory study [SBJR⁺08], long-term health care costs will rise by 141.9% from 2005 to 2030 to 17.8 billion Swiss francs (keeping 2005 prices constant); this is equivalent to 2.8% of Swiss GDP. Though this may hint to the feasibility of the swap, the decision ultimately remains a political decision.

¹⁶Data as of end of 2008, according to the Swiss Secretariat for Economic Affairs.

Another challenge concerns the impact that a transaction of such magnitude could have on currency markets. The announcement of very large sums of money denominated in a certain currency flowing from one country to the other may significantly destabilize exchange rates. Given the large sums already exchanged by central banks through FX swaps when conducting their monetary policies, it is to be checked how much “capacity” the FX markets are still able to bare at the time of contract conclusion.

Finally, we have previously mentioned the risks entailed by rolling over a derivative contract. These are related to the agent’s regret in case the bid-ask spread in one period is far above or below the spread in the previous period; if applying backward induction, there is the risk that one of the counterparties will decide not to participate. Take, for example, the buyer of the swap: if the bid-ask spread in the second period is such that the minimum ask price lies above the maximum bid price quoted in the first period, then - through backward induction - the buyer would have been better off not entering the contract in the first place. This rollover risk cannot be hedged and, as such, is *per se* an interesting matter worthy of future analysis.

6. CONCLUSIONS AND FUTURE RESEARCH

The question we set off to answer is whether there is a rationale for a financial innovation whereby a developed economy would swap its longevity risk against the growth risk of a developing economy. Our answer is yes. Having identified the old-age dependency ratio as the appropriate underlying variable, we proceeded to price the innovative swap structure. We applied an exponential-utility-based pricing method and determined an interval of swap prices, any one of which makes an agreement between the two countries plausible. Yet this begs the question of which final price between maximum bid price and minimum ask price will actually be chosen. As this swap is an over-the-counter contract, liquidity issues are here irrelevant. So what is crucial to consider at this stage is the political strength of

the two countries; agreements of international finance are notoriously guided by political issues which cannot be captured by financial pricing models. Intuitively though, the final swap price will converge to the minimum ask price as the rich country's political strength relative to the poor country increases. Similarly, the final swap price will converge to the maximum bid price as the rich country's political strength relative to the poor country decreases. We could include some parameter to illustrate this basic intuition, but this would not induce any significant changes to the results derived in the paper.

In a following paper we will estimate the dynamics W and a for a selected group of countries and check which countries could benefit from such an intergenerational agreement. We aim to quantify the swap rates by considering the participation constraints under alternative scenarios of demographic changes. Other issues we aim to investigate in future work include the channels through which banks' product strategies are influenced by asymmetric demographic trends, as well as the extent to which developmental aid and microfinance innovations are related to trends in population ageing in donor countries.

REFERENCES

- [AB07] J. Alho and V. Borge. Global ageing and macroeconomic consequences of demographic uncertainty in a multi-regional model. Working paper, CEPPII, 2007.
- [Abe03] A. Abel. The effects of a baby boom on stock prices and capital accumulation in the presence of social security. *Econometrica*, 71:551–595, 2003.
- [AS01] S. Athanasoulis and R. Shiller. World income components: Measuring and exploiting risk sharing opportunities. *American Economic Review*, 91:103154, 2001.
- [BB01] D. Blake and W. Burrows. Survivor bonds: Helping to hedge mortality risk. *Journal of Risk and Insurance*, 68:339–348, 2001.
- [BCDM06] D. Blake, A. Cairns, K. Dowd, and R. MacMinn. Longevity bonds: Financial engineering, valuation, and hedging. *Journal of Risk and Insurance*, 73:647–672, 2006.
- [BCM06] N. Batini, T. Callen, and W. McKibbin. The global impact of demographic change. Working paper, International Monetary Fund, 2006.
- [BPS06] G. Becker, T. Philipson, and R. Soares. The quantity and quality of life and the evolution of world inequality. Working paper, NBER, 2006.
- [Bro02] R. Brooks. Asset-market effects of the baby boom and social security reform. *American Economic Review, Papers and Proceedings*, 192:402–406, 2002.
- [Bry06] R. Bryant. Asymmetric demography and macroeconomic interactions across national borders. In *Demography and Financial Markets*. Reserve Bank of Australia, 2006.
- [Bys08] H. Byström. The microfinance collateralized debt obligation: a modern Robin Hood? *World Development*, forthcoming, 2008.
- [DBCD06] K. Dowd, D. Blake, A. Cairns, and P. Dawson. Survivor swaps. *Journal of Risk and Insurance*, 73:1–17, 2006.
- [EW03] S. Eijffinger and W. Wagner. The feasible gains from international risk sharing. Working paper, CESifo, 2003.
- [FS08] C. Frei and M. Schweizer. Exponential utility indifference valuation in a general semi-martingale model. Working paper, NCCR FinRisk, 2008.
- [GBH⁺06] T. Groome, N. Blancher, F. Haas, J. Kiff, W. Lee, P. Mills, S. Nakagawa, P. Ramlogan, O. Khadarina, and Y. Kim. The limits of market-based risk transfer and implications for managing systemic risks. Working paper, International Monetary Fund, 2006.
- [GBR06] T. Groome, N. Blancher, and P. Ramlogan. Population ageing and financial markets. *Finance and Development*, 43:69–81, 2006.
- [GMU07] B. Gadanez, R. Moessner, and C. Upper. Economic derivatives. *BIS Quarterly Review*, pages 69–81, 2007.
- [Hep06] S. Hepburn. Discounting climate change damages. Overview for the Stern Review, 2006.

- [IL07] G. Isiklar and K. Lahiri. How far ahead can we forecast? Evidence from cross-country surveys. *International Journal of Forecasting*, 23:167–187, 2007.
- [Kra02] M. Kraemer. Western europe past its prime - Sovereign rating perspectives in the context of ageing populations. Technical report, Standard and Poor's, 2002.
- [MPSY06] O. Mitchell, J. Piggott, M. Sherris, and S. Yow. Financial innovation for an ageing world. In *Demography and Financial Markets*. Reserve Bank of Australia, 2006.
- [MS05] M. Mania and M. Schweizer. Dynamic exponential utility indifference valuation. *Annals of Applied Probability*, 15:2113–2143, 2005.
- [PGHK03] D. Pearce, B. Groom, C. Hepburn, and P. Koundouri. Valuing the future: Recent advances in social discounting. *World Economics*, 4:121–141, 2003.
- [Pon08] G. Ponthière. A study of the sensitivity of longevity-adjusted income measures. *Oxford Development Studies*, 36:339–361, 2008.
- [Pre75] S. Preston. The changing relation between mortality and level of economic development. *Population Studies*, 29:231–248, 1975.
- [SBJR⁺08] L. Seematter-Bagnoud, J. Junod, H. Jaccard Ruedin, M. Roth, C. Foletti, and B. Santos-Eggimann. Angebot und Inanspruchnahme ambulanter medizinischer Leistungen in der Schweiz - Prognosen bis zum Jahr 2030. Technical report, Swiss Health Observatory, 2008.
- [Shi04a] R. Shiller. *A New Financial Order*. Princeton University Press, 2004.
- [Shi04b] R. Shiller. Radical financial innovation. In J. Hakala and U. Wystup, editors, *Entrepreneurship, Innovation and the Growth Mechanism of the Free Market Economies*. Princeton University Press, 2004.
- [UNP07] UNPD. World population prospects - The 2006 revision. Technical report, United Nations Population Division, 2007.
- [Ush73] D. Usher. An imputation of the measure of economic growth for changes in life expectancy. *Studies in Income and Wealth*, 38:193–225, 1973.
- [Wei01] M. Weitzman. Gamma discounting. *American Economic Review*, 91:260–271, 2001.
- [Wei07] M. Weitzman. The Stern review of the economics of climate change. *Journal of Economic Literature*, 45:703–724, 2007.

APPENDIX A. PROOFS

This section contains detailed proofs to all propositions in the paper.

A.1. Proof of proposition 3.1.

Proof.

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[\sum_{t=0}^T D_t \left(\tilde{W}_t \right) * U \left(\tilde{W}_t, s_{ask}, a_t \right) \right] \geq \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[\sum_{t=0}^T D_t \left(\tilde{W}_t \right) * U \left(\tilde{W}_t, 0, 0 \right) \right] \\
& \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[- \sum_{t=0}^T e^{-\tilde{\rho}t - \tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t} e^{-\tilde{\lambda}(\tilde{W}_t + s_{ask} - a_t)} \right] \geq \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[- \sum_{t=0}^T e^{-\tilde{\rho}t - \tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t} e^{-\tilde{\lambda}\tilde{W}_t} \right] \\
& -e^{-\tilde{\lambda}s_{ask}} \sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[e^{-\tilde{\rho}t - \tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}(\tilde{W}_t - a_t)} \right] \geq - \sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\rho}t - \tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}\tilde{W}_t} \right]
\end{aligned}$$

so that

$$\begin{aligned}
e^{-\tilde{\lambda}s_{ask}} & \leq \frac{\sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\rho}t - \tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}\tilde{W}_t} \right]}{\sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[e^{-\tilde{\rho}t - \tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}(\tilde{W}_t - a_t)} \right]} \\
e^{-\tilde{\lambda}s_{ask}} & \leq \frac{\sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}\tilde{W}_t} \right]}{\sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[e^{-\tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}(\tilde{W}_t - a_t)} \right]} \\
-\tilde{\lambda}s_{ask} & \leq \ln \frac{\sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}\tilde{W}_t} \right]}{\sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[e^{-\tilde{\lambda} * (\tilde{W}_t - \tilde{W}_{t-1})t - \tilde{\lambda}(\tilde{W}_t - a_t)} \right]} \\
s_{ask} & \geq \frac{1}{\tilde{\lambda}} \left(\ln \sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}} \times \mathbb{P}^a} \left[e^{-\tilde{\lambda}\tilde{W}_t(1+t) + \tilde{\lambda}\tilde{W}_{t-1}t + \tilde{\lambda}a_t} \right] - \ln \sum_{t=0}^T e^{-\tilde{\rho}t} \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda}\tilde{W}_t(1+t) + \tilde{\lambda}\tilde{W}_{t-1}t} \right] \right),
\end{aligned}$$

which gives the minimum price the seller is willing to receive for the swap.

From the buyer's side:

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[\sum_{t=0}^T D_t(W_t) * U(W_t, s_{bid}, a_t) \right] \geq \mathbb{E}_0^{\mathbb{P}^W} \left[\sum_{t=0}^T D_t(W_t) * U(W_t, 0, 0) \right] \\
\mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[- \sum_{t=0}^T e^{-\rho t - \lambda*(W_t - W_{t-1})t} e^{-\lambda(W_t - s_{bid} + a_t)} \right] & \geq \mathbb{E}_0^{\mathbb{P}^W} \left[- \sum_{t=0}^T e^{-\rho t - \lambda*(W_t - W_{t-1})t} e^{-\lambda W_t} \right] \\
-e^{\lambda s_{bid}} \sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\rho t - \lambda*(W_t - W_{t-1})t - \lambda(W_t + a_t)} \right] & \geq - \sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\rho t - \lambda*(W_t - W_{t-1})t - \lambda W_t} \right]
\end{aligned}$$

so that

$$\begin{aligned}
e^{\lambda s_{bid}} & \leq \frac{\sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\rho t - \lambda*(W_t - W_{t-1})t - \lambda W_t} \right]}{\sum_{t=0}^T \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\rho t - \lambda*(W_t - W_{t-1})t - \lambda(W_t + a_t)} \right]} \\
e^{\lambda s_{bid}} & \leq \frac{\sum_{t=0}^T e^{-\rho t} \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda*(W_t - W_{t-1})t - \lambda W_t} \right]}{\sum_{t=0}^T e^{-\rho t} \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda*(W_t - W_{t-1})t - \lambda(W_t + a_t)} \right]} \\
s_{bid} & \leq \frac{1}{\lambda} \left(\ln \sum_{t=0}^T e^{-\rho t} \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda W_t(1+t) + \lambda W_{t-1}t} \right] - \ln \sum_{t=0}^T e^{-\rho t} \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda W_t(1+t) + \lambda W_{t-1}t - \lambda a_t} \right] \right),
\end{aligned}$$

which gives the maximum price the buyer is willing to pay for the swap. \square

A.2. Proof of proposition 4.1.

Proof.

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\Lambda_t} \right] \\
& = \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda[(1+t)\epsilon_t + t \sum_{i=1}^{t-1} \beta^i \epsilon_{t-i}]} \right] \\
& = \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda[(1+t)(\sigma \xi_t + \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^\alpha) + t \sum_{i=1}^{t-1} \beta^i (\sigma \xi_{t-i} + \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^\alpha)]} \right] \\
& = \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda(1+t)\sigma \xi_t - \lambda(1+t)\sigma \sum_{j=0}^m \eta_j \xi_{t-j}^\alpha - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \xi_{t-i} - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^\alpha} \right] \\
& = \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda(1+t)\sigma \xi_t - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \xi_{t-i}} \right], \quad \text{since } \eta_j = 0, \forall j \\
& = \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda(1+t)\sigma \xi_t} \right] \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda t \sum_{i=1}^{t-1} \beta^i \sigma \xi_{t-i}} \right] \\
& = e^{-\frac{\lambda^2(1+t)^2 \sigma^2}{2}} e^{-\frac{\lambda^2 t^2 (\sum_{i=1}^{t-1} \beta^i)^2 \sigma^2}{2}} \\
& = e^{-\frac{\lambda^2}{2} \sigma^2 [(1+t)^2 + t^2 (\sum_{i=1}^{t-1} \beta^i)^2]},
\end{aligned}$$

where we have used that

$$\mathbb{E}[X^c] = e^{c\mu + \frac{c^2\sigma^2}{2}},$$

for X log-normal, and

$$\xi, \tilde{\xi}, \xi^a \sim \mathcal{N}(0, 1).$$

□

Proof.

$$\begin{aligned} & \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\Lambda_t^a} \right] \\ = & \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda \left[(1+t)\epsilon_t + t \sum_{i=1}^{t-1} \beta^i \epsilon_{t-i} + \epsilon_t^a \right]} \right] \\ = & \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda \left[(1+t)(\sigma\xi_t + \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a) + t \sum_{i=1}^{t-1} \beta^i (\sigma\xi_{t-i} + \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a) + \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a + \sigma^a \xi_t^a \right]} \right] \\ = & \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda(1+t)\sigma\xi_t - \lambda(1+t)\sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma\xi_{t-i} - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a - \lambda \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a - \lambda \sigma^a \xi_t^a} \right] \\ = & \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda(1+t)\sigma\xi_t - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma\xi_{t-i} - \lambda \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a - \lambda \sigma^a \xi_t^a} \right], \quad \text{since } \eta_j = 0, \forall j \\ = & \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda(1+t)\sigma\xi_t} \right] \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda t \sum_{i=1}^{t-1} \beta^i \sigma\xi_{t-i}} \right] \mathbb{E}_0^{\mathbb{P}^a} \left[e^{-\lambda \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a} \right] \mathbb{E}_0^{\mathbb{P}^a} \left[e^{-\lambda \sigma^a \xi_t^a} \right] \\ = & e^{-\frac{\lambda^2(1+t)^2\sigma^2}{2}} e^{-\frac{\lambda^2 t^2 (\sum_{i=1}^{t-1} \beta^i)^2 \sigma^2}{2}} e^{-\frac{\lambda^2 k \varsigma_t^2}{2}} e^{-\frac{\lambda^2 (\sigma^a)^2}{2}} \\ = & e^{-\frac{\lambda^2}{2} \left[\sigma^2 \left[(1+t)^2 + t^2 (\sum_{i=1}^{t-1} \beta^i)^2 \right] + k \varsigma_t^2 + (\sigma^a)^2 \right]}. \end{aligned}$$

□

Proof.

$$\begin{aligned} & \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\Lambda}_t} \right] \\ = & \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} \left[(1+t)\tilde{\epsilon}_t + t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\epsilon}_{t-i} \right]} \right] \\ = & \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} \left[(1+t)(\tilde{\sigma}\tilde{\xi}_t + \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-j}^a) + t \sum_{i=1}^{t-1} \tilde{\beta}^i (\tilde{\sigma}\tilde{\xi}_{t-i} + \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-i-j}^a) \right]} \right] \\ = & e^{-\frac{\tilde{\lambda}^2(1+t)^2\tilde{\sigma}^2}{2}} e^{-\frac{\tilde{\lambda}^2 t^2 (\sum_{i=1}^{t-1} \tilde{\beta}^i)^2 \tilde{\sigma}^2}{2}} \\ = & e^{-\frac{\tilde{\lambda}^2}{2} \tilde{\sigma}^2 \left[(1+t)^2 + t^2 (\sum_{i=1}^{t-1} \tilde{\beta}^i)^2 \right]}. \end{aligned}$$

□

Proof.

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\tilde{\Lambda}_t^a} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\tilde{\lambda} \left[(1+t)\tilde{\epsilon}_t + t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\epsilon}_{t-i} - \tilde{\epsilon}_t^a \right]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\tilde{\lambda} \left[(1+t) \left(\tilde{\sigma} \tilde{\xi}_t + \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-j}^a \right) + t \sum_{i=1}^{t-1} \tilde{\beta}^i \left(\tilde{\sigma} \tilde{\xi}_{t-i} + \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-i-j}^a \right) - \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a - \sigma^a \xi_t^a \right]} \right] \\
&= e^{-\frac{\tilde{\lambda}^2 (1+t)^2 \tilde{\sigma}^2}{2}} e^{-\frac{\tilde{\lambda}^2 t^2 \left(\sum_{i=1}^{t-1} \tilde{\beta}^i \right)^2 \tilde{\sigma}^2}{2}} e^{-\frac{\tilde{\lambda}^2 k \varsigma_t^2}{2}} e^{-\frac{\tilde{\lambda}^2 (\sigma^a)^2}{2}} \\
&= e^{-\frac{\tilde{\lambda}^2}{2} \left[\tilde{\sigma}^2 \left[(1+t)^2 + t^2 \left(\sum_{i=1}^{t-1} \tilde{\beta}^i \right)^2 \right] + k \varsigma_t^2 + (\sigma^a)^2 \right]}.
\end{aligned}$$

□

A.3. Proof of proposition 4.2.

Proof.

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\Lambda_t} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda \left[(1+t)\epsilon_t + t \sum_{i=1}^{t-1} \beta^i \epsilon_{t-i} \right]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda \left[(1+t) \left(\sigma \xi_t + \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a \right) + t \sum_{i=1}^{t-1} \beta^i \left(\sigma \xi_{t-i} + \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a \right) \right]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda (1+t) \sigma \xi_t - \lambda (1+t) \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \xi_{t-i} - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W} \left[e^{-\lambda (1+t) \sigma \xi_t} e^{-\lambda (1+t) \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a} e^{-\lambda t \sum_{i=1}^{t-1} \beta^i \sigma \xi_{t-i}} e^{-\lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a} \right] \\
&= e^{\frac{1}{2} \left[-\lambda (1+t) \sigma - \lambda (1+t) \sigma \sum_{j=0}^m \eta_j - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j \right]^2}.
\end{aligned}$$

where we have used that $\prod_{i=1}^n X_i \sim \log \mathcal{N} \left(n\mu, \sum_{i=1}^n \sigma_i^2 \right)$, for $X_i \sim \log \mathcal{N} \left(\mu, \sigma_i^2 \right)$, $i = 1, \dots, n$, independent log-normally distributed variables with the same mean parameter μ and possibly varying volatility σ_i . In this case, $\prod_{i=1}^n X_i \sim \log \mathcal{N} (0, 1)$, as $X_i \sim \log \mathcal{N} (0, 1)$, $i = 1, \dots, n$; thus, $\mathbb{E}[X^c] = e^{\frac{c^2}{2}}$. □

Proof.

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\Lambda_t^a} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda \left[(1+t)\epsilon_t + t \sum_{i=1}^{t-1} \beta^i \epsilon_{t-i} + \epsilon_t^a \right]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda \left[(1+t) \left(\sigma \xi_t + \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a \right) + t \sum_{i=1}^{t-1} \beta^i \left(\sigma \xi_{t-i} + \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a \right) + \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a + \sigma^a \xi_t^a \right]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda (1+t) \sigma \xi_t - \lambda (1+t) \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \xi_{t-i} - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a - \lambda \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a - \lambda \sigma^a \xi_t^a} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\lambda (1+t) \sigma \xi_t} e^{-\lambda (1+t) \sigma \sum_{j=0}^m \eta_j \xi_{t-j}^a} e^{-\lambda t \sum_{i=1}^{t-1} \beta^i \sigma \xi_{t-i}} e^{-\lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j \xi_{t-i-j}^a} e^{-\lambda \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a} e^{-\lambda \sigma^a \xi_t^a} \right] \\
&= e^{\frac{1}{2} \left[-\lambda (1+t) \sigma - \lambda (1+t) \sigma \sum_{j=0}^m \eta_j - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma - \lambda t \sum_{i=1}^{t-1} \beta^i \sigma \sum_{j=0}^m \eta_j - \lambda \sum_{i=1}^k \varsigma_t - \lambda \sigma^a \right]^2}.
\end{aligned}$$

□

Proof.

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\Lambda}_t} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} \left[(1+t)\tilde{\epsilon}_t + t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\epsilon}_{t-i} \right]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} \left[(1+t) \left(\tilde{\sigma} \tilde{\xi}_t + \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-j}^a \right) + t \sum_{i=1}^{t-1} \tilde{\beta}^i \left(\tilde{\sigma} \tilde{\xi}_{t-i} + \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-i-j}^a \right) \right]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} (1+t) \tilde{\sigma} \tilde{\xi}_t - \tilde{\lambda} (1+t) \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-j}^a - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} \tilde{\xi}_{t-i} - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-i-j}^a} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^{\tilde{W}}} \left[e^{-\tilde{\lambda} (1+t) \tilde{\sigma} \tilde{\xi}_t} e^{-\tilde{\lambda} (1+t) \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-j}^a} e^{-\tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} \tilde{\xi}_{t-i}} e^{-\tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \xi_{t-i-j}^a} \right] \\
&= e^{\frac{1}{2} \left[-\tilde{\lambda} (1+t) \tilde{\sigma} - \tilde{\lambda} (1+t) \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} - \tilde{\lambda} t \sum_{i=1}^{t-1} \tilde{\beta}^i \tilde{\sigma} \sum_{j=0}^m \tilde{\eta}_j \right]^2}.
\end{aligned}$$

□

Proof.

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\bar{\Lambda}_t^a} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\bar{\lambda}[(1+t)\bar{\epsilon}_t + t \sum_{i=1}^{t-1} \bar{\beta}^i \bar{\epsilon}_{t-i} - \bar{\epsilon}_t^a]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\bar{\lambda}[(1+t)(\bar{\sigma}\bar{\xi}_t + \bar{\sigma} \sum_{j=0}^m \bar{\eta}_j \xi_{t-j}^a) + t \sum_{i=1}^{t-1} \bar{\beta}^i (\bar{\sigma}\bar{\xi}_{t-i} + \bar{\sigma} \sum_{j=0}^m \bar{\eta}_j \xi_{t-i-j}^a) - \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a - \sigma^a \xi_t^a]} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\bar{\lambda}(1+t)\bar{\sigma}\bar{\xi}_t - \bar{\lambda}(1+t)\bar{\sigma} \sum_{j=0}^m \bar{\eta}_j \xi_{t-j}^a - \bar{\lambda}t \sum_{i=1}^{t-1} \bar{\beta}^i \bar{\sigma}\bar{\xi}_{t-i} - \bar{\lambda}t \sum_{i=1}^{t-1} \bar{\beta}^i \bar{\sigma} \sum_{j=0}^m \bar{\eta}_j \xi_{t-i-j}^a + \bar{\lambda} \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a + \bar{\lambda}\sigma^a \xi_t^a} \right] \\
&= \mathbb{E}_0^{\mathbb{P}^W \times \mathbb{P}^a} \left[e^{-\bar{\lambda}(1+t)\bar{\sigma}\bar{\xi}_t} e^{-\bar{\lambda}(1+t)\bar{\sigma} \sum_{j=0}^m \bar{\eta}_j \xi_{t-j}^a} e^{-\bar{\lambda}t \sum_{i=1}^{t-1} \bar{\beta}^i \bar{\sigma}\bar{\xi}_{t-i}} e^{-\bar{\lambda}t \sum_{i=1}^{t-1} \bar{\beta}^i \bar{\sigma} \sum_{j=0}^m \bar{\eta}_j \xi_{t-i-j}^a} e^{\bar{\lambda} \sum_{i=1}^k \varsigma_t \epsilon_{t-i}^a} e^{\bar{\lambda}\sigma^a \xi_t^a} \right] \\
&= e^{\frac{1}{2} \left[-\bar{\lambda}(1+t)\bar{\sigma} - \bar{\lambda}(1+t)\bar{\sigma} \sum_{j=0}^m \bar{\eta}_j - \bar{\lambda}t \sum_{i=1}^{t-1} \bar{\beta}^i \bar{\sigma} - \bar{\lambda}t \sum_{i=1}^{t-1} \bar{\beta}^i \sigma \sum_{j=0}^m \eta_j + \bar{\lambda} \sum_{i=1}^k \varsigma_t + \bar{\lambda}\sigma^a \right]^2}.
\end{aligned}$$

□

ZÜRCHER KANTONALBANK

E-mail address: miret.padovani@zkb.ch

ZÜRCHER KANTONALBANK AND SWISS FINANCE INSTITUTE

E-mail address: paolo.vanini@zkb.ch