Mergers and Asset Prices in a Firm Network Economy

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Mergers and Asset Prices in a Firm Network Economy

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Abstract

We examine merger activity and its effect on asset pricing in a firm network economy. Mergers create internal capital markets which change the cash flow risk structure of the merging firms. We propose a solution concept for coalitional games without the superadditivity axiom, which extends the Shapley value, and apply it to the merging activity of firms in a network. The possibility of a merger increases the equity value of standalone firms. Higher network dependence generally increases merger activity but nevertheless causes lower firm equity values. Recession and expansion, as measured by the average debt/total assets ratio, generally decrease the number of coalitions in a network, generating an inverted U-shaped curve of merger activity.

Keywords: Mergers, coalitional games without the superadditivity axiom, asset pricing in coalitions, network dependence models, buyer-supplier networks.

JEL Classification: G34, C71, C78.
1 Introduction

The existence and the scope of firms has long puzzled the economic theory. In a classical general equilibrium setting, where firms are motivated by profit generation, only a single firm should exist. Even though the mergers’ literature was an active and stimulating area of financial research, a general theory of merger activity and waves has eluded financial theory. The motivation for merger waves is even included in the Brealey and Myers (2000) list of ten unsolved questions of modern financial theory.

The first to provide a coherent theory of firm integration (i.e. a takeover of a supplier firm by its main buyer) were the articles by Grossman and Hart (1986) and Hart and Moore (1990). They explain the firm scope and the reasons for mergers in an incomplete contract/property rights paradigm. These articles provide a rationale for a merger when either one firm’s incentives or investment decisions depend sufficiently on the property rights of the other firm’s production. More recently, the papers by Lambrecht (2004), Morellec and Zhdanov (2005) and Lambrecht and Myers (2006) proposed a real options merger formation framework based on the comparison between the underlying real activity of the individual and the merged firm. The feature of all of these models is that eventually only a single firm would exist.

The models stated above all have the following shortcomings. Firstly, they are limited to two firms and hence the results oscillate wildly between no mergers and the merger of all firms in the economy. An economic environment of multiple firms enlarges the merger possibility and the coalition formation opportunity set. Secondly, they are infinite horizon models. While mergers can produce long-term benefits, they can end up in default over the short run.

Our paper extends the literature on mergers by removing some of the limitations of previous papers. Specifically, we look at the merger as a creation of internal capital markets. Internalization of capital markets is an important part of a firm’s strategy and the functioning of markets in general. It is the essence of the existence of firms itself, as emphasized by

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1In their models, the merger is motivated by the increase of a specific firm characteristic, such as a sales price, driven by the geometric Brownian motion, to a high enough level. This is a property of the geometric Brownian motion process. Other processes governing the real economic activity would most probably generate different merger predictions.

Internal capital markets change the risk structure of the firm. Consider a two firm example. Firm one is the buyer of goods and firm two is the supplier. In addition to the buy-supply relationship of the two firms, they both also generate external cash flows. From the perspective of firm one, the buy relationship increases or decreases the cash flow volatility (depending on the correlation with the external cash flows). The same is true for firm two with the only difference that the buy orders are replaced with supply orders. Both firms are therefore exposed to the fluctuations of cash flows through their buy-supply relationship. Now suppose that the two firms merge. What was before a classical goods market now becomes an internal capital market. Neither firm one nor firm two is exposed to the fluctuations in the asset dynamics originating from buy-supply orders. The risk structure of asset fluctuations has changed. The cost of a merger is the vulnerability of the merged firm to the aggregated external cash flows.

The merger problem is even harder when there are more than two firms in the economy. In this case, firms can form coalitions and internalize certain goods markets. As in a two firm example above, the risk structure of coalitional cash flows changes. We assume that firms in a coalition continue to operate as independent entities and therefore do not generate externalities.

We use the term coalition very broadly to describe the “cooperation of firms in forming internal capital markets”. Coalition formation affects the credit risk structure of the individual firms. It includes among others classical mergers, mergers of equals, acquisitions or hostile takeovers. Cooperations and alliances are in our setting not examples of coalitions.

The risk structure of cash flows is not the only important factor in determining merger conditions. Our model does not account for merger synergies, market concentration effects, coalitional externalities, growth potential, tax advantages or resource transfers. Moreover, there exists considerable empirical literature on the tradeoff between the benefits of mergers and over-diversification. Ravenscroft and Scherer (1988) argue that the productivity declined in the years following the merger, while others, such as Healy, Palepu, and Ruback (1992)
found an average increase in subsequent corporate returns.

The studies above illustrate that the value of the coalition does not necessarily add value to the firms forming the coalition. Therefore classical coalition formation results, such as the Shapley (1953) value or the results in Maskin (2003), which all assume the superadditivity of the value function, are not appropriate in this environment. We therefore develop a theory of coalition formation without the superadditivity axiom.

Since the analysis of agency conflicts within the firm is not the main purpose of this paper, we assume that the bargaining parties are managers on behalf of shareholders and that their incentives are aligned. After (and if) the merger occurs, one of the managers of the merging firms is paid out and leaves all the managerial decisions to the other one. The merged firm managerial structure is then the same as before the merger. Since no distinction is made between the firm and its shareholders (managers), we will say with the slight abuse of language that the bargaining agents are firms themselves.

The theory that we propose functions as follows. The firms enter the bargaining process sequentially, i.e. firm $i$ starts the bargaining process when firms $1, \ldots, i-1$ have already formed the coalitions. The coalitions formed can not be re-negotiated and can not break apart - our model does not apply to spinoffs. Firm $i$ joins the coalition which maximizes the total welfare and is paid the (marginal) contribution to the share price increase (Vickrey principle) to the coalition it joined. The process is repeated for all possible permutations of $N$ firms, i.e. for all bargaining orders. We prove that by adding the superadditivity axiom we obtain the Shapley value again. Therefore our proposal can be considered as an extension of the Shapley concept to coalitional games without the superadditivity axiom. The superadditivity axiom is a very restrictive assumption implying that the value of any coalition is necessarily greater than the value of its parts. Games with externalities, mergers for diversification and many others are all examples where the superadditivity axiom fails. The axioms of the theory can be implemented algorithmically.

We then apply this bargaining theory to a model of firm merger in a network of firms, where

\footnote{In words, the superadditivity axiom assumes that the value of the coalition is always greater than the value of individuals forming the coalition.}
economic dependencies between firms are given at the beginning and can not be changed thereafter$^3$. Firms issue buy orders, as described by network dependencies that generate the asset dynamics as in Brumen and Vanini (2006). We assume that the equity value maximization is the firm’s/coalition’s sole objective function. In order to capture the tradeoff between the cash flow volatility and the possibility of default we use the Leland and Toft (1996) model of equity pricing. By relying on the buyer/supplier network theory, the model has the potential to incorporate all three groups of mergers, vertical, horizontal and conglomerate and give predictions as to which merger is more likely in specific situations.

The results show that firm values in a network should not be considered in isolation. We term the difference between the equity values of the firm not accounting for merger possibilities and in an economy with possible mergers “merger corrections”. Merger possibilities increase the equity value of the firms. Higher network dependence decreases the number of coalitions formed, i.e. it encourages mergers but generates lower firm equity value of the merging firm. Our model therefore predicts that firms suffering high levels of cash flow volatility should merge and exactly the opposite should happen for firms of very low cash flow volatility. Recession (expansion), as measured by the average debt/total assets ratio, decrease the number of coalitions in a network, i.e. it fosters the merger activity generating a typical inverted U-shaped merger activity curve. Our model is in line with the paper of Bernile, Lyandres, and Zhdanov (2007) documenting the existence of mergers also in severe recession.

The paper closest related to ours is Jovanovic and Rousseau (2002) which establishes a theory of merger and acquisition activity in relation to firm’s (Tobin’s) Q value and technological changes. Relating to their paper, we conclude that the mergers can occur even if there is no change in technological efficiency of individual or the merged firm. In our analysis the merger can serve only hedging purposes, i.e. it hedges the firm against volatile cash flows before default without any structural or technological change. The merger between YouTube and Google in late 2006 and an attempt of EUREX to merge with the LSE were at least partially of this kind.

$^3$An equivalent statement is that the cost of breaking an economic relationship is too high, see Grossman and Hart (1986).
Contrary to the articles by Lambrecht (2004) and Morelec and Zhdanov (2005), our model predicts multiple coalitions (as opposed to a single one) and cyclical merger activity in line with Lambrecht (2004). The papers by Scharfstein and Stein (2000), Inderst and Laux (2006) and others take a more strategic approach to the formation of internal capital markets on behalf of division managers and the incentives that form them. A distantly related paper is Habib and Mella-Barral (2006) which investigates different type of firm-firm connections, such as partnerships and mergers and similarly to our paper identifies conditions for a specific form of firm relationship. Our paper also provides an explanation for managers’ decision about the “currently unprofitable mergers”, as in Gorton and Rosen (2005), but which can generate profits in the long run.

This paper is structured as follows. Section 2 develops the general theory of coalitional games without the superadditivity axiom which is then applied as a merger solution concept. Section 3 illustrates the proposed merger theory in a two firm example. In Section 4 the merger theory in a general network environment is presented. Section 5 sums up the results.

2 Theory of Coalition Formation without Externalities and the Superadditivity Axiom

The proposed theory establishes a solution concept to cooperative games without externalities and the superadditivity axiom. The axioms are inspired heavily by Maskin (2003), but also differ from it substantially. The theory does not incorporate externalities, but it relaxes the superadditivity assumption of the value function\(^4\). Since the superadditivity axiom does not hold, the grand coalition will not necessarily form. The worth of a coalition of two firms may be worth less than the sum of individual entities. An example are mergers to diversify assets. While they can be beneficial, Brealey and Myers (2000) give an example of Kaiser Industries, a holding company for Kaiser Steel, Kaiser Aluminum and Kaiser Cement which

\(^4\)Under the superadditivity of the value function we mean that \(v(S_1) + v(S_2) \leq v(S_1 \cup S_2)\) for any coalitions \(S_1 \cap S_2 = \emptyset\).
traded at significant discount until it sold off its holdings. If the value function also satisfies
the superadditivity assumption, the solution concept coincides with the Shapley value. We
therefore consider the proposed solution concept as a (particular) extension of the Shapley
value to games with general value functions.

We denote by $\pi = \{1, \ldots, n\}$ the set of integers smaller than $n + 1$. A $N$-player transferable
utility game $(N, v)$ without externalities is given by

- A set of players $\overline{N}$.
- A function $v$, which assigns a worth $v(S)$ to a coalition $S \in \mathcal{P}$, given a partition $\mathcal{P}$ of $\overline{N}$.

We fix the number of players $N$ and focus on coalitional games that are normalized to $v(\emptyset) = 0$.
We call a partition $\mathcal{P}^n$ of $\pi$, $n < N$ a partial partition. Given a partial partition $\mathcal{P}^{i-1}$, let
$\varphi_i(\mathcal{P}^{i-1})$ be the prediction function of player’s $i$ payoff and $\psi(\mathcal{P}^{i-1})$ be the prediction of a
partition of $\overline{N}$ given that the partition of $i-1$ is $\mathcal{P}^{i-1}$. We require that $\{\varphi_i\}_{i=1}^N$ and $\psi$ satisfy
the following set of axioms.

(NA) Non-negotiation commitment: Let $j, k \in i, i < N$, and partial partition $\mathcal{P}^i$ be given
where $j \in S \in \mathcal{P}^i$ and $k \notin S$. If $j \in S^* \in \psi(\mathcal{P}^i)$ then also $k \notin S^*$.

The axiom NA guarantees that the players that have been allocated to separate coalitions in
the bargaining process do not “renegotiate” with other players entering the bargaining process
after him to form a new coalition or join a different one than the already assigned. Maskin
interprets this axiom as “cutting the telephone lines”.

(BC) Binding coalitions: Let $i < N$ and the partial partition $\mathcal{P}^i$ be given. If $S \in \mathcal{P}^i$ then
there exists $S' \in \psi(\mathcal{P}^i)$ such that $S \subset S'$.

The axiom BC assures that as the bargaining process proceeds the already formed coalitions
do not break apart, i.e. players in a coalition remain together in that coalition till the end of
the bargaining process.
For any $S \in \mathcal{P}^{i-1} \cup \emptyset$ and $\hat{S} \in \mathcal{P}^{i-1} \cup \emptyset'$ we define

$$
\Phi^i(S, \hat{S} | \mathcal{P}^{i-1}) = \begin{cases} 
v(S^N(S)) - \sum_{j>i} t_j(S^{j-1}(S)|\psi^{j-1}(\mathcal{P}^{i-1}, \hat{S} \cup i)) & S \neq \hat{S} \\
v(S^N(S \cup i)) - \sum_{j>i} t_j(S^{j-1}(S \cup i)|\psi^{j-1}(\mathcal{P}^{i-1}, S \cup i)) & S = \hat{S} 
\end{cases}
$$

where $t_j(S|\mathcal{P})$ is defined recursively in the VP axiom below (equation (4)). We used the following notation: $\psi^j(\mathcal{P}^i)$ ($j \geq i$) is the partition of $\overline{J}$ such that the elements of $\psi^j(\mathcal{P}^i)$ are members of $\psi(\mathcal{P}^i)$ without players $N \setminus J$. $S^i(S)$ is the unique element of $\psi^j(\mathcal{P}^i)$ such that $S \subset S^i$. The existence of $S^i(S)$ is guaranteed by the BC axiom. We note that $\Phi^N(S, \hat{S} | \mathcal{P}^{N-1}) = v(S)$ in case $S \neq \hat{S}$ and $\Phi^N(S, S | \mathcal{P}^{N-1}) = v(S \cup N)$ otherwise. The quantity $\Phi^i(S, S'|\mathcal{P}^{i-1})$ is the worth of the coalition $S \in \mathcal{P}^{i-1}$ if the first $i - 1$ players form a partial partition $\mathcal{P}^{i-1}$, i.e. they have already formed selected coalitions, and $i$ is allocated to $S'$. $\Phi^i$ captures the optimal anticipation by player $i$ of future players $j \geq i$ behaving optimally themselves.

(CA) Competitive allocation: Let $\mathcal{P}^{i-1}$ be a partial partition. Then $i$ is allocated to $S^o$, such that

$$
S^o = \arg\max_{\hat{S} \in \mathcal{P}^{i-1}} \sum_{S \in \mathcal{P}^{i-1}} \Phi^i(S, \hat{S} | \mathcal{P}^{i-1}),
$$

i.e. $(S^o \cup i) \in \psi^i(\mathcal{P}^{i-1})$ if and only if $S^o$ is the maximizer in (2).

CA axiom states that the player $i$ joins the coalition which generates maximum total welfare of all coalitions. This is the “efficiency axiom”.

(CW) Competitive wages: Let $\mathcal{P}^{i-1}$ be a partial partition and assume that player $i$ is competitively allocated (i.e. according to axiom CA) to coalition $S^o \in \mathcal{P}^{i-1}$. Then the payoff to player $i$ is

$$
\varphi_i(\mathcal{P}^{i-1}) = \Phi^i(S^o, S^o | \mathcal{P}^{i-1}) - \Phi^i(S^o, S^{oo} | \mathcal{P}^{i-1})
$$

where $S^{oo} \in \mathcal{P}^{i-1}$ is any coalition $S^{oo} \neq S^o$ (The choice of $S^{oo}$ is arbitrary since the game does not have externalities.)
The CW axiom speaks about the value assigned to a player in the bargaining process. The player receives the difference between the coalition’s worth with that player \( \Phi^i(S^o, S^{oo} | P^{i-1}) \) and the one when the player joins any other coalition \( S^{oo} - \Phi^i(S^o, S^{oo} | P^{i-1}) \). The player is awarded his marginal contribution to coalition’s worth. The choice of \( S^{oo} \) is well defined, since the game does not have externalities. The choice of \( S^{oo} \) in games with externalities is a major obstacle in defining the solution concept for cooperative games with externalities.

(VP) **Vickrey payments:** Let \( P^{i-1} \) be a partial partition and player \( i \) be competitively allocated to \( S^o \in P^{i-1} \). Then for all \( S \in P^{i-1} \) the following holds

\[
t_i(S | P^{i-1}) = \begin{cases} 
\Phi^i(S^o, S^o | P^{i-1}) - \Phi^i(S^o, S^{oo} | P^{i-1}) & S = S^o \\
0 & S \neq S^o 
\end{cases}
\]

where \( S^{oo} \) is as in the CW axiom.

The last axiom states that only the coalition which the player joins awards that player with the exact amount by which the coalition’s value increases. This is in stark contrast to games with externalities where externalities can induce payments from other coalitions.

The bargaining process theory functions as follows. Players enter the bargaining process sequentially, i.e. player \( i \leq N \) begins negotiation after players \( 1, \ldots, i-1 \) have already formed the appropriate coalitions. Player \( i \) bargains for its payoff (axiom CW) and joins the selected coalition (CA). The coalition awards him his contribution according to the VP axiom. The stability of the coalition formation is guaranteed by the NA and BC axioms. The axioms NA-VP are constructive and can be implemented on a computer. The algorithm that implements them is unfortunately highly recursive and resource consuming.

The following theorem provides the existence of a payoff prediction function \( \varphi \) and a coalition prediction function \( \psi \) that satisfy above axioms.

**Theorem 2.1.** For every \( N \)-player transferable utility game \((N, v)\) there exist payoff prediction function \( \varphi \) and coalition prediction function \( \psi \) as defined above, which satisfy axioms NA, BC, CA, CW and VP.
To construct the payoff prediction function we randomize over the order of players entering the bargaining process. Let \( \pi \in \mathcal{S}^N \) be a permutation in a set of all permutations of \( N \) elements. We define

\[
\varphi_i = \frac{1}{N!} \sum_{\pi \in \mathcal{S}^N} \varphi_i^\pi,
\]

where \( \varphi_i^\pi \) is obtained as before when the order of players entering the bargaining process is \( \pi \). The predictor of the coalition structure is a uniform distribution over all the coalition predictions, i.e. \( \psi_i = \psi_i^\pi (\pi \in \mathcal{S}^N) \) with probability \( \frac{1}{N!} \).

The justification for the selection of axioms above comes additionally from the following theorem.

**Theorem 2.2.** Let \( v \) be a coalitional game of \( N \) players. The following holds.

(a) For every \( S \in \psi(\emptyset) \) \( \sum_{i \in S} \varphi_i(\psi^i-1(\emptyset)) = v(S) \).

(b) For every partial partition \( \mathcal{P}^i-1 \) we have that \( \sum_{S \in \mathcal{P}^i-1} t_i(S|\mathcal{P}^i-1, S^o) = \varphi_i(\mathcal{P}^i-1) \), where \( S^o \in \psi^i(\mathcal{P}^i-1) \).

(c) If for some \( i \) and for every subset \( S \) of \( N \) players \( v(S \cup \{i\}) = v(S) \) holds, then \( \varphi_i(\mathcal{P}^i-1) = 0 \).

(d) If in addition to axioms NA, BC, CA, CW and VP, the superadditivity of the value function \( v \) is also assumed, then \( \varphi_i \) in equation (5) equals the Shapley value of player \( i \) and the grand coalition forms.

Part (a) of the theorem states that the payoffs to all coalition members precisely equal the worth of that coalition, i.e. the worth of the coalition is divided among the forming players. It is the marginal contribution of every player (the CW axiom) that determines the percentage of the coalition’s worth the player receives. Part (b) states that the payoff to player \( i \) is the sum of all Vickrey payments by all existing coalitions and is obvious from the definition of the Vickrey payments - only the coalition that the player joins transfers value to that player.
Intuitively, this is the consequence of the no externality assumption of the game. Part (c) is an extension of the Shapley axiom tocoalitional games without the superadditivity axiom. It states that if the player \( i \) has zero marginal contribution to any coalition, its payoff should be zero. Part (d) proves that the established theory is a generalization of the Shapley value concept.

3 The two firm case

In the previous chapter we defined the solution concept for games without the superadditivity axiom, i.e. the value of a coalition does not necessarily exceeds the sum of the values of its parts. This result has broad empirical and theoretical support. In this chapter we apply developed theory to obtain merger results in a simple two firm network.

We assume the setting of Brumen and Vanini (2006) of two dependent firms, a buyer (firm 1) and a supplier (firm 2), see Figure 1(a). The buyer firm issues buy orders modelled by a Poisson process \( N_1 \) (with intensity \( \lambda_1 \)) at which time it transfers a nominal amount proportional to \( E_{12} \cdot P_1 = V_{12} \) of its total assets to the supplier. We make the same assumption as in Brumen and Vanini (2006) where buy orders act only as the creator of future orders from other firms and therefore decrease the asset value temporarily. \( E_{12} \) is the number of links between firms 1 and 2 and denotes the strength of business relationship between the firms. \( P_1 \) is the proportionality factor describing the proportion of the buyer’s assets transferred to the supplier. In addition to network buyer-supplier cash flows, the two firms have external cash flows, irrespective of the network. The dependence\(^5\) between the external cash flows and the buy orders is gathered in a matrix \( B \). The element \( B_{11} \) describes the dependence between the buy orders of firm 1 and the external cash flows. Additional Poisson process \( N_2 \) (with intensity \( \lambda_2 \)) drives the external cash flows of firm 2. External cash flows are depicted dashed

\(^5\)The word dependence is used, since the buy/supply environment is not Gaussian, i.e. the order flows follow a geometric Poisson process. One can imagine the matrix \( B \) as a correlation between external cash flows and buy orders of firms.
in Figure 1(a). The dynamics of the firms’ asset values can be written as

\[
\begin{align*}
    dA_1 &= (B_{11} - V_{12})A_1 dN_1 \\
    dA_2 &= V_{12}A_1 dN_1 + B_{22}A_2 dN_2
\end{align*}
\]

We assume that both firms have issued equity and zero-coupon debt with principals \(D_1, D_2\) and maturity \(T\). We denote by \(S_1, S_2\) the equity values. In a two firm network, the dependency value is defined as the number of economic links between the firms, i.e. the value of \(E_{12}\). By Proposition 4.2. of Brumen and Vanini (2006) the firm’s equity values can be expressed as follows.

**Proposition 3.1.** The equity price of firm \(i\) with principal value \(D_i\) and maturity \(T\) is given by

\[
S_i = S_i(\sigma_i), \quad \sigma_1 = (B_{11} + V_{11})\sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\frac{V_{21}A_1(0)}{A_2(0)}} \lambda_1 + B_{22}^2 \lambda_2, \quad V_{11} = -P_1E_{12}, \quad V_{21} = -V_{11}
\]

and \(S_i\) is the Leland/Toft equity value as given in Leland and Toft (1996), equation (9).

The choice of the Leland and Toft (1996) model correctly reflects the tradeoff between the increased stock price due to increased volatility of cash flows and the negative effect due to increased likelihood of default.

![Figure 1](image)

**Figure 1:** Example of firms and cash flows in a two firm buyer - supplier network - 1(a). The network cash flow directions are denoted in solid and the external cash flows in dashed lines. A merged firm 1(b) possesses only external cash flows. Cash flows in a network 1(a) as seen by individual firms 1 and 2 are presented respectively in figures 1(c) and 1(d).

Now consider the case when the two firms merge. We assume additionally that there are no agency conflicts and the shareholder value maximization is the only motive of the firm.
The merged firm operates the individual units as independent entities, with the difference that the transfer of goods between the firms is now an internal capital market and the only risks that the merged firm faces are those of the volatility of the external cash flows $\mathbf{B}$, see Figure 1(b). The dynamics of the merged company is then given as $A_{12} = A_1 + A_2$, where $A_1$ and $A_2$ are defined as in (6)-(7) and the equity value of the merged company is $S_{12} = S_{12}(\sigma_{12})$, where $A_{12}(0) = A_1(0) + A_2(0) \text{, } D_{12} = D_1 + D_2 \text{, } \sigma_{12} = \sqrt{B_{11}^2 \lambda_1 + B_{22}^2 \lambda_2}$. We notice that the single most important parameter of the merged firm that drives the stock values is the volatility difference between the individual firms and the merged one. Moreover, the parameters $V_{12}$ and $V_{11}$ do not enter the volatility equation.

The firms’ bargaining in a merger process is done as in Section 2. Since there are only two firms in a network, the axioms NA and BC simply state that the merged firm does not fall apart. In the case of two firms, the only decision is whether the second firm entering the bargaining process joins the coalition of the first firm, i.e. it merges with it or not. The decision is done according to the CA axiom which states that the merger is beneficial if and only if the value of the merged firm exceeds the joint value of both firms. In this case the firms’ values are determined by the CW axiom and coincides by Theorem 2.2(d) with Shapley (1953).

**Proposition 3.2.** In a two firm example the firms merge if and only if $S_{12} \geq S_1 + S_2$. In this case, share prices of firms 1 and 2 are $\frac{1}{2}(S_1 + S_{12} - S_2)$ and $\frac{1}{2}(S_2 + S_{12} - S_1)$ respectively. If $S_1 + S_2 \geq S_{12}$ then firms 1 and 2 do not merge and are worth $S_1$ and $S_2$ respectively.

Figure 2 shows the equity values of the merged firm and the sum of equity values of each individual firm. In a two firm environment, the merged firm equity does not depend on the network dependency value, therefore a straight line for the value of the merged firm. The merger is rejected at low levels of network dependency. Here the volatility in the buy-supply cash flows contributes positively to the share prices of both firms, i.e. it captures the upside potential while minimizing the default cost of both buyer and the supplier firm. At high levels of network dependency parameter the default effect of volatile cash flows starts to dominate. It is here where the firms merge and form an internal capital market in the buyer-supplier...
Figure 2: A representation of a merger indicator with respect to the dependency value of the network. The dependency value of the network was defined in Brumen and Vanini (2006), Section 4, and in the case of two firms reduces to the number of connections $E_{12}$ between firm one and two. Proposition 3.2 indicates that the firms merge when $S_1 + S_2 \leq S_{12}$. The model parameters are as follows: the buy order proportionality factor $P = 0.0005$, the buy order intensities were $\lambda = (120, 100)'$, $B = \text{diag}(0.011, -0.0038)'$, recovery rate 0.2, $A_0 = (100, 100)'$ and the principal debt values are $D_0 = (80, 80)'$.

both firms are now hedged against the fluctuations in the buy-supply orders and the volatility comes only from the external cash flows. In this range of network dependency values, the positive effect of volatility dominates and the merger of the firms is preferable for both firms. Figure 2 also shows that increased network dependence in the buyer-supplier chain raises the value of the merger, i.e. the difference between the merged firm value and the total value of individual firms increases with increasing network dependence. Different values of network parameters move or scale the merger indicator function in Figure 2 but the shape remains fairly stable.

We next examine the behavior of the network dependency cut-off values, i.e. the values of the network dependency parameter $E_{12}$ when the merger becomes profitable, as depending on the average firm D/A ratio in the economy. The example gives an indication of the business cycle effect on the merger activity. Figure 3 shows that the cutoff value of the network dependency decreases with increasing average D/A ratio up to a certain point (the average D/A ratio of around 0.67 in Figure 3). As the economy progresses into recession, i.e. the D/A
Figure 3: The relationship between the network dependency cutoff values $E_{12}^*$, i.e. the value of the network dependence parameter $E_{12}$ where merger becomes rational, and the average debt to total assets $D/A$ ratio. The network dependence parameter takes values between 0 and 100. Other parameters of the model are given in the caption below Figure 2.

Increasing the average D/A ratio increases the probability of default for every single firm in the network. A merger therefore becomes more attractive. After the D/A ratio increases above certain level, the probability of default is already so high that only additional increase in cash flow volatility benefits stockholders of any one company in the network. The network dependency cutoff value $E_{12}^*$ displays an U-shaped curve and the merger activity an inverted U-shaped line. The maximum amount of mergers does not occur in expansions nor in severe recessions, even though in both states mergers do occur. The existence of mergers even in severe recessions has been empirically documented in Bernile, Lyandres, and Zhdanov (2007) and is intuitively similar to the asset substitution problem.
4 A general model

We now turn to the case of \( N \) firms in a general buyer-supplier network. We use the same network structure and notation as in Brumen and Vanini (2006). A network consists of \( N \) firms who issue buy orders from their suppliers. Buy orders depend on business relationships between firm which are described by the adjacency matrix \( E \in \mathbb{R}^{N \times N} \). For example, \( E_{12} = 2E_{23} \) means that the firm 1 buys at every issue of a buy order twice as much (proportional to its asset value) from firm 2 as does firm 2 from firm 3. Buy orders of firm \( i \) arrive with intensity \( \lambda_i \) independently of all other firms (\( \lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^{N \times N} \)). Every buy order transfers a proportional nominal amount \( P_i \) of firm’s assets to all of its suppliers (\( P = \text{diag}(P_1, \ldots, P_N) \in \mathbb{R}^{N \times N} \)). The suppliers are characterized by firms for which \( E_{ij} > 0, j = 1, \ldots, N \). In addition to network generated cash flows, firms receive external cash flows, where the correlation between the network generated cash flows and the external ones is given by the matrix \( B \in \mathbb{R}^{N \times N} \). Under certain technical assumptions (see Brumen and Vanini (2006), Theorem 3.1 for precise statements) the asset value process \( A = (A_1, \ldots, A_N) \in \mathbb{R}^N \) of all the firms in the network can under the risk-neutral measure be approximated by a multivariate geometric Brownian motion

\[
dR = r_1 dt + \alpha \lambda^{1/2} dW, \tag{8}
\]

where \( R \) is the vector of returns of \( A \), i.e. \( dR_i = \frac{dA_i}{A_i} \) and \( \alpha_{ij} = (B_{ij} + \frac{V_{ij}A_j(0)}{(1+B_{ij})A_i(0)}, B_{ii} + V_{ii}, B_{ij}), \) where \( V = E' \cdot P \). The asset volatility of firm \( 1 \leq i \leq N \) in this setting is given by

\[
\sigma_i = \sqrt{\sum_{j<i} \left( B_{ij} + \frac{V_{ij}A_j(0)}{(1+B_{ij})A_i(0)} \right)^2 \lambda_j + (B_{ii} + V_{ii})^2 \lambda_i + \sum_{j>i} B_{ij}^2 \lambda_j.} \tag{9}
\]

We define as before the firm coalition to describe the cooperation of firms in forming internal capital markets. Classical mergers, mergers of equals, acquisitions or hostile takeovers are in our setting all examples of coalition formation. We make the following assumptions. The objective function of a firm coalition is its equity value maximization. There are no agency
conflicts within the coalitions and no tax advantages of a bigger coalition. Finally, firms in a coalition operate as subdivisions of that coalition. Therefore, there are no production efficiency gains and no externalities of firm mergers. A merger changes the volatility of cash flows (and therefore the equity prices) through the creation of internal capital markets, but does not effect any other coalition. We use Leland and Toft (1996) equity pricing model to determine the equity value of coalitions.

The bargaining and coalition formation process described in Section 2 is done at the beginning of the period. After the coalition formation, no subsequent mergers or coalition breakups are allowed. We first specify the value of coalitions. Let \( LT(V, P, \sigma) \) be the Leland-Toft equity value of the firm with firm market value \( V \) at the beginning of the period, total principal \( P \) and asset volatility \( \sigma \) as defined in Proposition A.3 of the Appendix. Assume \( S \subset \mathcal{N} \) is a coalition. The value \( v(S) \) of this coalition is given by

\[
v(S) = LT \left( \sum_{i \in S} A_i(0), \sum_{i \in S} D_i, \sqrt{\sum_{j \in S} \left( \sum_{i \in S} \alpha_{i,j} \right)^2 \lambda_j} \right),
\]

i.e. the initial coalition asset market value is the sum of all assets in the coalition and the debt principal of the coalition is the sum of principals of individual firms. We interpret the value \( v(i) = LT(A_i(0), D_i, \sigma_i) \) for any \( 1 \leq i \leq N \) as the standalone firm value with \( \sigma_i \) as in (9). The axioms NA, BC, CA and VP all retain their meaning in coalitions. The bargaining process theory proceeds as follows. Firms enter the bargaining process sequentially, i.e. firm \( i \leq N \) begins negotiation after firms \( 1, \ldots, i-1 \) have already formed the appropriate coalitions. Firm \( i \)'s value is determined by the CW axiom, taking into account future coalition building, and joins the selected coalition according to the CA axiom. The stability of the coalition formation is guaranteed by the NA and BC axioms.

As an illustration we consider a network of 3 firms. A larger network complicates computations considerably but does not offer any new economic insights. In this network firm 1 is a retailer, i.e. a buyer from both firms 2 and 3, firm 2 is an intermediary (a buyer from firm 3 and a supplier of the retailer) and firm 3 is a producer (supplier to both firms). The
The parameter values described before are given in Table 1. The parameters in Table 1 are chosen to reflect realistic values of such a small economy. The $D/E$ ratios of firms 1, 2 and 3 are all around 0.7 and the asset volatilities are between 0.2 and 0.4 for all three firms.

We answer the following two questions. What is the effect of network dependency on merging activity and economic surplus generation? Secondly, what effect does the recession/expansion, measured in terms of average debt to total asset $D/A$ ratio of all firms in a network have on merger activity? For the case of network and $D/A$ parameters in Tables 2 3 where the average number of coalitions is 3, no merging has occurred. All firms operate separately. In cases of networks when 2 coalitions have formed, two firms have merged and the third one operates independently. We deduce which firms have merged by looking at their values with and without the merger effects. The difference between a firm’s value in isolation (without merger effects) and its merged value proxies for the Leland and Toft (1996) volatility/default cash-flow tradeoff due to the creation of internal capital markets. The increase in firm value is attributed to the positive effect between the tradeoff of firm default probability and stock increase due to upward potential of volatile cash flows.

The effect of different levels of network dependencies on coalition formation is shown in Table 2. Looking at the values of the firms under merger possibilities we notice that there exists an optimal level of network dependency for all three firms. The optimal levels are not the same. For the retailer (firm 1), this value is around $M = 4$. The intermediary and the supplier prefer low network dependency, i.e. their optimal lies at $M = 1$. Table 2 reveals that mergers occur with low and high network dependency values. In our example, the mergers

<table>
<thead>
<tr>
<th>$E$</th>
<th>$\text{diag}(P)$</th>
<th>$\lambda$</th>
<th>$\text{diag}(B)$</th>
<th>$D$</th>
<th>$A_0$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 2 2</td>
<td>0.0005</td>
<td>50</td>
<td>0.015</td>
<td>70</td>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>0 0 4</td>
<td>0.0007</td>
<td>60</td>
<td>0.003</td>
<td>80</td>
<td>110</td>
<td>1</td>
</tr>
<tr>
<td>0 0 0</td>
<td>0.0003</td>
<td>65</td>
<td>$-0.006$</td>
<td>90</td>
<td>120</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Model parameters for the network of three firms. $E$ is the network dependency matrix, $P$ the proportionality matrix and $B$ the correlation between network and external cash flows, as explained in the beginning of Section 4. $\lambda$ is the vector of intensities of buy orders of firms in the network. $D$ is the principal of the zero-coupon bonds issued by firms in the network. Vector $A_0$ shows the firms’ market asset values at time 0 and $T$ is the time period.
Table 2: The dependence of firm values and the number of coalitions formed with respect to the dependency value of the network. The network dependency of the whole network increases with ascending $M$, i.e. a network dependency matrix in row $M$ is $E(M) = ME$, where $E$ is given in Table 1. The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. The next three columns (5-7) are the values of individual firms, not incorporating the merger effects. The last column gives the average number of coalitions formed. Other parameters are given in Table 1.

<table>
<thead>
<tr>
<th>Network dependency</th>
<th>Firm values with mergers</th>
<th>Standalone firm values</th>
<th>Merger surplus</th>
<th>Avg. # of coalitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1.0</td>
<td>9.8</td>
<td>31.2</td>
<td>16.7</td>
<td>9.8</td>
</tr>
<tr>
<td>2.0</td>
<td>29.3</td>
<td>15.8</td>
<td>8.6</td>
<td>29.3</td>
</tr>
<tr>
<td>3.0</td>
<td>31.0</td>
<td>0.6</td>
<td>0.6</td>
<td>31.0</td>
</tr>
<tr>
<td>4.0</td>
<td>31.0</td>
<td>0.3</td>
<td>2.7</td>
<td>31.0</td>
</tr>
<tr>
<td>5.0</td>
<td>31.0</td>
<td>0.0</td>
<td>4.1</td>
<td>31.0</td>
</tr>
<tr>
<td>6.0</td>
<td>21.3</td>
<td>0.1</td>
<td>3.9</td>
<td>21.3</td>
</tr>
<tr>
<td>7.0</td>
<td>4.6</td>
<td>0.7</td>
<td>2.9</td>
<td>4.6</td>
</tr>
<tr>
<td>8.0</td>
<td>2.0</td>
<td>2.8</td>
<td>1.9</td>
<td>0.9</td>
</tr>
<tr>
<td>9.0</td>
<td>2.5</td>
<td>4.9</td>
<td>1.1</td>
<td>0.4</td>
</tr>
<tr>
<td>10.0</td>
<td>3.3</td>
<td>7.0</td>
<td>0.5</td>
<td>0.4</td>
</tr>
</tbody>
</table>

occurred for values of $M = 2, 3, 8, 9, 10$. Next, merger net surplus generated for low network dependency values ($M = 2, 3$) is higher than for high network dependencies ($M = 8, 9, 10$). Also, the type of merger for low dependencies is different that for those with high ones. For $M = 2, 3$ the intermediary and the supplier merge. For $M = 8, 9, 10$, the retailer and the intermediary merge. In the cases of very high network dependency, the retailer is exposed most. It hedges itself by merging with the intermediary. The only exposure that the merged firm now has is to the supplier and the external cash flows. This is an example of upward integration. For $M = 2, 3$ the intermediary and the supplier have merged. The reason for this type of merger is that this decreases their joint default exposure to the retailer. The latter is is an example of downward market integration. The overall structure of the results depends on the parameter values chosen. We have computed the merger results for parameters which give realistic asset volatility values.

Similar to the case of two firms we now consider the dependence of merger activity and firms’ values in relation to the network average D/A ratio. D/A ratio can be considered as
Table 3: The dependence of the firm values and number of coalitions with respect to the average D/A ratio of the firms in the economy. The first column does not reflect the real D/A ratio, but only a decrease in the D/A ratio of all firms. The D/A ratio of firm $i$ is given by the formula $A_i(0) + D/A \cdot 10$, and where $A_i(0)$ is given in Table 1. The next three columns (2-4) represent the value of firms 1-3 incorporating the merger effects. The next three columns (5-7) are the individual firm values, not incorporating the merger effects. The last column gives the average number of coalitions with the selected parameter values. All other parameter values are given in Table 1.

ratio has a much larger economic impact - in terms of merger surplus generated - on networked firms as the variation in the network dependency value. Since the firm value in the Leland and Toft (1996) model depends heavily on the leverage parameter, this result confirms economic intuition. Furthermore, we get merger activity in recessions ($D/A = 1$ in Table 3) as well as in expansion ($D/A$ high), again confirming the results of Bernile, Lyandres, and Zhdanov (2007). For both parameter values we have downward integration effects, that is the intermediary and the supplier merge. Contrary to the results in Table 2, $D/A$ ratio only impacts the size of the surplus, but not the merging firms.

4.1 Merger effects in different types of networks

We now consider the likelihood and the type of merger, i.e. is it a merger of buyers, suppliers or a vertical merger, depending on different types of networks. In particular we consider three networks depicted in Figure 4. The network in Figure 4(a) depicts two suppliers to a single
(a) Network of multiple suppliers $S_1$ and $S_2$ and a single buyer $B$, denoted NT 1.

(b) Network of multiple buyers $B_1$ and $B_2$ and a single supplier $S$, denoted by NT 2.

(c) Network of a buyer $B$, an intermediary $I$ and a supplier $S$, denoted by NT 3.

**Figure 4:** Different classes of buyer-supplier networks. $B_i$ denotes the buyers, $S_i$ the suppliers. $I$ is the intermediary of the buyer-supplier chain in network 4(c).

Buying firm, whereas 4(b) shows two buyers and a single supplier. Figure 4(c) shows a vertical network of a buyer, an intermediary and a supplier. The adjacency matrices of networks 4(a), 4(b) and 4(c) are

$$E_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

respectively. The three types of networks are distinctively different. In network 4(a) the default of the buyer causes a cash-flow shock to both of its suppliers, differently as in 4(b) where the default of one of the buyers causes only a cash inflow reduction. The third network shares familiarities with both 4(a) and 4(b). The default of the buyer has the same effect on the intermediary as in 4(b) on the supplier, and the supplier default in 4(c) has the same effect on the intermediary as in 4(a) to the buyer. Network 4(c) therefore shares similarities with both 4(a) and 4(b).

In Table 4 we vary the number of business relationships (links) between the individual firms. We present only the comparison of merger activity across different types of networks. The complete tables for every single network are presented in the Appendix (Tables 5, 6 and 22).
The results show that the merger in all three types of networks is never of a conglomerate type - that is, there is never a merger between the two suppliers and two buyers. This confirms a low probability of mergers for diversification and confirming the intuition in Brealey and Myers (2000). All three networks also display a common feature that a merger generates total economic surplus only for higher network dependency values. For lower network dependency the increase in volatility of cash flows is beneficial, while for higher values of network dependencies (and higher asset volatilities), it increases the default probability. In all three types of network the merger is always between vertically connected firms indicating that a buyer-supplier merger creates an internal capital market which has a positive market value. This effect is similar to Hart (1995). In the table above there was never a merger between all three types of firms (the network values of \( M \) correspond to realistic volatility values of firm cash flows). This is most likely the consequence of the structure of external cash flows. All three networks also generate similar maximum economic surplus - 8.6 for network type NT 1, 12.6 for NT 2 and 11.2 for NT 3. While for networks of types NT 1 and NT 3 the merger occurs at roughly the same amount of network dependency parameter (\( M = 8 \) and 9), for NT 3 the merger occurs much earlier, at \( M = 4 \).
5 Conclusions

The paper examined the effect of network dependencies and volatility structure of firm’s cash flows on firm equity values and merger formation process. Merger of two firms creates an internal capital market and changes the risk-structure of cash flows that the firm generates. This can either create or destroy firm value. For that purpose we develop a theory of coalition formation without the superadditivity axiom and apply it to the merger formation process. Firm equity value in a network should not be considered in isolation. “Merger corrections” increase the equity value of the firms accounting for the potential of a value creating merger.

Accounting for mergers, there exists an optimal value of network integration, different for all firms in the network. The results also indicate that there exists an $U$-shaped dependence of merger activity with respect to economic expansion/recession, confirming previous empirical results. Network dependence in general decreases equity values and increases the number of coalitions. Possible model extensions are coalitional formation with externalities, resource transfers and endogenous network formation.
A Appendix: Proofs of Theorems

Proof. (of Theorem 2.1.) To prove the theorem it suffices to prove the existence of the function \( \Phi^i \) for every \( i = 1, \ldots, N \). We proceed by induction on the player top down, i.e. we start with player \( N \) and proceed downwards to player 1. The basis for the induction is given in the definition: \( \Phi^N(S_1, S_2|P^{N-1}) = v(S_1 \cup N) \) if \( S_1 = S_2 \) and \( v(S_1) \) otherwise. We now assume that we have constructed \( \Phi^i, i = k + 1, \ldots, N \). \( \Phi^k \) is then defined by equation (1), where \( t_j, j > k \) is defined by \( \Phi^j \) (which was already constructed) and \( S^N(S) \) is constructed using the CA axiom. This proves the inductive step.

The proof of Theorem 2.2 is facilitated by the series of Lemmas and the notation is adopted from there.

Lemma A.1. \( \Phi^i(S, \hat{S}|P^{i-1}) \) is independent of \( P^{i-1} \) for all \( i = 1, \ldots, N \).

Proof. We prove by the downward induction on the player number \( k \). The induction basis for player \( N \):

\[
\Phi^N(S, \hat{S}|P^{N-1}) = \begin{cases} 
v(S) & S \neq \hat{S} \\
v(S \cup N) & S = \hat{S} 
\end{cases}
\]

Since the coalitional game does not possess externalities, the value function \( v \) is independent of \( P^{N-1} \). Now we assume that all \( \Phi^i, i > k \) are independent of \( P^{i-1} \) respectively. By the CA and VP axioms and the definition (1) of \( \Phi^i \) the induction step follows.

Lemma A.2. Let \( i+1 \) be competitively allocated to \( S^o \) when the partial partition is \( (P^{i-1}, \hat{S} \cup i) \). Then the following relationships hold.

(a) If \( S \neq \hat{S} \in P^{i-1} \) and \( S^o \neq S \) then \( \Phi^i(S, \hat{S}|P^{i-1}) = \Phi^{i+1}(S, S^o|(P^{i-1}, \hat{S} \cup i)). \)

(b) If \( S \neq \hat{S} \) and \( S^o = S \) then \( \Phi^i(S, \hat{S}|P^{i-1}) = \Phi^{i+1}(S, S|(P^{i-1}, \hat{S} \cup i)) - t_{i+1}(S|(P^{i-1}, \hat{S} \cup i)). \)

(c) If \( S \neq S^o \) then \( \Phi^i(S, S^o|P^{i-1}) = \Phi^{i+1}(S \cup i, S^o|P^{i-1}, S \cup i). \)

(d) If \( S = S^o \) then \( \Phi^i(S, S^o|P^{i-1}) = \Phi^{i+1}(S \cup i, S \cup i|P^{i-1}, S \cup i) - t_{i+1}(S \cup i|P^{i-1}, S \cup i). \)
Proof. To prove part (a) we write

$$
\Phi^i(S, \hat{S}|P^i) = v(S^N(S)) - \sum_{j>i} t_j(S^{j-1}(S)|\psi^{j-1}(P^i, \hat{S} \cup i))
$$

$$
= v(S^N(S)) - \sum_{j>i+1} t_j(S^{j-1}(S)|\psi^{j-1}(P^i, \hat{S} \cup i, S^o \cup (i+1)))
\quad - t_i(S^i(S)|\psi^i(P^{i-1}, \hat{S} \cup i)).
$$

Since $S \neq S^o$ we have that $t_j(S^i(S)|\psi^i(P^{i-1}, \hat{S} \cup i)) = t_j(S|\psi^i(P^{i-1}, \hat{S} \cup i)) = 0$, which proves part (a).

To prove part (b) we compute

$$
\Phi^i(S, \hat{S}|P^i) = v(S^N(S)) - \sum_{j>i} t_j(S^{j-1}(S)|\psi^{j-1}(P^i, \hat{S} \cup i))
$$

$$
= v(S^N(S \cup (i+1))) - \sum_{j>i+1} t_j(S^{j-1}(S \cup (i+1))|\psi^{j-1}(P^i, \hat{S} \cup i, S \cup (i+1)))
\quad - t_{i+1}(S^i(S)|\psi^i(P^{i-1}, \hat{S} \cup i))
\quad = \Phi^{i+1}(S, S|P^{i-1}, \hat{S} \cup i) - t_{i+1}(S|\psi^{i}(P^{i-1}, \hat{S} \cup i)),
$$

since $t_{i+1}(S^i(S)|\psi^i(P^{i-1}, \hat{S} \cup i)) = t_{i+1}(S|\psi^i(P^{i-1}, \hat{S} \cup i))$, which proves part (b).

Part (c) and (d) are proved in a similar manner.

$$
\square
$$

Proof. (of Theorem 2.2.) We fix an ordering of players entering the bargaining process as $\pi = (1, 2, \ldots, N)$. By averaging over all permutations $\pi \in S^N$ we obtain the desired result. On occasions we will suppress certain function arguments when it is obvious from the context what they are.

To prove (a) we first establish the following identity.

$$
\sum_{i \in S} \varphi_i(P^{i-1}) = \Phi^k(S, S^o|P^{k-1}),
$$

(11)

where $S, S^o \in P^{k-1}$ and $S^o$ is the competitive allocation (by the axiom CA) of player $k$. We
prove (11) by induction on the player $k$ getting into the bargaining process. We first establish the basis of induction.

$$\varphi_1(\emptyset) = \Phi^1(\emptyset, \emptyset|\mathcal{P}^0) - \Phi^1(\emptyset, \emptyset'|\mathcal{P}^0) = \Phi^1(\emptyset, \emptyset|\mathcal{P}^0),$$

where the identities follow from the fact that the maximum in the CW axiom (3) for player 1 is attained at $S^\infty = \emptyset'$ and the fact that $\Phi^1(\emptyset, \emptyset'|\mathcal{P}^0) = 0$.

We now prove the inductive step. Let $2 \leq k < N$ and $\mathcal{P}^{k-1}$ be fixed. We first assume that $S \neq S^\infty$ in (11) and differentiate between two cases: $k + 1$ is competitively allocated to (i) $S^\infty \neq S$ and (ii) $S^\infty = S$. For the case (i) we have $\sum_{i \in S} \varphi_i(\mathcal{P}^{i-1}) = \Phi^k(S, S^\infty|\mathcal{P}^{k-1}) = \Phi^{k+1}(S, S^\infty|\mathcal{P}^{k-1}, S^\infty \cup k)$ by Lemma A.2(a). This proves the inductive step in the case (i).

Now consider the case (ii) $S^\infty = S$. Here

$$\sum_{i \in S \cup (k+1)} \varphi_i(\mathcal{P}^{i-1}) = \Phi^k(S, S^\infty|\mathcal{P}^{k-1}) + \varphi_{k+1}(\mathcal{P}^{k})$$

$$= \Phi^{k+1}(S, S|\mathcal{P}^{k-1}, S^\infty \cup k) - t_{k+1}(S|\mathcal{P}^{k-1}, S^\infty \cup k) \varphi_{k+1}(\mathcal{P}^{k-1}, S^\infty \cup k)$$

$$= \Phi^{k+1}(S, S|\mathcal{P}^{k-1}, S^\infty \cup k)$$

using Lemma A.2 and the CW and VP axioms.

We now turn to the induction step when $S = S^\infty$. The case $S^\infty \neq (S \cup k)$ is almost identical to (i) above and will not be repeated. We prove only the last case $S = S^\infty$ and $S^\infty = S \cup k$. Then

$$\sum_{i \in S \cup (k+1)} \varphi_i(\mathcal{P}^{i-1}) = \Phi^k(S, S|\mathcal{P}^{k-1}) + \varphi_{k+1}(\mathcal{P}^{k-1}, S^\infty \cup k)$$

$$= \Phi^{k+1}(S \cup k, S \cup k|\mathcal{P}^{k-1}, S \cup k) - t_{k+1}(S \cup k|\mathcal{P}^{k-1}, S \cup k)$$

$$+ \varphi_{k+1}(\mathcal{P}^{k-1}, S^\infty \cup k)$$

$$= \Phi^{k+1}(S \cup k, S \cup k|\mathcal{P}^{k-1}, S \cup k)$$

27
by Lemma A.2(d) and the CW and VP axioms. Altogether, this proves the induction step.

Since the induction hypothesis holds also for $k = N$, we have proven (a).

Part (b) is obvious from the construction of the Vickrey payments. To prove (c) we compute

$$
\varphi_k(\mathcal{P}^{k-1}) = \Phi^k(S^o, S^o|\mathcal{P}^{k-1}) - \Phi^k(S^o, S^{oo}|\mathcal{P}^{k-1}) \\
= v(S^N(S^o \cup k)) - \sum_{j>k} t_j(S^{j-1}(S^o \cup k)|\psi^{j-1}(\mathcal{P}^{k-1}, S^o \cup k)) \\
- v(S^N(S^o)) + \sum_{j>k} t_j(S^{j-1}(S^o)|\psi^{j-1}(\mathcal{P}^{k-1}, S^{oo} \cup k)),
$$

where $k$ is competitively allocated to $S^o$ and $S^o \neq S^{oo} \in \mathcal{P}^{k-1}$. By assumption we have that $v(S^N(S^o \cup k)) = v(S^N(S^o))$ and by Lemma A.1 we have that for all $j > k$ we have $t_j(S^{j-1}(S^o \cup k)|\psi^{j-1}(\mathcal{P}^{k-1}, S^o \cup k)) = t_j(S^{j-1}(S^o)|\psi^{j-1}(\mathcal{P}^{k-1}, S^{oo} \cup k))$. This concludes the proof.

To prove (d) we show that the axioms we defined imply the axioms which define the Shapley value. The Pareto optimality condition of the Shapley value is implied by (a) of this theorem. The anonymity of the value function in the Shapley axioms is implied by the averaging over $\varphi_i$ for different permutations. The dummy axiom is implied by (c). It remains to prove the linearity of the Shapley value, i.e. we prove that $\varphi_i^{v+v'}$ (here we make the dependence on the value function explicit) constructed from $v + v'$ equals the sum of $\varphi_i^v + \varphi_i^{v'}$. To prove this we show that $\Phi^i_{v+v'}(S_1, S_2|\mathcal{P}^{i-1}) = \Phi^i_v(S_1, S_2|\mathcal{P}^{i-1}) + \Phi^i_{v'}(S_1, S_2|\mathcal{P}^{i-1})$. This is done by induction on $i$. The case of $\Phi^N$ is proven from the assumption since

$$
\Phi^N_{v+v'}(S_1, S_2|\mathcal{P}^{N-1}) = \begin{cases} 
    v(S_1) + v'(S_1) & S_1 \neq S_2 \\
    v(S_1 \cup N) + v'(S_1 \cup N) & S_1 = S_2
\end{cases}
$$

which evidently proves the base for induction. The inductive step is a consequence of induction assumption and the equation (1).

\[ \square \]

**Proof.** (of Proposition 3.2) We first assume that the firms enter the bargaining process in
order \{1, 2\}. The only possible coalition structure after firm 1 has entered is \(\mathcal{P}^1 = \{\{1\}, \emptyset\}\).

Since in a network of two firms firm 2 enters last, we can compute \(\Phi^2\) for different coalitions in \(\mathcal{P}^1\) (we omit the notation \(\mathcal{P}^1\) from \(\Phi^2\)):

\[
\begin{align*}
\Phi^2(\{1\}, \emptyset) &= S_1 \\
\Phi^2(\{1\}, \{1\}) &= S_{12} \\
\Phi^2(\emptyset, \{1\}) &= 0 \\
\Phi^2(\emptyset, \emptyset) &= S_2
\end{align*}
\]

Therefore the optimal allocation of firm 2 is to a coalition according to CA axiom is the \(S^o\) which maximizes the following

\[
S^o = \arg \max_{\hat{S} \in \{\{1\}, \emptyset\}} \left[ \Phi^2(\{1\}, \{1\}) + \Phi^2(\emptyset, \{1\}) + \Phi^2(\{1\}, \emptyset) + \Phi^2(\emptyset, \emptyset) \right]
\]

\[
= \arg \max_{\hat{S} \in \{S_{12} + 0, S_1 + S_2\}} [S_{12} + 0, S_1 + S_2]
\]

Therefore \(S^o(\mathcal{P}^1) = \{1\}\) if \(S_1 + S_2 \leq S_{12}\) nad \(S^o(\mathcal{P}^1) = \emptyset\) if \(S_1 + S_2 > S_{12}\). This is exactly what the condition in Proposition 3.2 states - if \(S_{12} \geq S_1 + S_2\), firm 2 joins the coalition of firm 1, i.e. firms merge. If \(S_{12} < S_1 + S_2\), firm 2 joins the coalition \(\emptyset\) and operates as an independent entity. We further calculate the values \(\varphi_2\) of firm 2:

\[
\begin{align*}
\varphi_2(\mathcal{P}^1) &= \Phi^2(\{1\}, \{1\}) - \Phi^2(\emptyset, \emptyset) = S_{12} - S_1 \text{ when } S_1 + S_2 < S_{12} \\
\varphi_2(\mathcal{P}^1) &= \Phi^2(\emptyset, \emptyset) - \Phi^2(\emptyset, \{1\}) = S_2 \text{ when } S_1 + S_2 > S_{12}
\end{align*}
\]

We next compute just the needed Vickrey payments: \(t_2(\{1\}|\mathcal{P}^1, \{1, 2\}) = S_{12} - S_1\) when \(S_1 + S_2 < S_{12}\) and \(t_2(\{1\}|\mathcal{P}^1, \{1, 2\}) = 0\) in the other case. Therefore we get \((\mathcal{P}^0 = \{\emptyset\})\)

\[
\varphi_1(\mathcal{P}^0) = S_{12} - (S_{12} - S_1) = S_1 \text{ and } \varphi_1(\mathcal{P}^0) = S_1 - t_2(\{1\}|\mathcal{P}^1, \{1, 2\}) = S_1
\]

respectively for the two cases above. By reversing the order of firms in the bargaining process, we get that the values of firms 1 and 2 are \(\frac{1}{2}(S_1 + S_{12} - S_2)\) and \(\frac{1}{2}(S_2 + S_{12} - S_1)\) in the case \(S_1 + S_2 < S_{12}\) (exactly the Shapley values) and \(S_1\) and \(S_2\) in the opposite case.

\(\square\)

The following proposition is a restatement of the debt and equity pricing results in Leland
and Toft (1996) and is repeated here for coherence.

**Proposition A.3** (Leland-Toft (1996)). Let the dynamics of firm assets be \( \frac{dA}{A} = \mu dt + \sigma dW \). The firm issued zero-coupon debt with maturity \( T \) and principal \( P \) which is retired uniformly over the interval \([0, T]\). The firm defaults when \( A \) falls below the default boundary \( V_B \), determined below. The costs of bankruptcy are \( \alpha V_B \). Then the value of the equity in this model, denoted by \( LT \) (mnemonic for Leland-Toft Equity value), is

\[
LT(V, P, \sigma) = V - \alpha V_B \left( \frac{V}{V_B} \right)^{-x} - D,
\]

where the value of \( D \) is given by

\[
D = P \left( \frac{1 - e^{-rT}}{rT} - I(T) \right) + (1 - \alpha)V_B J(T),
\]

where

\[
I(T) = \frac{1}{rT} (G(T) - e^{-rT} F(T))
\]

\[
J(T) = \frac{1}{z \sigma \sqrt{T}} \left( - \left( \frac{V}{V_B} \right)^{-\sigma^2 t} N(q_1(T)) q_1(T) + \left( \frac{V}{V_B} \right)^{-\sigma^2 t} N(q_2(T)) q_2(T) \right)
\]

and the constants are given by

\[
F(t) = N(h_1(t)) + \left( \frac{V}{V_B} \right)^{-2a} N(h_2(t))
\]

\[
G(t) = \left( \frac{V}{V_B} \right)^{-\sigma^2 t} \left( N(q_1(t)) q_1(t) + \left( \frac{V}{V_B} \right)^{-\sigma^2 t} N(q_2(t)) q_2(t) \right)
\]

\[
z = \frac{\sqrt{(a \sigma^2)^2 + 2r \sigma^2}}{\sigma^2}
\]

\[
q_1(t) = \frac{-b - \sigma^2 t}{\sigma \sqrt{t}} \\
q_2(t) = \frac{-b + \sigma^2 t}{\sigma \sqrt{t}} \\
h_1(t) = \frac{-b - a \sigma^2 t}{\sigma \sqrt{t}} \\
h_2(t) = \frac{-b + a \sigma^2 t}{\sigma \sqrt{t}} \\
\]

\[
a = \frac{r - \delta - \sigma^2/2}{\sigma^2} \\
b = \log \left( \frac{V}{V_B} \right)
\]
and the default boundary $V_B$ is given by

$$V_B = \frac{AP/(rT)}{1 + \alpha x - (1 - \alpha)B}$$

with

$$A = 2ae^{-rT}N(a\sigma\sqrt{T}) - 2zN(z\sigma\sqrt{T}) - \frac{2}{\sigma\sqrt{T}}n(z\sigma\sqrt{T}) + \frac{2e^{-rT}}{\sigma\sqrt{T}}n(a\sigma\sqrt{T}) + z - a$$

$$B = -\left(2z + \frac{2}{z\sigma^2T}\right)N(z\sigma\sqrt{T}) - \frac{2}{\sigma\sqrt{T}}n(z\sigma\sqrt{T}) + z - a + \frac{1}{z\sigma^2T}$$
### Table 5: The dependence of firm values and the number of coalitions formed with respect to the dependency value of the network for network depicted in Figure 4(a). The network dependency of the whole network increases with ascending $M$, i.e. a network dependency value in row $M$ is $2M$ with adjacency matrix $M \cdot E_1$, where $E_1$ is given in equation (10). The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. The next three columns (5-7) are the values of individual firms, not incorporating the merger effects. The last column gives the average number of coalitions formed. Other parameters are given in Table 1.

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Table 6: The dependence of firm values and the number of coalitions formed with respect to the dependency value of the network for network depicted in Figure 4(a). The network dependency of the whole network increases with ascending $M$, i.e. a network dependency value in row $M$ is $2M$ with adjacency matrix $M E_2$, where $E_2$ is given in equation (10). The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. The next three columns (5-7) are the values of individual firms, not incorporating the merger effects. The last column gives the average number of coalitions formed. Other parameters are given in Table 1.

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Table 7: The dependence of firm values and the number of coalitions formed with respect to the dependency value of the network for network depicted in Figure 4(a). The network dependency of the whole network increases with ascending $M$, i.e. a network dependency matrix in row $M$ is $2M$ with adjacency matrix $M E_3$, where $E_3$ is given in equation (10). The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. The next three columns (5-7) are the values of individual firms, not incorporating the merger effects. The last column gives the average number of coalitions formed. Other parameters are given in Table 1.

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References


