Pricing American Options Under Stochastic Volatility and Stochastic Interest Rates

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First version: April 2006
Current version: March 2008

This research has been carried out within the NCCR FINRISK project on “Financial Econometrics for Risk Management”
Abstract

We introduce a new analytical approach to price American options. Using an explicit and intuitive proxy for the exercise rule, we derive tractable pricing formulas using a short-maturity asymptotic expansion. Depending on model parameters, this method can accurately price options with time-to-maturity up to several years. The main advantage of our approach over existing methods lies in its straightforward application to models with stochastic volatility and stochastic interest rates. We exploit this advantage by providing an analysis of the impact of volatility mean-reversion, volatility of volatility, and correlations on the American put price.

Key words: American options, stochastic volatility, stochastic interest rates, asymptotic approximation.

JEL Classification: G12.
1 Introduction

The valuation of American options is a challenging task, even under the Black-Scholes model (see Detemple (2005) for an extensive review). Several semi-analytical approximations for American option prices have been proposed in the literature (Barone-Adesi and Whaley (1987), Broadie and Detemple (1996), Bunch and Johnson (2001), Ju (1998)). Although these approaches are fast and accurate, they cannot easily be extended beyond the Black-Scholes model.

It has been firmly established that the Black-Scholes model is not consistent with quoted option prices. The literature advocates the introduction of stochastic volatility to reproduce the implied volatility smile observed in the market. The introduction of a second stochastic factor enormously complicates the pricing of American options. Presently, this can only be done by means of numerical schemes, which involve solving integral equations (Kim (1990), Huang, Subrahmanyam and Yu (1996), Sullivan (2000), Detemple and Tian (2002)), performing Monte Carlo simulations (Broadie and Glasserman (1997), Longstaef and Schwartz (2001), Rogers (2002), Haugh and Kogan (2004)), or discretizing the partial differential equation (Brennan and Schwartz (1977), Clarke and Parrott (1999), Ikonen and Toivanen (2007)).

The early exercise premium of the American put option depends on the cost of carry determined by interest rates. Consequently, the volatility of interest rates does affect the decision to exercise this option at any point in time. This fact is recognized in the literature dealing with models with stochastic interest rates (Ho, Stapleton and Subrahmanyam (1997), Menkveld and Vorst (2001), Detemple and Tian (2002)). This literature, however, considers only two-factor extensions of the Black-Scholes model assuming that the volatility of the underlying is constant.

Numerical approaches are complicated to implement when both volatility and interest rates are stochastic. Presently, the only feasible way of fast computing the option price in this case is to numerically solve the partial differential equation (PDE). From the analogy with standard tree approaches, a PDE solver amounts to reconnect three trees: a binomial tree for the stock price and two trinomial trees for mean-reverting stochastic volatility and stochastic interest rates. As a result,
the implementation of a PDE solver is case specific \(^2\) and may raise stability issues (especially when computing option Greeks).

In this paper we propose a new analytical approach that is both computationally tractable and general enough to be successfully applied to a three-factor model. Our approach is based on the idea of substituting the optimal exercise rule with a simple (suboptimal) exercise rule for which an approximate solution is easy to find and fast to compute. Similar ideas have already been explored in the literature in the context of the Black-Scholes model (Broadie and Detemple (1996), Carr (1998), Ju (1998)). Our proxy rule is to exercise the option as soon as its moneyness measured in standard deviations reaches some specified level. The rationale is that a put should be exercised when it can be considered sufficiently in-the-money (large moneyness). The option price under this rule appears to have a regular asymptotic behavior near maturity with an asymptotic expansion available in a closed form for a broad class of models. The American option price is then approximated by the maximum over these option prices. We provide several numerical experiments and comparisons showing that our method performs well with respect to computational time and accuracy.

Taking advantage of our computationally efficient method, we study the effect of introducing stochastic volatility and stochastic interest rates on the American put price and its components: the European put price and the early exercise premium. Under realistic model parameters, the strongest impact seems to be generated by a stochastic volatility reinforced by a negative correlation with the underlying asset price (leverage effect). Here the change in the American put price is mainly driven by the change in the European put price. However, the impact on the early exercise premium should not be ignored, especially near the early exercise boundary (see Section 3.5 for a simple financial argument). The impact of stochastic interest rates is negligible unless they are negatively correlated with the underlying asset price. The European put price is always adversely affected by such a correlation. However, this effect is not present in the case of in-the-money American puts since the possibility of an early exercise reduces the expected life-time of the put thus diminishing the impact of stochastic interest rates.

\(^2\)It is feasible in the case of a three-factor affine specification, but next to impossible in the general setting.
The paper is organized as follows. In Section 2 we describe our approach in the context of the Black-Scholes model. We provide a motivation for our approach, discuss intuitively its main features, and compare it with other available methods. We also study numerically the accuracy of approximation of the option delta. In Section 3 we generalize our approach to incorporate multifactor models with stochastic volatility and stochastic interest rates. We run several numerical experiments to show that our approach is competitive with existing methods, and investigate the impact of stochastic volatility and stochastic interest rates on the components of the American put price. Section 4 concludes the paper. Technical proofs and results are gathered in Appendices. All the Matlab codes used in this paper are available from the authors on request.

2 Black-Scholes model

In this section we consider the Black-Scholes model where the stock price follows a log-normal diffusion process and develop analytical approximation for the American put. This section is aimed at presenting intuitively our approach in a simple setting before looking at richer settings.

2.1 Short-maturity asymptotics for American option prices

An American put option with strike price $K$ and maturity date $T$ is a derivative that gives its owner the right to receive $\max(K - S_t, 0)$ at any point in time $t \leq t' \leq T$. Under the Black-Scholes model the price $P(S, t)$ of this option satisfies the partial differential equation (PDE):

$$P_t + (r - \delta)SP_s + \frac{1}{2} \sigma^2 S^2 P_{ss} - rP = 0,$$

with boundary conditions:

$$P(\infty, t) = 0,$$

$$P(S, T) = \max(K - S, 0),$$

$$P(S(T - t), t) = \max(K - S(T - t), 0),$$

$$P(\overline{S}(T - t), t) = \max(K - \overline{S}(T - t), 0),$$
Here subscripts denote differentiation with respect to time $t$ and stock price $S$; $r$ and $\delta$ are the interest rate and the dividend yield; $\overline{S}(\tau)$ is the early exercise price, which depends on the option time-to-maturity $\tau = T - t$. Boundary condition (2) is the so-called "smooth pasting" condition. Note that the European put also satisfies PDE (1) with the boundary conditions with zero early exercise price $\overline{S}(\tau) = 0$. The unique solution for the American option price is then determined by the additional condition $P(S; t) \geq \max(K - S, 0)$.

Solving PDE (1) given the five conditions above is a non-trivial task. The only known analytical solution to this problem is found by Zhu (2006) in the form of a Taylor series expansion. While the emphasis of that paper is to show the existence of an exact analytical solution, such a solution does not have a clear advantage over some fast numerical schemes from a computational point of view. The convergence is rather slow, and the expansion terms are given recursively by a lengthy analytical formula.

A number of approximation methods exist in the literature. In particular, the behavior of $\overline{S}(\tau)$ near maturity (small $\tau$) has attracted lots of attention as a promising way to derive an analytical formula (Allobaidi and Mallier (2001), Barles et al. (1995), Chevalier (2005), Dewynne et al. (1993), Evans, Keller and Kuske (2002), Goodman and Ostrov (2002), Lamberton and Villeneuve (2003); see also Levendorski (2007) and references therein for further general results). However, this approach has not produced a sufficiently accurate approximation under realistic model parameters (see the numerical examples in Mallier (2002)). Although a high-order short-maturity asymptotic approximation of $\overline{S}(\tau)$ is available in an analytical form (see Allobaidi and Mallier (2004)), accuracy is still an issue.

We now show why a direct short-maturity analysis does not yield an applicable formula for American options. This will motivate the introduction of a new method of option pricing. Consider the choice to exercise a put option now or wait till maturity. For illustration we consider the case

$P_S(S(T-t), t) = -1$. (2)
of zero dividend. If the option is exercised now then the option holder receives \( K - S_t \) (provided that \( K > S_t \)). The expected payoff of the option at maturity is equal to the European put price \( P^E \). Put-call parity:

\[
P^E = Ke^{-rT} - S_t + C^E,
\]

implies that the early exercise premium is given by:

\[
K(1 - e^{-rT}) - C^E.
\]

The early exercise decision may be interpreted as equivalent to giving up a call option while putting money in a bank account. If \( K \) is sufficiently greater than \( S_t \) (option is said to be deep in-the-money) then the European call option price \( C^E \) is small. In this case the interest rate income \( K(1 - e^{-rT}) > 0 \) exceeds the call price meaning that the early exercise has a positive premium.

To measure the moneyness of the put option we introduce a convenient parameterization, denoting:

\[
\theta = \frac{\ln\left(\frac{K}{S_t}\right)}{\sigma \sqrt{\tau}}.
\]

This ratio is called normalized moneyness, and is frequently used in the empirical literature on option pricing (see e.g. Bates (2000), Carr and Wu (2003)). It measures the distance between log of stock price and log of strike price in terms of standard deviations. \(^5\) Intuitively, the decision to exercise the put option early should be based on comparison of \( \theta \) with some time-independent critical level \( \bar{\theta} \), which is likely to be between 1 and 2. \(^6\)

Although this reasoning seems to be plausible, it is not entirely correct. Indeed, the short-maturity asymptotics of \( S(\tau) \) (see e.g. Barles et al. (1995)) implies that \( \ln\left(\frac{K}{S(\tau)}\right) \sim \sqrt{\tau \ln(1/\tau)} \), and therefore

\[
\bar{\theta}(\tau) = \frac{\ln\left(\frac{K}{S(\tau)}\right)}{\sigma \sqrt{\tau}} \sim \sqrt{\ln(1/\tau)}.
\]

\(^5\)Strictly speaking, we should take the ratio of the strike price to the forward price to take into account the drift in log \( S \). The two definitions, however, are equivalent when time-to-maturity is small.

\(^6\)For example, if \( \theta > \bar{\theta} = 1.65 \) then there is a 95% chance that the call option in (4) will end out-of-the-money. This implies that the call price is likely to be relatively small.
As we may observe from (6), when \( \tau \) is very small, no matter how large is the gap between the stock price and the strike price, it is still not optimal to exercise the option early. To explain why our intuition does not work near expiration, note that the call option price \( C^E \sim \sqrt{\tau} \) for given fixed \( \theta \) (see Medvedev and Scaillet (2007) and (18)). Consequently, when time-to-maturity is very small, the interest income \( K(1 - e^{-r\tau}) \sim \tau \) is only second order relevant, and the American put converges to a European put with \( \mathcal{S} = 0 \) and \( \mathcal{B} = \infty \).

The above reasoning suggests that the exact asymptotics (6) is most likely to be accurate in a region where the early exercise premium becomes negligible. This means that a direct short-maturity asymptotic analysis is not able to deliver a good approximation under realistic model parameters. In this paper we show that it is still possible to rely on a short-maturity asymptotic analysis if we modify the initial problem. Ideally, the solution to the modified problem should be very close to the American option price, and have regular asymptotics when time-to-maturity goes to zero. This can be done by imposing an intuitive exercise rule based on the normalized moneyness parameter \( \theta \).

### 2.2 Modified problem

Let us consider a modified version of problem (1), with the smooth pasting condition (2) replaced by an explicit exercise rule. The new problem is defined by the same PDE

\[
P_t + (r - \delta)S P_S + \frac{1}{2} \sigma^2 S^2 P_{SS} - r P = 0, \tag{7}
\]

with boundary conditions:

\[
P(\infty, t) = 0, \tag{8}
\]

\[
P(S, T) = \max(K - S, 0), \tag{9}
\]

\[
P(\mathcal{S}(T - t), t) = \max(K - \mathcal{S}(T - t), 0), \tag{10}
\]
where $\overline{S}(T-t)$ satisfies $\overline{S}(T-t) = Ke^{-\sigma y\sqrt{T-t}}$.\footnote{The modified problem is closely related to a barrier option pricing problem. There are few examples of boundaries for which the distribution of the first passage time of a Brownian motion is known in a closed form, but they do not suit our setting.}

The unique solution to this problem is the price of an option that is exercised as soon as the normalized moneyness reaches the boundary $y$. As we have already noted, this proxy for the optimal exercise rule is intuitively appealing since it is based on a normalized measure of moneyness. Hence, we expect the solution to the modified problem to be close to the true American option price.

To derive proper asymptotic expansions, we rewrite PDE (7) in terms of $(\theta, \tau)$ instead of $(S, t)$. Using the definition of $\theta$ in (5), and setting $P(\theta, \tau) \equiv P\left(Ke^{-\sigma \theta \sqrt{T-\tau}}, T-\tau\right)$, we make the following substitutions in (7): $P_t = -P_\tau + \frac{\theta}{2\tau} P_\theta$, $P_S = -\frac{1}{\sigma S \sqrt{\tau}} P_\theta$, and $P_{SS} = \frac{1}{\sigma^2 S^2 \tau} P_{\theta\theta} + \frac{1}{\sigma S^2 \sqrt{\tau}} P_\theta$. Simplifying, we obtain:

$$\theta P_\theta + P_{\theta\theta} + \frac{1}{\sigma} \left[\sigma^2 + 2(\delta - r)\right] P_\theta \sqrt{\tau} - 2(P_\tau + rP)\tau = 0.$$ \hspace{1cm} (11)

As we shall see in the next section, there is a unique solution to (11) satisfying boundary conditions (8) and (10) in the form:

$$P(-\infty, \tau) = 0,$$ \hspace{1cm} (12)

$$P(y, \tau) = K \max\left(1 - e^{-\sigma y \sqrt{\tau}}, 0\right) = K(1 - e^{-\sigma y \sqrt{\tau}}),$$ \hspace{1cm} (13)

and which has regular asymptotics near maturity of the form:

$$P(\theta, \tau) = \sum_{n=1}^{\infty} P_n(\theta) \tau^{\frac{n}{2}}.$$ \hspace{1cm} (14)

The third condition (9) follows from the form of the asymptotics (14). Indeed, when $\tau = 0$ and $\theta$ is held fixed we have $S = 0$ and $\max(K - S, 0) = 0$. Thus, $P(\theta, 0) = 0$, a condition implied by (14).

Let us denote the solution to the modified problem (11) with conditions (12), (13), and (14) by $P(\theta, \tau; y)$. The American put price can be approximated from below by:

$$P(\theta, \tau) \approx P(\theta, \tau; \overline{y}(\theta, \tau)) = \max\{P(\theta, \tau; y) : y > 0\}.$$ \hspace{1cm} (15)
From the discussion in the previous section, we expect that the value-maximizing boundary $\bar{y}$ should be somewhere between 1 and 2 in case $\delta = 0$. In addition, when $y$ goes to infinity, the solution to the modified problem converges to the European option price denoted by $P(\theta, \tau; \infty)$. Note also that if $r = 0$ and $\delta > 0$, it is not optimal to exercise the American put before its expiration, and $\bar{y} = \infty$. The two numerical examples in Figures 1a and 1b with $r = 0.05$, $\delta = 0$ and $r = 0$, $\delta = 0.05$ confirm these statements. In both figures we plot the solution $P(0, 1; y)$ to the modified problem corresponding to an at-the-money put option ($\theta = 0$) with 1 year to maturity ($\tau = 1$).

### 2.3 Asymptotic expansion

In the next Proposition we describe the series representation of the general solution to (11) without boundary condition (13). Then we show how a unique solution is determined by requiring (13). To make the presentation more compact, let us introduce some notation. The set of polynomials in $\theta$ of the form:

$$a_n\theta^n + a_{n-2}\theta^{n-2} + a_{n-4}\theta^{n-4} + \ldots + a_m\theta^m, \quad m = \text{mod}(n, 2)$$

is denoted by $\Pi^1(n, \theta)$, and the subset for which $a_n = 1$ is denoted by $\Pi^0(n, \theta)$. Furthermore, let us set: 

$$\Phi(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-\frac{s^2}{2}} ds, \quad \phi(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}.$$

**Proposition 1** Consider partial differential equation (11) with boundary condition (12) and the regular asymptotic expansion (14) in the vicinity of $(0,0)$. For any solution to this problem there exist constants $C_1, C_2, \ldots$ such that for each $n$:

$$P_n(\theta) = C_n \left[ p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta) \right] + p_n^1(\theta)\Phi(\theta) + q_n^1\phi(\theta),$$

where $p_n^0 \in \Pi^0(n, \theta)$, $p_n^1 \in \Pi^1(n-2, \theta)$, $q_n^0 \in \Pi^0(n-1, \theta)$, and $q_n^1 \in \Pi^1(n-3, \theta)$ with coefficients depending on model parameters and $C_1, C_2, \ldots, C_n$.

**Proof.** See Appendix A. $\blacksquare$
Proposition 1 describes the form of the asymptotic expansion of the general solution (14) with appropriate asymptotic given by (12). To determine a unique \( N \)th order expansion, we need to select \( N \) constants \( C_n, n = 1, \ldots, N \). Let us show how to do this using a second order expansion of equation (14) as an illustration. Using Proposition 1 we find by substitution:

\[
P(\theta, \tau; y) = C_1 [\theta \Phi(\theta) + \phi(\theta)] \sqrt{\tau} \\
+ \left[C_2 ((\theta^2 + 1) \Phi(\theta) + \theta \phi(\theta)) + \frac{\sigma C_1}{2} \Phi(\theta)\right] \tau + O(\tau \sqrt{\tau}).
\]

(16)

The coefficients \( C_1 \) and \( C_2 \) are uniquely determined by imposing the early exercise condition (13).

The short-maturity expansion of the payoff function is:

\[
P(y, \tau; y) = K \left[1 - \exp(-\sigma y \sqrt{\tau})\right] \\
= \sigma y K \sqrt{\tau} - \frac{\sigma^2 y^2 K}{2} \tau + O(\tau \sqrt{\tau}).
\]

(17)

Equalizing expansion (16) at \( \theta = y \) to expansion (17) allows us to identify the missing coefficients. The expressions for these coefficients can be found in Appendix B, where we present the short-maturity expansion of \( P(\theta, \tau; y) \) up to the 4th order. Recall that the European put price is obtained by setting \( y = \infty \). We can check that when \( y \to \infty \) we have \( C_1(\infty) = \sigma K, \quad C_2(\infty) = -\frac{K \sigma^2}{2}, \) and:

\[
P(\theta, \tau; \infty) = \sigma K (\theta \Phi(\theta) + \phi(\theta)) \sqrt{\tau} - K \left[\frac{\sigma^2}{2} (\theta^2 \Phi(\theta) + \theta \phi(\theta)) + r \Phi(\theta)\right] \tau + O(\tau).
\]

(18)

This is exactly the asymptotics of the European put implied by the put-call parity and results in Medvedev and Scaillet (2007) for the European call. It follows that the European put (call) converges to zero at the speed of \( \sqrt{\tau} \) when \( \tau \) goes to zero for given fixed \( \theta \). Observe also that the leading order in expansion (18) coincides with the put price under an arithmetic Brownian motion specification for the stock price (Bachelier formula). Both types of models are equivalent near maturity.
2.4 The early exercise price

In the previous section we dealt with pricing of the American put. However, we did not address the issue of how to decide on the early exercise of the option. Formally, an approximation for the true early exercise level of moneyness $\theta = \frac{\log(\frac{K}{S})}{\sigma \sqrt{\tau}}$ can be found by solving the following fixed-point problem for given $\tau$:

$$\theta = \overline{y}(\theta, \tau). \quad (19)$$

It is important that the solution to this fixed-point problem is unique, for otherwise the early exercise decision is not well-defined. Uniqueness also guarantees that we obtain the exact early exercise price provided we know $\overline{y}(\theta, \tau)$.

**Proposition 2** Assume that function $\overline{y}(\theta, \tau)$ is uniquely defined and problem (19) has a unique solution for $\theta \geq 0$. Then this solution coincides with $\theta$.

Let us provide some justification for the assumption made in the proposition. Consider the first and the second conditions of the maximization problem (15):

$$P_y(\theta, \tau; \overline{y}(\theta, \tau)) = 0, \quad P_{yy}(\theta, \tau; \overline{y}(\theta, \tau)) < 0. \quad (20)$$

Differentiating the first order condition in (20), we find:

$$\frac{d\overline{y}(\theta)}{d\theta} = -\frac{P_{\theta y}(\theta, \tau; \overline{y}(\theta, \tau))}{P_{yy}(\theta, \tau; \overline{y}(\theta, \tau))}. \quad (21)$$

Note that $P_{\theta}$ is always positive as it represents the sensitivity of the option price to changes in moneyness. When $y$ goes up, the early exercise price decreases, implying that it is likely that the option will be exercised early. As a consequence, the option price is likely to become less sensitive to changes in moneyness, meaning that $P_{\theta}$ is decreasing in $y$ or $P_{\theta y} < 0$. Recalling the second order condition in (20) and using equality (21) we conclude that $\frac{d\overline{y}(\theta, \tau)}{d\theta} < 0$. The uniqueness of solution to (19) follows if we note that $\overline{y}(0, \tau) > 0$ since $P(\theta, \tau; 0) = 0$. 


In Figure 2 we illustrate the uniqueness of solution to (19) for different time-to-maturity and volatility levels. Observe that the intersection between the 45-degree line (solid line $\theta$) and a corresponding downward sloping curve (dotted line $\bar{y}(\theta)$) is always unique. Then, by virtue of Proposition 2, it yields the exact early exercise level of moneyness of the American put.

Let us conclude this section by noting that, in practice, it is not necessary to solve the fixed point problem (19). The decision to exercise the option should be based on comparison of $\theta$ to the value-maximizing boundary $\bar{y}(\theta, \tau)$. If $\theta > \bar{y}(\theta, \tau)$ then the option should be exercised.

2.5 Performance of the approximation

In this section we perform several numerical experiments to study the accuracy of the approximation of the American put option introduced in the previous section. The approximation error has two possible sources: the asymptotic expansion and the suboptimal exercise rule. Hereafter we find that the convergence of the asymptotic expansion is extremely fast, meaning that the major source of the approximation error is the suboptimal exercise rule. This error also appears to be small.

2.5.1 Convergence of the asymptotic expansion

To illustrate the speed of convergence of the asymptotic expansion we compute the difference between the true American put price and the approximation. For the numerical experiment we assume $r = 0.05$, $\delta = 0$, $\sigma = 0.2$. The optimal $\bar{y}$ is found using a simple search algorithm. We start with $y = 1.5$ and then move in the direction of increasing price with a step of 0.1. When this preliminary search is terminated, we refine the search with a smaller step of 0.01. This procedure allows us to find $\bar{y}$ with a precision of 0.01.

Figure 3 shows absolute approximation errors of our method with option prices $P(\theta, \tau; y)$ being computed either with expansion (14) truncated at different orders ($N = 2, 3, 4, 5$), or with a 2000-step binomial tree ("Tree"). This allows us to better assess the speed of convergence of the asymptotic expansion, since the error in our method with a binomial tree instead of an asymptotic expansion is driven by the suboptimal exercise rule only. The true American put price is evaluated using a 2000-step binomial tree.
Observe that the convergence of the short-maturity asymptotic expansion is very fast at all maturities. The 4th order expansion, given explicitly in Appendix B as an example, already appears to be sufficiently close to the tree-based approximation. The higher order expansion terms appear to be negligibly small relative to the error stemming from the suboptimal exercise rule.

2.5.2 Comparison with existing methods

In this section we compare our approach with other analytical approximations developed for the Black-Scholes model. We perform the analysis using model parameters chosen in the corresponding paper.

Broadie and Detemple (1996) suggest simple lower and upper bounds on the American call price. The lower bound is computed as the maximum over prices of call options that are exercised as the price level reaches some critical value (capped call options). The difference between our approach and that of Broadie and Detemple (1996) is that we use this rule for the normalized moneyness rather than the price of the underlying asset. We believe that expressing the suboptimal exercise rule in terms of normalized moneyness is more appealing, and we expect our approach to be more accurate. Although the price of the capped option in Broadie and Detemple (1996) admits an exact analytical expression, our setup is nevertheless given by an asymptotic expansion with a fast convergence rate, as shown in the previous section.

Table 1 reports lower bounds on American call prices from Broadie and Detemple (1996) (Tables 1 and 2), along with our results. To gauge the early exercise premium we also give European option prices. We compute American call prices using the put-call symmetry (see Broadie and Detemple (2004)). The call option price is equal to the put option price with $S$ replaced by $K$ and, vice-versa, and $r$ replaced by $\delta$, and vice-versa.

The first part of Table 1 reports option values corresponding to time-to-maturity equal to 6 months. Here the convergence of the asymptotic expansion is sufficiently fast and the 4th order expansion is largely sufficient. The accuracy of our approximation is clearly superior to the lower

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9 The upper bound is derived from the lower bound using a formula for the early exercise premium. The same procedure can be applied using our approximation, which also provides a lower bound.

10 At least in the domain where time-to-maturity is not large.
bound of Broadie and Detemple (1996). The relative error does not exceed 0.2%, which is more than sufficient for applications. In the second part of Table 1 we compare different approximations of option prices with long time-to-maturity (3 years). The convergence of the series here is much slower. This is to be expected since we rely on a short-maturity expansion. Accuracy is still reasonably good even if we limit ourselves, for example, to a 5th order expansion with a relative error not exceeding 0.5%. Our approximation based on the 5th order expansion has again a better accuracy than the lower bound of Broadie and Detemple (1996).

With respect to computational efficiency, our method is equivalent to the lower bound approximation of Broadie and Detemple (1996). Both methods involve similar maximization procedure, and formulas have comparable complexities. To give an idea of the computational speed, a Matlab code requires only 0.002 seconds to compute an approximation with a 4th order expansion. This is comparable to a 35-step binomial tree on the same computer. Note that Broadie and Detemple (1996) approach may still be preferable for pricing long maturity options. In this case, our approximation requires higher order expansion, which increases the computational time.

Bunch and Johnson (2000) propose an alternative fast method for American option pricing based on an analytical approximation of the early exercise boundary. Table 2 reports option values from Bunch and Johnson (2000) (Table II) and our results based on a 4th order asymptotic expansion (see Appendix B). The accuracy of our approximation is comparable with that of Bunch and Johnson (2000), and is roughly equivalent to a 300-step tree.

2.5.3 Approximation of the option delta

In practical applications it is important that not only option prices but also their Greeks are computed accurately. Our approximation has the advantage of being based on an analytical formula. As a consequence, option Greeks can be directly computed by taking corresponding derivatives of the approximation formula and evaluating them at \( y = y(\theta) \). Here we consider the accuracy of the approximation of the option delta, which is the price derivative of the option price.

Figure 4 shows absolute errors in the delta of one-year options corresponding to different strike

\footnote{See Zhu and He (2007) for a modification of this approach better suited for approximating long term options.}
prices. Here we use the same model parameters as in Figure 3: \( r = 0.05, \delta = 0, \sigma = 0.2 \). The "true" option delta is found as a numerical derivative of 2000-step tree option price with price increment of 0.1. Observe that the convergence of option delta is as fast as that of the option price (see Figure 3). Therefore the approximation is accurate: the error of the option delta approximation based on 4th order expansion does not exceed 0.005.

### 3 Multifactor models

The true power of the method introduced in the previous sections lies in its broad applicability: it can be extended to a more general model with stochastic volatility and stochastic interest rates. Let us consider the following risk-neutral dynamics:

\[
\begin{align*}
    dS_t &= (r_t - \delta)S_t dt + \sigma_t S_t dW^{(1)}_t, \\
    d\sigma_t &= a(\sigma_t) dt + b(\sigma_t) dW^{(2)}_t, \\
    dr_t &= \alpha(r_t, t) dt + \beta(r_t) dW^{(3)}_t,
\end{align*}
\]

with \( dW^{(i)}_t dW^{(j)}_t = \rho_{ij} \). Model (22) nests most models used in applications.

#### 3.1 Modified problem and its solution

We proceed in the same manner as before. The PDE for the put option price \( P(S, \sigma, r, t) \) is:

\[
0 = P_t + P_S S(r - \delta) + P_\sigma a(\sigma) + P_r \alpha(r, t) + \frac{1}{2} P_{SS} S^2 \sigma^2 \\
+ \frac{1}{2} P_{\sigma\sigma} \beta^2(\sigma) + \frac{1}{2} P_{rr} \beta^2(r) + P_{S\sigma} \sigma Sb(\sigma) \rho_{12} \\
+ P_{Sr} \sigma S\beta(\sigma) \rho_{13} + P_{\sigma r} b(\sigma) \beta(\sigma) \rho_{23} - rP,
\]

with the boundary conditions given in (1).

As in the Black-Scholes case, we go from \( P(S, \sigma, r, t) \) to \( P(\theta, \sigma, r, \tau) \), and derive a recursive system of PDEs for short-maturity asymptotics of \( P \) (see details in Appendix D). Proposition 2
describes the general solution to (36) in the form of expansion (14) having appropriate behavior at infinity. We do not provide the proof of the proposition, which is lengthy but straightforward. The result may be verified by a direct substitution of the solution with unknown coefficients into (36).

**Proposition 3** Consider partial differential equation (36) with $\rho_{23} = 0$, the boundary condition:

$$P(-\infty, \sigma, r, t) = 0,$$

and regular asymptotic expansion:

$$P(\theta, \sigma, r, \tau) = \sum_{n=1}^{\infty} P_n(\theta, \sigma, r) \tau^\alpha,$$

with $(\theta, \tau)$ in the vicinity of $(0, 0)$. For any solution to this problem there exist functions $C_1(\sigma, r), C_2(\sigma, r), \ldots$ such that for each $n$:

$$P_n(\theta, \sigma, r) = C_n(\sigma, r) [p_n^0(\theta, \sigma, r) \Phi(\theta) + q_n^0(\theta, \sigma, r) \phi(\theta)]$$

$$+ p_n^1(\theta, \sigma, r) \Phi(\theta) + q_n^1(\theta, \sigma, r) \phi(\theta),$$

where $p_n^0 \in \Pi^0(n, \theta)$, $p_n^1 \in \Pi^1(n - 2, \theta)$, $q_n^0 \in \Pi^0(n - 1, \theta)$, and $q_n^1 \in \Pi^1(3n - 5, \theta)$ with coefficients depending on model parameters and $C_1(\sigma, r), C_2(\sigma, r), \ldots, C_n(\sigma, r)$.

In Proposition 3 we assume that the spot interest rate is not correlated with the volatility of the underlying. Economic intuition does not provide any insight regarding the sign of this correlation, and this assumption greatly simplifies the approximation formula. It is probably manageable to derive the form of the solution for the general problem with arbitrary correlations. However, practical implementation is likely to be significantly more complicated in that case.

It is important, however, to incorporate non-zero correlations between volatilities and the stock prices, and between interest rates and the stock prices. The negative correlation between stock prices and their volatilities has long been recognized as a main source of the implied volatility skew. Contrary to that, the empirical literature on European option pricing tends to assume no
correlation between stock prices and interest rates. Bakshi, Cao and Chen (1997) admit that this is a potentially limiting assumption (see footnote on page 2009), since economic theory clearly suggests a negative correlation. They claim, however, that using a more general model does not lead to better pricing and hedging results. This may not be the case for American options, which are more sensitive to interest rates due to the early exercise feature. As we will see in the numerical analysis below both correlations have a sizable effect on the early exercise premium.

To derive the approximation for the American put we proceed in the same manner as in the Black-Scholes case. We impose the early exercise condition (13) to solve for coefficients \( C_i(\sigma, r) \), \( i = 1, 2, \ldots, n, \ldots \). The recursive formulas for these coefficients can be easily obtained from a symbolic calculus software and copy-pasted in a code for option pricing. The American put price is approximately given by

\[
P(\theta, \tau) \simeq P_1(\theta, \tau; \overline{y}(\theta, \tau)) \sqrt{\tau} + P_2(\theta, \tau; \overline{y}(\theta, \tau)) \tau + \ldots,
\]

where

\[
\overline{y}(\theta, \tau) = \arg \max_y \{ P(\theta, \tau; y) ; y > 0 \}.
\]

### 3.2 Approximation of the price versus approximation of the early exercise premium

The American put option price is the sum of the European put price and the early exercise premium. The approximation error can be decomposed in a similar way. If two errors are relatively independent then the accuracy of our approximation can be improved in models where European put option prices are available in a closed form. A well-known example is the class of affine multi-factor models, where European options can be valued quickly and accurately via the inverse Fourier transform (Duffie et al. (2000)).

Recall that the European put price is the solution to the modified problem with \( \overline{\theta} = \infty \). Consequently, the early exercise premium is approximately given by \( P(\theta, \tau; \overline{y}(\theta, \tau)) - P(\theta, \tau; \infty) \), where
$\bar{y}(\theta, \tau)$ is defined in (25). Now suppose that the price of the European put is available. Then the American put price may be approximated by

$$
P(\theta, \tau) \simeq P(\theta, \tau; \infty) + [P_1(\theta; \bar{y}(\theta, \tau)) - P_1(\theta; \infty)] \sqrt{\tau} + [P_2(\theta; \bar{y}(\theta, \tau)) - P_2(\theta; \infty)] \tau + ... \quad (26)
$$

To distinguish between the two approximations presented so far, we refer to (24) as approximation 1 and to (26) as approximation 2.

Let us use a numerical example to check whether approximation 2 performs better indeed. For a numerical experiment, we assume $r = 0.05$, $\delta = 0$, $\sigma = 0.2$. The volatility is stochastic with the risk-neutral dynamics of the variance $v_t = \sigma^2_t$ given by the square-root process:

$$
dv_t = \kappa_v (0.04 - v_t)dt + 0.2\sqrt{v_t}dW_t^{(2)} \quad \text{with} \quad \rho_{12} = -0.5, \quad (27)
$$

with mean-reversion parameter $\kappa_v = 0$ (no mean-reversion) or $\kappa_v = 2$ (fast mean-reversion).

This model belongs to the affine class and has been first proposed by Heston (1993). Figures 5 and 6 plot absolute errors of our method based on expansion (24) (panel (a)) and expansion (26) (panel (b)) along with a benchmark with a 200,000-path Monte-Carlo simulation. The error of our method based on the simulation should be mainly determined by the suboptimal exercise rule. The values of the American put are obtained using the Monte-Carlo-based approach of Longstaff and Schwartz (2001) with 200,000 sample paths, 500 time steps and 50 exercise dates.

Observe that the convergence of the asymptotic expansion under stochastic volatility is much slower than in the Black-Scholes model (compare Figures 5 and 6 to Figure 3). In the case of no mean-reversion in stochastic volatility (Figure 5), the convergence speed still appears to be quite good, even for options with one year to maturity. In this case, there is no noticeable difference between approximations 1 and 2, and the 5th order expansion seems to be sufficiently accurate. In

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\[12\] And, of course, the error of the Monte-Carlo method, which, on average, should be equal to zero.

\[13\] Beside regression-based methods, other Monte-Carlo based approaches are available to price American options such as stochastic mesh methods (Broadie and Glasserman (1997)), and duality methods (Haugh and Kogan (2004), Rogers (2002)); see the book of Glasserman (2003) for further details.

\[14\] To be precise, for each option price we run 10 Monte-Carlo simulations with 20,000 (10,000 plus 10,000 antithetic) paths each and then compute the average.
the case of fast mean-reversion in volatility, approximation 1 based on expansion (24) shows bad convergence if time-to-maturity exceeds 6 month. On the contrary, approximation 2 based on (26) performs as well as in the no mean-reversion case.

Mean-reversion in volatility means that, over a sufficiently long period of time, the average volatility converges to a constant level independent of the spot volatility (see Lewis (2000)). As a result, an option price with sufficiently long time-to-maturity does not depend on the spot volatility. On the contrary, the short-maturity asymptotic expansion is a function of the spot volatility, hence, it cannot yield an accurate approximation in this case. This is especially true for the European put price that directly depends on the distribution of the average volatility \(^{15}\) over the life-time of the option. The American put gives the possibility of an early exercise; therefore, its price is less closely related to the average volatility over the whole period up to maturity. The early exercise premium, being the American put stripped of the part that is related to the average volatility, has better chances to be well approximated by a short-maturity expansion. This is why we observe an improvement of the asymptotic convergence when using approximation 2 instead of approximation 1.

### 3.3 The early exercise price

An approximation for the early exercise price can be found in the same way as in the Black-Scholes case by solving the fixed-point problem (19). If the solution to this problem is unique then it gives the exact early exercise price provided that we have an accurate \(\varphi(\theta, \tau)\). In the Black-Scholes case the fast convergence of the asymptotic expansion guarantees that \(\varphi(\theta)\) can be computed almost exactly. As we have observed in the previous section, the convergence of the asymptotic expansion in the general case is not always satisfactory. In particular, the speed of mean reversion of the volatility may significantly affect the results.

Figure 7 illustrates these points using the numerical examples of the previous section. In the absence of mean-reversion (Figure 7a), function \(\varphi(\theta)\) computed using a 5th order expansion

\(^{15}\)To be more precise, the European option price depends on the joint distribution of the integrated variance and the stock price.
monotonically decreases, and intersects the 45-degree once. In the case of fast mean-reverting stochastic volatility (Figure 7b), the approximated $\bar{y}(\theta)$ is not always monotonically decreasing. The monotonicity is distorted when time-to-maturity increases, leading to no intersection with the 45 degree line in the high volatility case.

3.4 Comparison with existing methods

In this section we provide several numerical examples to assess the accuracy of both approximations. First, we consider two models where either volatility or interest rates are stochastic. For each model we select the most recent paper that advocates an alternative computational method. For different model parameters we provide three American put prices: the one computed under our analytical approach, the one computed under the alternative approach, and the one computed under a Monte-Carlo approach. The third price is considered as the "true" one since we use a large number of simulated paths. We conclude the section by reporting our results for the model with both stochastic volatility and stochastic interest rates. No reference paper exists for this general case as of the time of this writing.

3.4.1 Stochastic volatility

The literature on the pricing of American options under the two-factor model with stochastic volatility is relatively limited. The typical approach relies on solving numerically the PDE satisfied by the American option (see Ikonen and Toivanen (2007) and references therein). We take the same numerical examples based on the following Heston model specification:

\[
\begin{align*}
\frac{dS_t}{S_t} &= rS_t + \sqrt{v_t}S_t dW_t^{(1)}, \\
\frac{dv_t}{v_t} &= \kappa_v(\bar{v} - v_t)dt + \sigma_v \sqrt{v_t}dW_t^{(2)},
\end{align*}
\]  

with $r = 0.1$, $\bar{v} = 0.16$, $\sigma_v = 0.9$, $\kappa_v = 5$, $\rho_{12} = 0.1$, and two levels of the spot volatility parameter $\sigma = \sqrt{\bar{v}} = 0.25$ and 0.5.

The model parameters used in the literature are not very "typical". For example, the empirical
estimates of the volatility of volatility, and the rate of volatility mean reversion are usually much lower in magnitude (see e.g. Bakshi, Cao and Chen (2000)). Since the size of model parameters might affect the accuracy of the asymptotic expansion we consider two sets of parameters. In Table 3, the second and the fourth columns correspond to the model parameters adopted in Ito and Toivanen (2006), and others. The third and the fifth columns correspond to half the original volatility of volatility parameter and to half the original rate of mean reversion.

Table 3 compares our approximations with those of: (1) Longstaff and Schwartz (2001) based on Monte-Carlo simulations with 200,000 sample paths (100,000 plus 100,000 antithetic), 500 time steps and 50 exercise dates; and (2) Ito and Toivanen (2006). The European put prices for the same model parameters are computed using a closed form formula.

Approximation 2, based on a 5th order expansion, yields accurate values for all combinations of model parameters. Approximation 1 performs less impressively for the first set of model parameters with large volatility of volatility and fast rate of mean-reversion. On the other hand, the expansion yields reasonably accurate approximation for more realistic parameter values. A Matlab code takes less than 0.1 seconds to compute an option price using a 5th order expansion with the search algorithm described in Section 2.5.1. The computational time required by the Ito and Toivanen (2006) algorithm for the same magnitude of errors is at least twice as much.

3.4.2 Stochastic interest rates

A separate line of literature deals with pricing American options under stochastic interest rates. The simple and flexible Hull and White (1990) specification is widely used to capture non-flat stochastic term structure. The dynamics of the short-term interest rate is

\[ dr_t = (\eta(t) - \gamma r_t) dt + \beta dW_t \]

where \( \eta(t) \) is some deterministic function that is calibrated from the current term structure (see Hull and White (1990)). In applying our method, we use the expansion

\[ \eta = \eta(T) - \eta'(T) \tau + O(\tau^2), \]

which allows to rewrite expansion (34) with the following coefficients: \( \alpha_0(r) = \eta(T) - \gamma r \), and \( \alpha_1(r) = -\eta'(T) \).

To check the accuracy of our approximations we use the model parameter values of Menkveld

\footnote{We thank Jari Toivanen for providing us with the results of Ito and Toivanen (2006) pricing method for other values of model parameters.}
and Vorst (2001). They consider two types of non-flat interest rate term structure: upward sloping and downward sloping. They also vary the correlation between the underlying and the spot interest rate. We use the same model parameters $\beta = 0.01$, $\gamma = 0.1$ for the comparison. We consider only the case of zero correlation since the put prices appear not to be very sensitive to this parameter for the chosen small value of the volatility of interest rate parameter $\beta$. Table 4 shows the at-the-money put option prices for different combinations of the spot volatility and shapes of the term structure. As in the previous section we provide the results based on Longstaff and Schwartz (2001) algorithm with 200,000 sample paths, 500 time steps, and 50 exercise dates as true option prices. Our approximations are again computed using expansions up to the 5th order with the search algorithm described in Section 2.5.1. Observe that both approximations are sufficiently accurate and in most cases are closer to the Monte-Carlo results than the values obtained by Menkveld and Vorst (2001).

### 3.4.3 Stochastic volatility and stochastic interest rates

In this section we assess the accuracy of our method in the case of a model with stochastic volatility and stochastic interest rates. Since there is no reference paper for that case, we compare our approximations only with the outcome of the Monte-Carlo based approach. We assume an affine specification, and borrow model parameter values from Bakshi, Cao and Chen (2000). The risk-neutral dynamics is:

\[
\begin{align*}
    dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\
    dv_t &= 1.58(0.03 - v_t)dt + 0.20\sqrt{v_t}dW_t^{(2)}, \\
    dr_t &= 0.26(0.04 - r_t)dt + 0.08\sqrt{r_t}dW_t^{(3)}, \\
    \rho_{12} &= -0.26, \quad \rho_{13} = \rho_{23} = 0.
\end{align*}
\]

Table 5 reports the European and American put values for different combination of the spot volatility and time-to-maturity. The American put price is computed using Longstaff and Schwartz (2001) approach with 200,000 sample paths, 500 time steps, and 50 exercise dates. The European
put price is computed using a closed form expression. Our approximations are computed using expansions up to the 5th order with the search algorithm described in Section 2.5.1.

Our method yields reasonably accurate option prices. Approximation 2 seems to be more accurate than approximation 1. To give an idea of the computational advantage of our method, the Matlab code with Longstaff and Schwartz (2001) algorithm takes about 15 minutes to compute one option price. Our approximation takes only 0.1 seconds.

3.5 Effect of stochastic volatility and stochastic interest rates

In this section we take the advantage of our fast pricing algorithm to study the effects of stochastic volatility and stochastic interest rates on the American put price. In particular, we are interested in their effect on the early exercise premium. A simple alternative approximation approach to pricing American options in a multifactor setting is to compute the European option price using known closed form solution while adding the early exercise premium evaluated under the Black-Scholes model.\footnote{We thank Liuren Wu for pointing out this approach sometimes used in practice.} In this section we explain why such an approach may result in an economically significant mispricing. Apart from illustrating the advantage of our approach for option pricing, this study provides new insights on the key determinants of the American put price.

The effect of a change in model parameters on the American put price ($\Delta P$) can be decomposed into the effect on the early exercise premium ($\Delta \text{EEP}$) and the European put price ($\Delta P^E$):

$$
\Delta P = \Delta \text{EEP} + \Delta P^E.
$$

(29)

Note that for $S$ below the early exercise price, the American put price should be equal to its payoff. Therefore, for sufficiently small $S$ (or large $K$) $\Delta P = 0$. This implies $\Delta \text{EEP} = -\Delta P^E$, which means that the effect on the early exercise premium is comparable with the impact on the European put price. If we believe that a change of model has an economically significant impact on the European put price, then we should accept that the impact on the early exercise premium cannot be neglected. In this section we illustrate the effect of volatility mean-reversion, volatility
of volatility, volatility of interest rate, and correlations on the elements of identity (29).

We consider the following affine model specification:

\[
\begin{align*}
    dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_t^{(1)} , \\
    dv_t &= k_v (0.04 - v_t) dt + \sigma_v \sqrt{v_t} dW_t^{(2)} , \quad dW_t^{(1)} dW_t^{(2)} = \rho_{12} dt , \\
    dr_t &= \sigma_r \sqrt{r_t} dW_t^{(3)} , \quad dW_t^{(1)} dW_t^{(3)} = \rho_{13} dt .
\end{align*}
\]

(30)

with \( r_0 = 0.05 , \sigma_0 = 0.2 \). To insure that our method is accurate, we consider options with 6 months to maturity.

Figure 8 illustrates the effect of introducing different model specifications on the American put price, the early exercise premium, and the European put price. Here the benchmark is the Black-Scholes model. Observe that in-the-money, the effect on the early exercise premium is comparable in magnitude and opposite in sign to the effect on the European put price, as expected. In particular, the European put price and the early exercise premium are greatly affected by the leverage effect \( (\rho_{12} < 0 \text{, see Figures 8c, 8d and 8f}) \), which results in the well-known phenomenon of the implied volatility skew. A fast mean-reversion in volatility \( (k_v = 2 \text{ instead of } k_v = 0) \) reduces the impact of stochastic volatility on both American and the European option price (compare Figure 8a to Figure 8b, and Figure 8c to Figure 8d).

The volatility of the interest rate has a negligible impact if correlation parameter \( \rho_{13} = 0.18 \).

However, a negative correlation results in a significant impact on the European put price as well as on the early exercise premium (Figure 8e). Indeed, negative correlation between the stock price and the spot interest rate means that the states where the European put yields larger payoffs (low price) are discounted heavier due to higher interest rates. As illustrated by Figure 8e, such an effect is not present in the case of in-the-money American puts. The possibility of early exercise reduces the expected lifetime of the put, thus diminishing the impact of stochastic interest rates.

We conclude this section with a last observation. As suggested by Figure 8, out-of-the-money American puts can be priced relatively accurately by assuming that the early exercise premium is

\[^{18}\text{We have not plotted the graph corresponding to this case.}\]
the same as in the Black-Scholes model. In terms of identity (29), this means that $\Delta P \simeq \Delta P^E$ for $S >> K$. There the early exercise premium is relatively small and its magnitude can be safely assumed unaffected.

4 Concluding remarks

In this paper we describe a new approach to pricing American options in a general setting with stochastic volatility and stochastic interest rates. Although the analytical approximation is based on a short-maturity asymptotic expansion, it performs extremely well in the Black-Scholes context with time-to-maturity up to several years. Under stochastic volatility, the convergence of the asymptotic expansion is adversely affected by fast mean-reversion in volatility. This problem is dealt with by considering the approximation of the early exercise premium instead of the American put. Then the convergence is achieved much faster: the approximation remains accurate for options with time-to-maturity up to one year. Using our method, we run several numerical experiments to study the effect of model specification on the American put. These experiments provide new insights on the behavior of the early exercise premium and the early exercise boundary in the presence of additional stochastic factors.

References


Quantitative Finance 6, pp. 229-242.

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Finance 7:10, pp. 1203-1227.
APPENDIX A. Proof of Proposition 1.

Substituting (14) into (11) we arrive at:

\[-nP_n + \theta P_{n\theta} + P_{n\theta\theta} + \frac{1}{\sigma} \left( \sigma^2 + 2(\delta - r) \right) P_{n-1\theta} - 2r P_{n-2\theta} = 0, \quad n = 1, 2...\]  \hspace{1cm} (31)

with \(P_0 = P_{-1} = 0\). The homogeneous solutions of equation (31) form a two dimensional space. One dimension is spanned by a polynomial solution which does not satisfy the boundary condition (8) at infinity. The other independent solution has the form:

\[P_n^0(\theta) = p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta).\]  \hspace{1cm} (32)

Let us substitute (32) in the homogeneous part of (31). After some rearrangements we find:

\[\left( \frac{d^2 p_n^0}{d\theta^2} + \theta \frac{dp_n^0}{d\theta} - np_n^0 \right) \Phi(\theta) + \left( -(n + 1)q_n^0 - \theta \frac{dq_n^0}{d\theta} + \frac{d^2 q_n^0}{d\theta^2} + 2 \frac{dp_n^0}{d\theta} \right) \phi(\theta) = 0.\]

It is easy to verify that PDE \(\frac{d^2 p_n^0}{d\theta^2} + \theta \frac{dp_n^0}{d\theta} - np_n^0 = 0\), has polynomial solution \(p_n^0(\theta) = \pi_n^0(\theta^n + \pi_n^1(\theta^{n-2} + \pi_n^2(\theta^{n-4} + ..., with \pi_n^0 = 1, \pi_n^1 = \frac{(n - 2i)(n - 2i - 1)}{2i + 2} \pi_n^0.\) The polynomial solution to \(-(n + 1)q_n^0 - \theta \frac{dq_n^0}{d\theta} + \frac{d^2 q_n^0}{d\theta^2} + 2 \frac{dp_n^0}{d\theta} = 0\), has the form

\[q_n^0(\theta) = \pi_n^0(\theta^{n-1} + \pi_n^1(\theta^{n-3} + \pi_n^2(\theta^{n-5} + ... with \]

\[\pi_n^0(n - 2i)(n - 2i - 2) \pi_n^0(n - 2i - 2) \]

\[\frac{2n - 2i - 2}{2n - 2i - 2} \]

Let us now find a particular solution \(P_n^1(\theta)\), which satisfies the boundary condition at infinity. Any solution of (31) with appropriate behavior at the boundary will be given by:

\[P_n(\theta) = C_n P_n^0(\theta) + P_n^1(\theta),\]

where \(C_n\) is some constant. Let us look for a particular solution \(P_n^1\) in the form \(P_n^1(\theta) = p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta).\) This implies that the general solution is:

\[P_n(\theta) = C_n \left[ p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta) \right] + p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta).\]  \hspace{1cm} (33)

30
Let us guess that polynomials $p_n^1$ and $q_n^1$ are as follows:

\[ p_n^1(\theta) = \pi_{n0}^1 \theta^n + \pi_{n1}^1 \theta^{n-2} + \pi_{n2}^1 \theta^{n-4} + \ldots, \]

\[ q_n^1(\theta) = \kappa_{n0}^1 \theta^{n-1} + \kappa_{n1}^1 \theta^{n-3} + \kappa_{n2}^1 \theta^{n-5} + \ldots \]

After substituting $P_{n-1}$ and $P_{n-2}$ in the form (33) into equation (31) for $P_n^1$ we will obtain a system of two equations:

\[
\frac{d^2 p_n^1}{d\theta^2} + \theta \frac{dp_n^1}{d\theta} - np_n^1 + \left( \sigma C_{n-1} \frac{dp_{n-1}^0}{d\theta} + \sigma p_{n-1}^1 - 2r C_{n-2} p_{n-2}^0 - 2r p_{n-2}^1 \right) = 0,
\]

\[
-(n+1) q_n^0 - \theta \frac{dq_n^0}{d\theta} + \frac{d^2 q_n^0}{d\theta^2} + 2 \frac{dp_n^1}{d\theta} + \left( \sigma C_{n-1} q_{n-1}^0 + \sigma C_{n-1} \frac{dq_{n-1}^0}{d\theta} \right)
- \sigma C_{n-1} q_{n-1}^0 + \sigma p_{n-1}^1 + \sigma \frac{dq_{n-1}^1}{d\theta} - \theta q_{n-1}^1 - 2r C_{n-2} q_{n-2}^0 - 2r C_{n-2} q_{n-2}^1 \right) = 0.
\]

These equations can be solved as before. In particular we may assume $\pi_{n0}^1 = \kappa_{n0}^1 = 0$ since we can safely subtract a homogeneous solution. Here we do not write down the lengthy recursive relationship. In practice the PDE for $P_n^1$ can be solved directly by the substitution of its guessed form.

**APPENDIX B. 4th order expansion of the solution to the modified problem under the Black-Scholes model.**

The solution to the modified problem has the 4th order short-maturity expansion:

\[
P(\theta, \tau; y) = \sum_{n=1}^{4} \tau^{n} \left\{ C_n \left[ p_n^0(\theta) \Phi(\theta) + q_n^0(\theta) \phi(\theta) \right] + p_n^1(\theta) \Phi(\theta) + q_n^1(\theta) \phi(\theta) \right\},
\]

where

\[
p_n^0(\theta) = \theta, \quad p_n^1(\theta) = 0, \quad q_n^0(\theta) = 1, \quad q_n^1(\theta) = 0,
\]

31
\[
\begin{align*}
p_0^0(\theta) &= \theta^2 + 1, \quad p_2^2(\theta) = \frac{1}{2\sigma} C_1 \left( \sigma^2 - 2 \mu \right), \quad q_2^0(\theta) = \theta, \quad q_2^1(\theta) = 0, \\
p_0^3(\theta) &= \theta^3 + 3\theta, \quad p_3^1(\theta) = \frac{1}{\sigma} \left[ C_2 \sigma^2 - 2 C_2 \mu - r C_1 \sigma \right] \theta, \quad q_3^0 = \theta^2 + 2, \\
q_3^1(\theta) &= \frac{1}{8\sigma^2} \left[ 8 C_2 \sigma^3 - 16 C_2 \sigma \mu - 8 r C_1 \sigma^2 - 4 C_1 \sigma^2 \mu + C_1 \sigma^4 + 4 C_1 \mu^2 \right], \\
p_4^0(\theta) &= \theta^4 + 6\theta^2 + 3, \\
p_4^1(\theta) &= \frac{1}{2\sigma} \left[ 3 C_3 \sigma^2 - 6 C_3 \mu - 2 r \sigma C_2 \right] \theta^2 + \frac{1}{4\sigma^2} \left[ 3 C_3 \sigma^3 + \sigma^4 C_2 - 6 C_3 \sigma \mu - \sigma \left( -3 C_3 \sigma^2 + 6 C_3 \mu + 2 r \sigma C_2 \right) 
+ 4 C_2 \mu^2 - 2 \sigma^3 r C_1 - 4 \sigma^2 C_2 \mu - 2 r \sigma^2 C_2 + 4 \sigma r C_1 \mu \right], \\
q_4^0(\theta) &= \theta^3 + 5\theta, \\
q_4^1(\theta) &= \frac{1}{48\sigma^3} \left[ 8 C_1 \mu^3 + 72 C_3 \sigma^4 - C_1 \sigma^6 - 48 r \sigma^3 C_2 + 6 C_1 \sigma^4 \mu - 12 C_1 \sigma^2 \mu^2 - 144 C_3 \sigma^2 \mu \right] \theta,
\end{align*}
\]

and

\[
\begin{align*}
C_1 &= \left( K y \sigma \right)^{-1} \left( \Phi_0 y + \phi_0 \right), \\
C_2 &= - \left( \Phi_0 C_1 \sigma^2 - 2 \Phi_0 C_1 \mu + K y^2 \sigma^3 \right) \left[ 2\sigma \left( \Phi_0 y^2 + \Phi_0 + \phi_0 y \right) \right]^{-1},
\end{align*}
\]
\[ C_3 = \left[ 24\sigma^2 \left( \Phi_0 y^3 + 3 \Phi_0 y + \phi_0 y^2 + 2 \phi_0 \right) \right]^{-1} \times \left( -24 \Phi_0 y \sigma^3 C_2 + 48 \Phi_0 y \sigma C_2 \mu + 24 \Phi_0 y \sigma^2 r C_1 \right. \\
\left. -24 \phi_0 C_2 \sigma^3 + 48 \phi_0 C_2 \sigma \mu + 24 \phi_0 r C_1 \sigma^2 \\
+12 \phi_0 C_1 \sigma^2 \mu - 3 \phi_0 C_1 \sigma^4 - 12 \phi_0 C_1 \mu^2 + 4 K y^2 \sigma^5 \right), \]

\[ C_4 = -\left[ 48\sigma^3 \left( \Phi_0 y^4 + 6 \Phi_0 y^2 + 3 \Phi_0 + \phi_0 y^3 + 5 \phi_0 y \right) \right]^{-1} \times \left( 72 \Phi_0 \sigma^4 y^2 C_4 - 144 \Phi_0 \sigma^2 y^2 C_3 \mu - 48 \Phi_0 \sigma^3 y^2 r C_2 + 48 \Phi_0 \sigma C_2 \mu^2 \\
+12 \Phi_0 \sigma^3 C_2 + 72 \Phi_0 \sigma^4 C_3 - 144 \Phi_0 \sigma^2 C_3 \mu - 48 \Phi_0 \sigma^3 r C_2 - 24 \Phi_0 \sigma^4 r C_1 \\
-48 \Phi_0 \sigma^3 C_2 \mu + 48 \Phi_0 \sigma^2 r C_1 \mu + 8 \phi_0 y C_1 \mu^3 + 72 \phi_0 y C_3 \sigma^1 - \phi_0 y C_1 \sigma^0 \\
-48 \phi_0 y r \sigma^3 C_2 - 12 \phi_0 y C_1 \mu^2 \sigma^2 - 144 \phi_0 y C_3 \sigma^2 \mu + 6 \phi_0 y C_1 \sigma^4 \mu + 2 K y^4 \sigma^7 \right), \]

\[ \mu = r - \delta, \ \Phi_0 = \Phi(y), \ \phi_0 = \phi(y). \]
APPENDIX C. Proof of Proposition 2

Given the uniqueness assumption, it is sufficient to show that \( \bar{\theta} \) solves (19). Suppose that the spot price is equal to the early exercise price, that is \( \theta = \bar{\theta} \). Then, by definition, the American put and the put option with suboptimal exercise strategy \( y = \bar{\theta} \) should both be equal to the put option payoff \( K - S \). Since the price of any put option with a suboptimal exercise strategy does not exceed the price of the American put, then \( y = \bar{\theta} \) is necessarily the value-maximizing suboptimal strategy. This proves that \( \gamma(\bar{\theta}) = \bar{\theta} \).

APPENDIX D. The PDEs for short-maturity asymptotics of \( P(\theta, \sigma, r, \tau) \)

Let us make the change of variables from \((S, t)\) to \(\theta = \frac{\log(S/K)}{\sigma \sqrt{T - t}}\) and \(\tau = T - t\) and make use of the following relationships:

\[
\begin{align*}
P_S &= -\frac{1}{\sigma S\sqrt{T}} P_\theta, \\
P_{SS} &= \frac{1}{\sigma^2 S^2 T} P_{\theta \theta} + \frac{1}{\sigma S \sqrt{T}} P_\theta, \\
P_t &= \frac{\theta}{2\tau} P_\theta - P_\tau, \\
P_\sigma &= P_\sigma - \frac{\theta}{\sigma} P_\theta, \\
P_{\sigma S} &= -\frac{1}{\sigma S \sqrt{T}} P_{\sigma \theta} + \frac{1}{\sigma^2 S \sqrt{T}} P_\theta + \frac{\theta}{\sigma^2 S \sqrt{T}} P_{\theta \theta}, \\
P_{\sigma \sigma} &= P_{\sigma \sigma} - \frac{2\theta}{\sigma} P_{\sigma \theta} + \frac{2\theta}{\sigma^2} P_\theta + \frac{\theta^2}{\sigma^2} P_{\theta \theta}, \\
P_r &= P_r, \\
P_{Sr} &= -\frac{1}{\sigma S \sqrt{T}} P_{\theta r}, \\
P_{rr} &= P_{rr},
\end{align*}
\]

\(\alpha(r, t) = \alpha(r, T - \tau) = \alpha_0(r) + \tau \alpha_1(r) + \tau^2 \alpha_2(r) + \ldots \) \hspace{1cm} (34)
This allows us to transform (23) into:

\[
0 = \frac{\theta}{2} P_0 + \frac{1}{2} P_{0\theta} - \tau P_r + \sqrt{\tau} \left[ \frac{1}{2\sigma} \left( \sigma^2 + 2(\delta - r) \right) P_0 + b \rho_{12} \left( -P_{\sigma\theta} + \frac{1}{\sigma} P_0 + \frac{\theta}{\sigma} P_{0\theta} \right) \right]
\]

\[
- \beta \rho_{13} P_{\theta r} + b \beta \rho_{23} \left( P_{\sigma r} - \frac{\theta}{\sigma} P_{\theta r} \right) + \tau \left[ a \left( P_{\sigma} - \frac{\theta}{\sigma} P_{\theta} \right) \right]
\]

\[
+ \frac{\beta^2}{2} \left( P_{\sigma\sigma} - \frac{2\theta}{\sigma} P_{\sigma\theta} + \frac{\theta^2}{\sigma^2} P_{\theta\theta} \right) + \frac{\beta^2}{2} P_{rr} - \tau r P + \alpha_0(r) P_r
\]

\[
+ \tau^2 \alpha_1(r) P_r + \tau^3 \alpha_2(r) P_r ...
\]  

Further, let us take an expansion of the option price near maturity of the form:

\[
P = P_1 \sqrt{\tau} + P_2 \tau + P_3 \tau \sqrt{\tau} + ...
\]  

Substituting this into (35), we obtain the following PDE for \( P_n \) (\( n > 0 \)):

\[
0 = P_{n\theta \theta} + \theta P_{n\theta} - n P_n + \frac{1}{\sigma} \left( \sigma^2 + 2(\delta - r) \right) P_{n-1\theta}
\]

\[
+ 2b \rho_{12} \left( -P_{n-1\sigma \theta} + \frac{1}{\sigma} P_{n-1\theta} + \frac{\theta}{\sigma} P_{n-1\theta \theta} \right)
\]

\[
- 2\beta \rho_{13} P_{n-1\theta r} + 2b \beta \rho_{23} \left( P_{n-1\sigma r} - \frac{\theta}{\sigma} P_{n-1\theta r} \right)
\]

\[
+ 2a \left( P_{n-2\sigma} - \frac{\theta}{\sigma} P_{n-2\theta} \right) + b^2 \left( P_{n-2\sigma\sigma} - \frac{2\theta}{\sigma} P_{n-2\sigma\theta} \right)
\]

\[
+ \frac{2\theta}{\sigma^2} P_{n-2\theta} + \frac{\theta^2}{\sigma^2} P_{n-2\theta\theta} \right) + P_{n-2rr} \beta^2 - 2r P_{n-2} + \alpha_0(r) P_{n-2}
\]

\[
+ \alpha_1(r) P_{n-4} + \alpha_2(r) P_{n-6} + ..., \]  

with \( P_m = 0 \) for \( m \leq 0 \).
Table 1. American call option prices and their approximations under the Black-Scholes model.

The table compares option price bounds of Broadie and Detemple (1996) with our approximation based on asymptotic expansions of different orders. Broadie&Detemple refers to option price lower bounds reported in Tables 1 and 2 of Broadie and Detemple (1996). “True value” is a 15’000-step binomial tree approximation computed in Broadie and Detemple (1996). Here all options have strike price $K = 100$. Time-to-maturity $\tau$, asset price $S$, interest rate $r$, volatility $\sigma$, and dividend rate $\delta$ are indicated in the table.

<table>
<thead>
<tr>
<th>Option parameters $r$, $\sigma$, $\delta$</th>
<th>$S$</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>European $\tau = 0.5$ years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>----------------------------------------</td>
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<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>European $\tau = 3$ years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2. Put option prices and their approximations under the Black-Scholes model.
The table compares the approach of Bunch and Johnson (2000) with our approximation based on a 4-th order asymptotic expansion.
Bunch & Johnson refers to results reported in Table II of Bunch and Johnson (2000). “True value” is the 10,000-step binomial tree approximation computed in Bunch and Johnson (2000). Here asset price $S = 40$ and interest rate $r = 0.0488$. Time-to-maturity $\tau$ and asset price volatility $\sigma$ are indicated in the table.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\sigma = 0.2$</th>
<th>$\sigma = 0.3$</th>
<th>$\sigma = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tau = 1/12$</td>
<td>$\tau = 1/3$</td>
<td>$\tau = 7/12$</td>
</tr>
<tr>
<td>European put</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$K = 35$</td>
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</tr>
<tr>
<td></td>
<td>0.006</td>
<td>0.196</td>
<td>0.417</td>
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<td>0.687</td>
<td>1.189</td>
</tr>
<tr>
<td></td>
<td>0.246</td>
<td>1.330</td>
<td>2.113</td>
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<tr>
<td>4th order</td>
<td>0.006</td>
<td>0.200</td>
<td>0.432</td>
</tr>
<tr>
<td></td>
<td>0.077</td>
<td>0.697</td>
<td>1.218</td>
</tr>
<tr>
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<td>0.247</td>
<td>1.345</td>
<td>2.152</td>
</tr>
<tr>
<td>Bunch &amp; Johnson</td>
<td>0.006</td>
<td>0.200</td>
<td>0.433</td>
</tr>
<tr>
<td></td>
<td>0.077</td>
<td>0.698</td>
<td>1.229</td>
</tr>
<tr>
<td></td>
<td>0.247</td>
<td>1.347</td>
<td>2.153</td>
</tr>
<tr>
<td>300-step tree</td>
<td>0.006</td>
<td>0.200</td>
<td>0.434</td>
</tr>
<tr>
<td></td>
<td>0.078</td>
<td>0.698</td>
<td>1.220</td>
</tr>
<tr>
<td></td>
<td>0.247</td>
<td>1.348</td>
<td>2.157</td>
</tr>
<tr>
<td>True value</td>
<td>0.006</td>
<td>0.200</td>
<td>0.433</td>
</tr>
<tr>
<td></td>
<td>0.077</td>
<td>0.698</td>
<td>1.220</td>
</tr>
<tr>
<td></td>
<td>0.247</td>
<td>1.346</td>
<td>2.155</td>
</tr>
<tr>
<td>$K = 40$</td>
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<td></td>
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<tr>
<td>European put</td>
<td>0.840</td>
<td>1.522</td>
<td>1.881</td>
</tr>
<tr>
<td></td>
<td>1.299</td>
<td>2.428</td>
<td>3.064</td>
</tr>
<tr>
<td></td>
<td>1.758</td>
<td>3.334</td>
<td>4.247</td>
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<tr>
<td>4th order</td>
<td>0.852</td>
<td>1.578</td>
<td>1.986</td>
</tr>
<tr>
<td></td>
<td>1.310</td>
<td>2.481</td>
<td>3.165</td>
</tr>
<tr>
<td></td>
<td>1.768</td>
<td>3.386</td>
<td>4.349</td>
</tr>
<tr>
<td>Bunch &amp; Johnson</td>
<td>0.853</td>
<td>1.581</td>
<td>1.992</td>
</tr>
<tr>
<td></td>
<td>1.310</td>
<td>2.484</td>
<td>3.171</td>
</tr>
<tr>
<td></td>
<td>1.769</td>
<td>3.389</td>
<td>4.354</td>
</tr>
<tr>
<td>300-step tree</td>
<td>0.853</td>
<td>1.581</td>
<td>1.990</td>
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<tr>
<td></td>
<td>1.310</td>
<td>2.482</td>
<td>3.169</td>
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<td>1.769</td>
<td>3.391</td>
<td>4.357</td>
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<tr>
<td>True value</td>
<td>0.852</td>
<td>1.580</td>
<td>1.990</td>
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<tr>
<td></td>
<td>1.310</td>
<td>2.483</td>
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<td></td>
<td>1.768</td>
<td>3.387</td>
<td>4.353</td>
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<tr>
<td>$K = 45$</td>
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<tr>
<td>European put</td>
<td>4.840</td>
<td>4.780</td>
<td>4.840</td>
</tr>
<tr>
<td></td>
<td>4.980</td>
<td>5.529</td>
<td>5.972</td>
</tr>
<tr>
<td></td>
<td>5.236</td>
<td>6.377</td>
<td>7.166</td>
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<tr>
<td>4th order</td>
<td>5.021</td>
<td>5.085</td>
<td>5.261</td>
</tr>
<tr>
<td></td>
<td>5.059</td>
<td>5.702</td>
<td>6.237</td>
</tr>
<tr>
<td></td>
<td>5.286</td>
<td>6.506</td>
<td>7.376</td>
</tr>
<tr>
<td>Bunch &amp; Johnson</td>
<td>5.002</td>
<td>5.091</td>
<td>5.265</td>
</tr>
<tr>
<td></td>
<td>5.062</td>
<td>5.708</td>
<td>6.244</td>
</tr>
<tr>
<td></td>
<td>5.289</td>
<td>6.512</td>
<td>7.385</td>
</tr>
<tr>
<td>300-step tree</td>
<td>5.000</td>
<td>5.088</td>
<td>5.267</td>
</tr>
<tr>
<td></td>
<td>5.060</td>
<td>5.706</td>
<td>6.246</td>
</tr>
<tr>
<td></td>
<td>5.286</td>
<td>6.511</td>
<td>7.383</td>
</tr>
<tr>
<td>True value</td>
<td>5.000</td>
<td>5.088</td>
<td>5.267</td>
</tr>
<tr>
<td></td>
<td>5.060</td>
<td>5.706</td>
<td>6.244</td>
</tr>
<tr>
<td></td>
<td>5.287</td>
<td>6.510</td>
<td>7.383</td>
</tr>
</tbody>
</table>
Table 3. Put option prices under stochastic volatility and their approximations

Option prices are computed for the Heston model:

\[ dS_t = rS_t dt + \sqrt{v_t} S_t dW^{(1)}_t, \]
\[ dv_t = \kappa (\bar{v} - v_t) dt + \sigma_v \sqrt{v_t} dW^{(2)}_t, \]

with \( r = 0.1, \kappa = 5 \) (or 2.5), \( \bar{v} = 0.16, \sigma_v = 0.9 \) (or 0.45), \( \rho_{12} = 0.1 \). Time-to-maturity \( \tau \) is 3 months and strike price \( K \) is 10. The model parameters are borrowed from Ito and Toivanen (2006), and the spot variance is denoted by \( v \). Monte-Carlo refers to Longstaff and Schwartz (2000) algorithm with 200,000 sample paths, 500 time steps and 50 exercise dates. Our approximation is computed with a 5-th order asymptotic expansion. Ito&Toivanen corresponds to results presented in Tables 1 and 2 of Ito and Toivanen (2006). European put price is computed using a closed form formula.

<table>
<thead>
<tr>
<th>Method</th>
<th>( v = 0.0625 )</th>
<th>( v = 0.25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \kappa = 5 )</td>
<td>( \kappa = 2.5 )</td>
</tr>
<tr>
<td>Approximation 1 (5-th order)</td>
<td>1.081</td>
<td>1.072</td>
</tr>
<tr>
<td>Approximation 2 (5-th order)</td>
<td>1.111</td>
<td>1.077</td>
</tr>
<tr>
<td>Ito&amp;Toivanen</td>
<td>1.108</td>
<td>1.077</td>
</tr>
<tr>
<td>Monte-Carlo</td>
<td>1.107</td>
<td>1.075</td>
</tr>
<tr>
<td>European put</td>
<td>1.048</td>
<td>1.009</td>
</tr>
</tbody>
</table>

| \( S = 10 \)                |
|-----------------------------|-------------------------------|-------------------------------|
| Approximation 1 (5-th order) | 0.534                          | 0.475                          | 0.811                          | 0.838                          |
| Approximation 2 (5-th order) | 0.521                          | 0.478                          | 0.796                          | 0.836                          |
| Ito&Toivanen                | 0.520                          | 0.479                          | 0.796                          | 0.837                          |
| Monte-Carlo                 | 0.521                          | 0.478                          | 0.793                          | 0.837                          |
| European put                | 0.502                          | 0.459                          | 0.770                          | 0.812                          |

| \( S = 11 \)                |
|-----------------------------|-------------------------------|-------------------------------|
| Approximation 1 (5-th order) | 0.208                          | 0.174                          | 0.463                          | 0.488                          |
| Approximation 2 (5-th order) | 0.214                          | 0.178                          | 0.448                          | 0.487                          |
| Ito&Toivanen                | 0.214                          | 0.178                          | 0.448                          | 0.487                          |
| Monte-Carlo                 | 0.214                          | 0.177                          | 0.451                          | 0.488                          |
| European put                | 0.208                          | 0.173                          | 0.436                          | 0.475                          |
The put option prices are computed for the constant volatility model with interest rates following the Hull and White model:

$$dr_t = (\eta r - \gamma r)dt + \beta dW^t_r.$$  

The model parameters are $\gamma = 0.1$, $\beta = 0.01$. Time-to-maturity $\tau$ is one year, stock price and strike price $S = K = 100$, the stock price volatility is denoted by $\sigma$. USTS refers to an upward sloping term structure, and DSTS refers to the downward sloping term structure. The model parameters and term structure specifications are borrowed from Menkveld and Vorst (2001). Monte-Carlo refers to the Longstaff and Schartz (2000) algorithm with 200,000 sample paths, 500 time steps, and 50 exercise dates. Our approximation is computed with a 5-th order asymptotic expansion. Menkveld & Vorst refers to the results reported in Table 5 of Menkveld and Vorst (2001). European put prices are also taken from the same table.

<table>
<thead>
<tr>
<th>Method</th>
<th>USTS</th>
<th>DSTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma = 0.5$</td>
<td>$\sigma = 0.3$</td>
</tr>
<tr>
<td>Approximation 1 (5th order)</td>
<td>18.00</td>
<td>10.36</td>
</tr>
<tr>
<td>Approximation 2 (5th order)</td>
<td>17.98</td>
<td>10.35</td>
</tr>
<tr>
<td>Merkvedl&amp;Vorst</td>
<td>17.93</td>
<td>10.33</td>
</tr>
<tr>
<td>Monte-Carlo</td>
<td>17.98</td>
<td>10.36</td>
</tr>
<tr>
<td>European</td>
<td>17.55</td>
<td>9.92</td>
</tr>
</tbody>
</table>
Table 5. Put option prices under stochastic volatility and stochastic interest rates and their approximations

Option prices are computed for an affine model with stochastic volatility and stochastic interest rates:

\[
\begin{align*}
    dS_t &= r_t S_t dt + \sqrt{\nu_t} S_t dW^{(1)}_t, \\
    d\nu_t &= 1.58(0.03 - \nu_t) dt + 0.2 \sqrt{\nu_t} dW^{(2)}_t, \\
    dr_t &= 0.26(0.04 - r_t) + 0.08 \sqrt{r_t} dW^{(3)}_t,
\end{align*}
\]

with \( \rho_{12} = -0.26, \rho_{13} = \rho_{23} = 0 \). The stock price is \( S = 100 \) and spot interest rate is \( r = 0.04 \). Time-to-maturity is denoted by \( \tau \), and the spot volatility is denoted by \( \sigma = \sqrt{\nu} \). Monte-Carlo refers to the Longstaff and Schwartz (2000) algorithm with 200,000 sample paths, 500 time steps, and 50 exercise dates. Our approximation is computed with a 5-th order asymptotic expansion.

<table>
<thead>
<tr>
<th>Method</th>
<th>( \tau = \frac{1}{12} )</th>
<th>( \tau = \frac{1}{4} )</th>
<th>( \tau = \frac{1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma = 0.2 )</td>
<td>( \sigma = 0.3 )</td>
<td>( \sigma = 0.4 )</td>
</tr>
<tr>
<td>Approximation 1 (5th order)</td>
<td>0.076 0.613 1.381</td>
<td>0.403 1.622 2.854</td>
<td>0.970 2.906 4.610</td>
</tr>
<tr>
<td>Approximation 2 (5th order)</td>
<td>0.076 0.612 1.357</td>
<td>0.403 1.619 2.822</td>
<td>0.970 2.904 4.575</td>
</tr>
<tr>
<td>American put (Monte-Carlo)</td>
<td>0.075 0.621 1.351</td>
<td>0.404 1.627 2.819</td>
<td>0.969 2.903 4.573</td>
</tr>
<tr>
<td>European put</td>
<td>0.075 0.604 1.323</td>
<td>0.402 1.601 2.760</td>
<td>0.967 2.878 4.489</td>
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<tr>
<td>K = 90</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Approximation 1 (5th order)</td>
<td>2.135 3.461 4.584</td>
<td>3.228 5.172 6.690</td>
<td>4.329 6.924 8.890</td>
</tr>
<tr>
<td>K = 100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K = 110</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
Figure 1. The solution to the modified problem

Both graphs plot the solution to the modified problem as a function of the early exercise level of the normalized moneyness $y$. Option parameters are: $S = K = 100$, $\tau = 1$. Volatility is at $\sigma = 0.2$. The solution is denoted by $P(0,1; y)$ ($\theta = 0$, $\tau = 1$). The American put price is denoted by $P$ and the European put price is denoted by $P(0,1; \infty)$. In case (b), the American put price is equal to the European put price, and the maximum of $P(0,1; y)$ is achieved at $y = \infty$. 

\begin{itemize}
\item \textbf{(a)} $r = 0.05$, $\delta = 0$
\item \textbf{(b)} $r = 0$, $\delta = 0.05$
\end{itemize}
Figure 2. Finding the early exercise price in the Black-Scholes model
Graphs show the value-maximizing boundary $\bar{y}(\theta)$ as a function of $\theta$ in the Black-Scholes model. Each one plots this function for three levels of the (annual) volatility parameter $\sigma$ for time-to-maturity of 1 month, 6 months or 12 months. The model parameters are $r = 0.05$, $\delta = 0$. The intersection of $\bar{y}(\theta)$ with the 45-degree line (the solid line) yields the optimal early exercise level $\bar{\theta}$ of $\theta$. 
Figure 3. Convergence of the asymptotic expansion in the Black-Scholes model
Each graph shows absolute approximation errors of our method based on different orders of asymptotic expansion ($N=2,3,4,5$) and for different time-to-maturity (up to 12 month). “Tree” refers to the errors of our method with option prices being computed using a binomial tree with 2000 steps. The “true” American put price is also computed using a 2000-step binomial tree. Black-Scholes model parameters are $r = 0.05$, $\sigma = 0.2$, $K = 100$. The time unit is one year.
Figure 4. Approximation of the American put option delta in the Black-Scholes model

The graph shows absolute approximation errors of the delta of the American put option. The approximation is obtained by differentiating the price approximation formula based on asymptotic expansion of different orders ($N=2, 3, 4, 5$), and then evaluating at $y = y_1(t)$. The “true” option delta is obtained by a numerical differentiation of the put price computed using a 2000-step binomial tree. Black-Scholes model parameters are $r = 0.05$, $\sigma = 0.2$, option strike price $K = 100$. The time unit is one year.
Figure 5. Convergence of asymptotic expansion under stochastic volatility in the absence of mean-reversion

Each graph shows approximation errors of our method based on different orders of asymptotic expansion ($N=2, 3, 4, 5$), and a Monte-Carlo simulation with 200,000 paths (MC) for a particular price level. The option strike price $K = 100$. The horizontal axis indicates the option time-to-maturity in months. The American put price is computed using the Longstaff and Schwartz (2001) algorithm with 200,000 paths, 500 time steps and 50 exercise dates. The model specification is (the time unit is one year):

$$dS_t = 0.05S_t dt + \sqrt{v_t} S_t dW_t^{(1)},$$

$$dv_t = 0.2 \sqrt{v_t} dW_t^{(2)} , dW_t^{(1)} dW_t^{(2)} = -0.5 dt, \quad v_0 = 0.04.$$
Figure 6. Convergence of asymptotic expansion under mean-reverting stochastic volatility

Each graph shows approximation errors of our method based on different orders of asymptotic expansion ($N = 2, 3, 4, 5$), and a Monte-Carlo simulation with 200,000 paths (MC) for a particular price level. The option strike price $K = 100$. The horizontal axis indicates the option time-to-maturity in months. The American put price is computed using the Longstaff and Schwartz (2001) algorithm with 200,000 paths, 500 time steps and 50 exercise dates. The model specification is (the time unit is one year):

$$dS_t = 0.05S_t dt + \sqrt{v_t} S_t dW^{(1)}_t,$$

$$dv_t = 2(0.04 - v_t) dt + 0.2\sqrt{v_t} dW^{(2)}_t, \quad dW^{(1)}_t dW^{(2)}_t = -0.5 dt, \quad v_0 = 0.04.$$
Figure 7. Finding the early exercise price under stochastic volatility
Graphs show the value-maximizing boundary $\tilde{y}(\theta)$ as a function of $\theta$ in the model with zero drift in stochastic volatility (panel (a), see Figure 5 for the model specification), and in the model with mean-reverting stochastic volatility (panel (b), see Figure 6 for the model specification). Each graph plots this function for three levels of the (annual) volatility parameter $\sigma$ for time-to-maturity 1 month, 6 months or 12 months. If unique, the intersection of $\tilde{y}(\theta)$ with the 45-degree line (the solid line) yields the optimal early exercise level $\tilde{\theta}$ of $\theta$. 
Figure 8. Impact of stochastic volatility and stochastic interest rates on the American put price and its components

Graphs show the impact of a change in model parameters on the early exercise premium (dEEP), the European put price (dPE) and the American put price (dP). The Black-Scholes model is considered as baseline model. The new model with additional stochastic factors has the following specification:

\[ dS_t = r_S dt + \sqrt{\nu_t} S_t dW_{t}^{(1)}, \]
\[ d\nu_t = \kappa (0.04 - \nu_t) dt + \sigma \nu_t \sqrt{\nu_t} dW_{t}^{(2)}, \]
\[ dr_t = \sigma r_t \sqrt{r_t} dW_{t}^{(3)}, \]
\[ dW_{t}^{(1)} dW_{t}^{(2)} = \rho_{12} dt, \quad dW_{t}^{(1)} dW_{t}^{(3)} = \rho_{13} dt, \quad \nu_0 = 0.04, \quad r_0 = 0.05. \]

Time-to-maturity is 6 months, strike price \( K = 100. \)