Fundamental Stock Price with Consumption CAPM and Money Illusion: An International Comparison

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CAPM and Money Illusion: An International Comparison

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Abstract

What is the fundamental value of a stock? Do stock prices deviate from this fundamental value? If yes, do they go back to their fundamental value? This paper proposes to answer these three questions by using a stock valuation model based on the Consumption-Capital Asset Pricing Model (C-CAPM) with and without money illusion. I first show how to use the C-CAPM to get a fundamental price that can be empirically estimated. This fundamental price is a function of expected future dividends and future consumption, of their future conditional variance and covariance and of agents’ risk aversion. Secondly, I estimate the C-CAPM fundamental price for the United States, the United Kingdom, Japan and Switzerland for a period going from 1965 through to 2006. I identify several periods in which stock prices deviate significantly from their fundamental value. Thirdly, I show that after a shock, the gap between the price and its fundamental value decreases with time, which suggests that stock prices go back to the C-CAPM fundamental value. Finally, I show that forecasts using the C-CAPM fundamental price are more accurate than forecasts based on the observed price only. Additionally, I show that the C-CAPM forecasts systematically outperform forecasts made with other fundamental models (e.g. models based on the price-to-dividend ratios).

Keywords: Fundamental stock price, Consumption CAPM, Money illusion, Out-of-sample forecasts.

1 Introduction

What is a stock really worth? Do stock prices deviate from this fundamental (or fair) value? And if they deviate, do they eventually go back to their fundamental value? Such questions have been prominent topics for decades in the finance profession. Many economic agents are interested in their answers: shareholders, who are comparing investment alternatives, traders, who are looking for speculation opportunities, or central bankers, who try to identify stock market imbalances of which unwinding could have an impact on the economy. In this paper, I study these three questions with a stock valuation model based on the Consumption-Capital Asset Pricing Model (C-CAPM) (Lucas 1978 and Breeden 1979) with and without money illusion. I first show how to use the no-arbitrage condition of the C-CAPM to express the fundamental stock price as a function of expected dividends and consumption, as well as of their covariance. I then estimate the fundamental stock price (and deviations from it) for four countries: the United States, the United Kingdom, Japan and Switzerland. Finally, I assess the out-of-sample forecast accuracy of the C-CAPM fundamental model. I find that forecasts based on the C-CAPM fundamental price significantly outperform forecasts based on the observed price or on indicators that are known to have some predictive power (e.g. the price-to-dividend ratio).

Not surprisingly given the interest that fundamental stock prices arouse, academics or practitioners have proposed several models to estimate them. A large majority of them are based on the discounted cash flow model (or net present value model), which states that the fundamental stock price is equal to the sum of the discounted expected payoffs of the stock.¹ This type of model requires a forecast of future cash flows generated by the stock, along with an appropriate discount rate. The most basic model in this category is the Gordon growth model (Gordon and Shapiro 1956, Gordon

¹Lee (1998), Dupuis and Tessier (2003), Zong, Darrat and Anderson (2003) and Borio and Lowe (2002) are some of the few exceptions to the net present value model (although the former three indirectly build their method on it). They all measure the fundamental price by separating the permanent component of stock prices from their temporary and non-fundamental component.
1962). In this model, stock’s payoffs are the dividends. Future discount rates, as well as the future dividend growth rate are both constant. Many authors have refined this model. The first line of innovation is to use dividend forecasts that are more realistic than a constant growth rate. For example, Shiller (1981 and 2005) uses ex-post realized dividends. Kaplan and Ruback (1995), Becchetti and Mattesini (2005) or Bagella, Becchetti and Adriani (2005) use a two-stage model, in which short-term forecasts are given by analysts and long-term forecasts are determined by the historical growth rate. A similar three-stage model is proposed by Panigirtzoglou and Scammell (2002). Yao (1997) separates increasing dividends from decreasing dividends. The second line of developments concentrates on the definition of future cash-flow. Ang and Liu (2001), Vuolteenaho (2002) or Dong and Hirshleifer (2005) use earnings instead of dividends; Black, Fraser and Groenewold (2003a,b) use profits. Cohen, Polk and Vuolteenaho (2003) or Pástor and Veronesi (2006) choose the market-to-book value ratio instead of the price-dividend ratio (PD ratios hereafter). The third line of improvement concerns the econometric methodology with the use of panel studies rather than the single times series (Lee, Myers and Swaminathan 1999, Becchetti and Adriani 2004 or Gentry, Jones and Mayer 2004). Lastly, some authors have studied the net present value fundamental model in a general equilibrium framework (Black, Fraser and Groenewold 2003a,b and Kinley 2004). In opposition to the dividends’ dynamic, the dynamic of the discount factor has received little attention. In general, the discount factor is constant and estimated by the CAPM. Campbell and Shiller (1987 and 1988a,b) have filled this gap by modelling the dynamic of both dividends and interest rates with a VAR model. They use the estimated joint dynamic to get a proxy of agents’ expectations. Their VAR approach is the starting point of impressive literature (which not necessarily devoted to the estimation of fundamental values).

The fundamental model presented in this paper is in the spirit of the VAR fun-

\footnote{An exhaustive survey of the literature on fundamental prices is beyond the scope of this paper. Only a selective list of the main innovations is presented here.}
fundamental model developed by Campbell and Shiller (CS hereafter). Like the models cited previously, it is based on the discounted cash-flow model, but it differs from them by using a stochastic discount factor (SDF) based on the no-arbitrage condition of the C-CAPM (Lucas 1978 and Breeden 1979). In most of the papers cited above, the SDF is given by the traditional CAPM. I use the C-CAPM for two main reasons: firstly, this model is based on sound economic arguments explaining consumption and investment decisions of a representative agent in a general model of production economy (see Breeden 1979 or Cox, Ingersoll and Ross 1985). Secondly, the model links consumption to asset prices. It is therefore well adapted to study the relation between the real economy and financial markets. These characteristics have made the C-CAPM one of the cornerstone of asset pricing (see e.g. Cochrane 2001). To summarize the model, in the C-CAPM, a rational representative agent splits her income between consumption and savings in a risky asset in order to maximize the utility of both her present and future consumption. The no-arbitrage equation states that the utility lost in investing one unit of consumption in an asset today must be equal to the expected utility of the additional future consumption obtained with the asset’s payoff. With this condition, the asset price is equal to the expected future payoffs discounted with the intertemporal marginal rate of substitution of the representative agent. This rate is a function of the marginal utility of present and future consumption and thus, indirectly, it is a function of present and future consumption. Consequently, the fundamental asset price is a function of the expected present value of future dividends and future consumption. The fundamentals variables (or fundamentals) are thus dividends and consumption instead of dividends and discount rates as in CAPM based models. To my knowledge, only Campbell and Shiller (1988a) and Lund and Engsted (1996, LE hereafter) have used the C-CAPM to compute a fundamental stock price.\footnote{The intertemporal marginal rate of substitution also depends on the form of the utility function and thus on the risk aversion of the representative agent.} Shiller (2005) also presents a fundamental price based on the C-CAPM. However, his computation is based on ex-post dividends and consumption growth rate and the coefficient of relative risk aversion is not estimated but arbitrarily set equal to 3.\footnote{Shiller (2005) also presents a fundamental price based on the C-CAPM. However, his computation is based on ex-post dividends and consumption growth rate and the coefficient of relative risk aversion is not estimated but arbitrarily set equal to 3.}
The model presented here differs from CS and LE in several ways: firstly, I compute two alternative fundamental models, one with real variables and the other with nominal variables. The latter use the implicit assumption that agents suffer from money illusion. I adopt this assumption after observing that both PD ratios, inflation and nominal consumption growth rates are subject to structural breaks, whereas real consumption growth rates are not. Stable consumption growth rates coupled with breaks in PD ratios are theoretically incompatible. Assuming money illusion is a way to cope with this problem (Modigliani and Cohn 1979). Several authors have documented the negative correlation between inflation and stock prices and most of the recent literature attributes it to money illusion (Ritter and Warr 2002, Campbell and Vuolteenaho 2004 and Cohen, Polk and Vuolteenaho 2005). I depart from this literature by studying the indirect impact of money illusion on asset prices via the discrepancy that using real and nominal fundamentals implies rather than by studying the direct link between inflation and stock prices. I find that money illusion seems to play a significant role in the United States, whereas it is less relevant for Japan. Both models perform fairly equally in the United Kingdom and do not work well in Switzerland.

Secondly, the fundamental price developed in this paper is based on a second-order Taylor approximation of the no-arbitrage condition. CS and LE stop at the first-order approximation. With an additional order of approximation the fundamental price becomes a function of the first and second moments of dividends and consumption, including their covariance. I estimate these second moments with a multivariate GARCH model. The advantage of this approach is to capture the impact of time-varying covariances on the fundamental price. This is new in the context of fundamental stock prices. Note that the second-order approximation presented in this paper is not restricted to the C-CAPM framework and can be used to compute the fundamental price derived from any other SDF model.

Thirdly, in each step of the computation of the fundamental price for time $t$, I have
been particularly careful to use only the information available at that time. Thus, I put myself in the same position as an investor, who computes the fundamental price at time \( t \). As a result, the fundamental price estimated here is a true *ex-ante* price. CS and LE compute an *ex-post* price by using the whole sample to estimate the fundamentals’ dynamic.

Finally, I assess the out-of-sample accuracy of forecasts based on the fundamental price and compare it with other simpler fundamental models. The ability of simple fundamental models to forecast future prices in the long term is now well documented. Campbell and Shiller (1998 and 2001) or Rapach and Wohar (2005), for example, show that price-dividends (PD) ratios can help to forecasts stock price movements (in-sample) for horizons of 6 up to 10 years. Recently, Rapach and Wohar (2006) bring evidence that this result also holds out-of-sample. One of the main results of this paper is to show that the C-CAPM fundamental price is able to *improve out-of-sample forecasts even for horizons shorter than 6 years* in the United States, the United Kingdom and Japan. In fact, in both the United States and the United Kingdom, the information contained in the C-CAPM fundamental price helps to improve the forecasts for all horizons. The improvement is particularly spectacular for the United Kingdom, where the fundamental model performs at least 20% better\(^5\) than the random walk with drift for all horizons longer than 3 months! No other simpler fundamental model tested in this paper is able to systematically outperform the forecasts of the C-CAPM model. In addition, I show that, in the United States and in the United Kingdom, the accuracy of the C-CAPM fundamental model increases when the price is far from its empirical value. This suggests that the tendency of the price to move back toward its fundamental value is stronger when the gap is wide.

The fact that the C-CAPM fundamental price is able to give out-of-sample forecasts is a sign that there is a link between market price and the fundamental price.

\(^5\)The accuracy is measured in terms of mean absolute error.
This forecast ability is observable in three out of four countries, Switzerland being the exception. I give an additional piece of evidence of this link by showing that the gap between the price and its fundamental value is mean-reverting in the long term in all countries.

The paper is structured as follow: Section 2 presents the fundamental price equation and the transformations that are necessary to estimate it. Section 3 describes the data. Section 4 documents the long term evolution of the SDF, its impact on the PD ratio and introduce the money illusion hypothesis. Section 5 describes the econometric methodology used to estimate the different coefficients of the fundamental equation. Section 6 presents the fundamental prices and the dynamic of the gap between the price and its fundamental value. Section 7 checks if prices eventually go back to their fundamental value and assesses the forecasting performance of the fundamental model. Section 8 assesses the value added of the C-CAPM fundamental model developed in this paper. Section 9 concludes.

2 The Fundamental Stock Price Equation

The equation for the fundamental stock price is based on the no-arbitrage equation derived from the C-CAPM (Section 2.1). From this equation, I express the fundamental price as the present value of expected future fundamentals, which are the dividends and the expected marginal utility of consumption (Section 2.2). Section 2.3 shows how to linearize the fundamental equation and to express the fundamental price as a linear function of expected fundamentals. Section 2.4 explains how to compute the expectations about fundamentals. Finally, Section 2.5 combines these elements to give the final equation for the fundamental price.
2.1 The C-CAPM no arbitrage equation

As mentioned in the introduction, the first step for computing the fundamental price of stocks is to chose one model, from which the fundamental equation will be derived. In this paper, I define the fundamental price as follows

**Definition 1** At time \( t \), the fundamental price of an asset is the equilibrium price resulting from the optimal choice made by a rational representative agent, who allocates her income between consumption and savings (in the asset) in order to maximize the utility of her present and expected future real consumption.

This definition corresponds to the maximization problem at the center of the C-CAPM (Lucas 1978 and Breeden 1979). The Euler equation given by the first order condition of this maximization problem is

\[
P_t = \mathbb{E}_t \left[ M_{t+1} (P_{t+1} + D_{t+1}) \right]
\]

with

\[
M_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}
\]

where \( P_t \) is the stock price at time \( t \), \( D_{t+1} \) is the dividend paid by the stock at the end of period \( t \), \( \beta \) is the subjective discount factor of the representative agent and \( U'(C_t) \) is the marginal utility of consumption \( C_t \) in period \( t \). \( M_{t+1} \) is called the stochastic discount factor (SDF). This equation is a no arbitrage equation, which states that the utility lost by reducing consumption of one unit in period \( t \) and investing it in the stock is equal to the discounted and expected increase in utility obtained from the extra payoff at time \( t+1 \).

To be able to compute SDF, I assume that

**Assumption 1** The representative agent has a power utility function
With a power utility function, the SDF is

\[ M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \]  

(3)

where \( \gamma \) is the coefficient of relative risk aversion of the representative agent.

Of course, many other utility functions are possible (e.g. exponential utility, habit formation, prospect utility, etc...). In particular, utility functions with time-varying risk aversion have attracted a lot of attention in the recent literature.\(^6\) Their success is mainly due to their ability to capture cyclical variation in the SDF. However, the goal of this paper is to estimate the long term fundamental value of stock prices rather than explaining their short term variations. In that sense, as mentioned by Campbell and Viceira (2002, p. 25), the constant relative risk aversion implied by power utility is "inherently attractive and is required to explain the stability of financial variables in the face of secular economic growth". However, if there is a need to use another utility function, the methodology proposed here is easily adaptable, by replacing the SDF with the adequate expression.\(^7\) The only requirement is that the utility function should give a SDF which is a linear function of observable variables. Habit formation functions or loss aversion functions, for example, have a SDF that fits into this framework (cf. Monnin 2007).

\(^6\)For example, habit formation models initiated by Constantinides (1990) or Campbell and Cochrane (1999) have been used extensively to empirically study time-varying risk aversion.

\(^7\)Most asset pricing models can be expressed in the SDF model. They only differ by the form taken by \( M_{t+1} \) (cf. Cochrane 2001).
2.2 Fundamental present value equation

By forward iteration of the future price in the no arbitrage equation (1), the fundamental price can be expressed as

\[ P_t = E_t \sum_{i=1}^{\infty} \left( \prod_{j=1}^{i} M_{t+j} \right) D_{t+i} \]  

This equation simply tells that the fundamental asset price in period \( t \) is equal to the expected present value of future dividends paid by the asset. The discount factor \( M_{t+i} \) used to compute the present value is a function of the expected marginal utilities of future consumption. Thus, the two fundamental variables driving the fundamental asset price are dividends and consumption.

Since prices and dividends are not stationary, it is convenient, for empirical purposes, to express the present value in equation (4) in terms of stationary variables. For that, as suggested by Cochrane (1992), divide both sides by dividends to get

\[ PD_t = E_t \sum_{i=1}^{\infty} \prod_{j=1}^{i} M_{t+j} \gamma_{t+j} \]  

where \( PD_t = P_t/D_t \) is the price dividend ratio (PD ratio) at time \( t \) and \( \gamma_t = D_t/D_{t-1} \) is the gross growth rate of dividends between \( t \) and \( t-1 \).

2.3 Linearization

The right hand side of equation (5) is clearly non linear, which is not convenient for empirical estimations. To cope with this problem, it is possible to linearize the fundamental price by taking the logarithm of equation (5) and by using a second order

\[ \text{Formally, the transversality condition } \lim_{i \to \infty} E_t \left[ \left( \prod_{j=1}^{i} M_{t+j} \right) P_{t+i} \right] = 0 \text{ is imposed to get equation (4). This condition rules out bubbles in the infinite horizon.} \]
Taylor expansion of the right hand side of this equation around its mean. This yields (see proof in Appendix A.1)

\[ pd_t = E_t (pd_t^*) + \frac{1}{2} V_t (pd_t^*) + R_t \]  

(6)

where \( pd_t = \ln PD_t \) and \( pd_t^* = \ln PD_t^* \) with \( PD_t^* = \sum_{i=1}^{\infty} \prod_{j=1}^{i} M_{t+j} \gamma_{t+j} \) being the PD ratio with all future fundamentals known with certainty (or, in other terms, the PD ratio with perfect forecasts). \( R_t \) is the remainder of the Taylor expansion, which is a function of third and higher expected moments of \( pd_t^* \).

Note that, by definition, \( PD_t^* \) is equal to

\[ PD_t^* = M_{t+1} \gamma_{t+1} (1 + PD_{t+1}^*) \]  

(7)

Taking the logarithm of this equation yields

\[ pd_t^* = m_{t+1} + \Delta d_{t+1} + \ln (1 + PD_{t+1}^*) \]  

(8)

where \( m_{t+1} = \ln M_{t+1} \), \( \Delta d_{t+1} = d_{t+1} - d_t \) and \( d_t = \ln D_t \). This expression can be linearized with a first-order Taylor approximation. As shown by Campbell, Lo and MacKinlay (1997), the last term of equation (8) can be approximated by (cf. Appendix A.2)

\[ \ln (1 + PD_{t+1}^*) \approx \kappa + \rho pd_{t+1}^* \]  

(9)

where \( \rho = 1/ (1 + \exp (-pd_t)) \) and \( \kappa = -\ln \rho - (1 - \rho) \ln (1/\rho - 1) \) are both linearization coefficients and \( pd_t \) is the average log PD ratio observed until time \( t \).

\(^9\)Note that the linearization coefficient \( \rho \) is time varying since it changes each time that the observed average log PD ratio \( \bar{pd}_t \) changes. To keep notation simple, I did not use a subscript for the time with \( \rho \), but the reader should keep in mind that \( \rho \) is reestimated at each period to compute the fundamental price. The reestimation of \( \rho \) is necessary to get a fundamental price that is computed only on the data observable at time \( t \).
this approximation in equation (8) gives

\[ pd_t^* \simeq \kappa + m_{t+1} + \Delta d_{t+1} + \rho pd_{t+1} \]  

(10)

Finally, by substituting \( pd_{t+1} \) forward, we get the following linear approximation of \( pd_t^* \)

\[ pd_t^* \simeq \sum_{i=1}^{\infty} \rho^{i-1} (\kappa + m_{t+i} + \Delta d_{t+i}) \]

(11)

If we now use the approximation (11) in equation (6), we get that

\[ pd_t = E_t \left( \sum_{i=1}^{\infty} \rho^{i-1} (m_{t+i} + \Delta d_{t+i}) \right) + \frac{1}{2} V_t \left( \sum_{i=1}^{\infty} \rho^{i-1} (m_{t+i} + \Delta d_{t+i}) \right) + c_t' \]

(12)

where \( c_t' = R_t + \kappa / (1 - \rho) \). The next step is to replace the log SDF \( m_{t+i} \) by its definition given in equation (3)\(^{10}\) and to express the fundamental log PD ratio in vector terms to simplify notation. For that, let us define the vector \( x_t = \begin{bmatrix} \Delta c_t & \Delta d_t \end{bmatrix}' \) which collects all the fundamentals. Using this notation, we can rewrite equation (12) as

\[ pd_t = \sum_{i=1}^{\infty} \rho^{i-1} g' E_t (x_{t+i}) + \frac{1}{2} V_t \left( \sum_{i=1}^{\infty} \rho^{i-1} g' x_{t+i} \right) + c_t \]

(13)

where \( g' = \begin{bmatrix} -\gamma & 1 \end{bmatrix} \) and \( c_t = R_t + (\kappa + \ln \beta) / (1 - \rho) \). Developing the conditional variance in the second term of the right hand side of this equation yields

\[ pd_t = \sum_{i=1}^{\infty} \rho^{i-1} g' E_t (x_{t+i}) + \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho^{i+j-2} g' E_t (\Lambda_{t+i,t+j}) g + c_t \]

(14)

\(^{10}\)Note that \( m_{t+i} = \ln M_{t+i} = \ln \beta - \gamma \Delta c_{t+1} \) where \( \Delta c_{t+1} = c_{t+1} - c_t \) with \( c_t = \ln C_t \).
where $A_{t+i,t+j} = \varepsilon_{t,t+i} \varepsilon_{t,t+j}'$ and $\varepsilon_{t,t+i} = x_{t+i} - E_t (x_{t+i})$ is the error of a forecast made at time $t$ about the fundamental $x_{t+i}$. The expression $E_t (A_{t+i,t+j})$ is the difference of the expected conditional covariance at time $t$ between the $t + i$ and $t + j$ forecast errors.\(^{11}\) Equation (14) simply states that the fundamental log PD ratio is a linear function of expected fundamentals (dividend growth and consumption growth) and of their expected conditional covariances and autocovariances.

2.4 Expectations about future fundamentals

As stated in equation (14), the fundamental log P/D ratio is a function of the representative agent’s expectations about fundamentals and their conditional covariance. Therefore, to estimate the fundamental price, we have to specify how does the representative agent forms her expectation. According to Definition 1, the representative agent is rational, which implies, by definition, that she will use all the relevant information available at time $t$ to make her forecasts. At time $t$, the relevant information set is constituted of all present and past fundamentals $x_t$ and of all present and past variables, which have some forecasting power for the fundamentals. The latter variables are collected in the $(p \times 1)$ vector $z_t$. Note that $z_t$ can include the observed log PD ratio if this one helps to predict future fundamentals. Additionally, we assume that

**Assumption 2** The representative agent forms her expectations about future fundamentals in two steps. In the first step, she estimates the dynamic of the variables in her information set with a VAR model, in which the conditional covariance is modeled with a multivariate GARCH. In the second step, she uses the estimated VAR M-GARCH to forecast future fundamentals and their conditional covariance.

The VAR part of the model is used to forecast the first part of the right hand side of equation (14) (future fundamentals). The GARCH part estimates the dynamic of

\(^{11}\)The term covariance refers both to variances and covariances.
the conditional covariance, which is then used to forecast the second part of the right hand side of equation (14) (future conditional variance). Concretely, assumption 2 implies that the representative agent uses the following model to make her forecasts about future fundamentals:

\[
y_t = A_0 + A_1 y_{t-1} + \ldots + A_j y_{t-j} + \varepsilon_t \quad (15)
\]

\[
\varepsilon_t \sim N(0, H_t) \quad (16)
\]

where \( y_t = [x_t \ z_t]' \) is a vector collecting present and past observations (until lag \( j \)), \( A_i \) are matrices of coefficients\(^{12} \) estimated at time \( t \) and \( \varepsilon_t \) is an error term, which is normally distributed with a time-varying covariance matrix \( H_t \). Assumption 2 also specifies that the covariance matrix \( H_t \) is modelled as a multivariate GARCH. The multivariate GARCH is a generalization of the univariate GARCH and it estimates time-varying covariances in addition to time-varying variances. A general formulation of the multivariate GARCH is the vech model of Bollerslev, Engle and Wooldridge (1988), which has the following specification

\[
h_t = K + Bh_{t-1} + Ce_{t-1} \quad (17)
\]

\[
e_t = h_t + u_t \quad (18)
\]

where \( h_t = \text{vech}\{H_t\} \), \( e_t = \text{vech}\{\varepsilon_t \varepsilon_t'\} \) where the vech operator converts the lower triangle of a symmetric matrix into a vector. The matrix \( B \) and \( C \) are matrices of coefficients\(^{13} \) and \( u_t \) is an error term which is normally distributed with a constant covariance matrix \( \Omega \). In this model, the covariance matrix is a linear function of its

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\(^{12}\) The coefficient matrices are time-varying since a rational agent update her estimation with each new observation. I did not use any subscript for the time to simplify the notation. The reader should however bear in mind that a different VAR-GARCH is estimated for each period using only the information available at that time.

\(^{13}\) The coefficient matrices are time-varying (cf. footnote 12).
last past values and of last past residuals.\textsuperscript{14} For more clarity, it is useful to express this VAR-GARCH model with its companion form

\begin{align*}
\mathbf{y}_t &= \mathbf{Ay}_{t-1} + \mathbf{e}_t \quad (19) \\
\mathbf{h}_t &= \mathbf{Bh}_{t-1} + \mathbf{u}_t \quad (20)
\end{align*}

where

\begin{align*}
\mathbf{y}_t &= \begin{bmatrix} y_t & y_{t-1} & \cdots & y_{t-j+1} & 1 \end{bmatrix}' \\
\mathbf{h}_t &= \begin{bmatrix} h_t & e_t & 1 \end{bmatrix}' \\
\mathbf{e}_t &= \begin{bmatrix} \varepsilon_t & 0 & \cdots & 0 \end{bmatrix}' \\
\mathbf{u}_t &= \begin{bmatrix} 0 & u_t & 0 \end{bmatrix}' \\
\mathbf{A} &= \begin{bmatrix} A_1 & \cdots & A_{j-1} & A_j & A_0 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \\
\mathbf{B} &= \begin{bmatrix} B & C & K \\ B & C & K \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}

If the representative agent uses this VAR-GARCH model to make her forecasts, her expectation about \( y_{t+i} \) and \( H_{t+i} \) will be

\begin{align*}
E_t (\mathbf{y}_{t+i}) &= \mathbf{A}' \mathbf{y}_t \quad (21) \\
E_t (H_{t+i}) &= \text{vech}^{-1} \{ \mathbf{B}' \mathbf{h}_t \} \quad (22)
\end{align*}

where the operator \( \text{vech}^{-1} \) converts a vector into the lower triangle of a symmetric matrix.

\textsuperscript{14}In the original vech model, the correlation matrix can be a function of more than one lag (and of other exogeneous variables). However, for simplicity and as it will be formally expressed in assumption 5, I restrict the model to one lag (with no exogeneous variables).
Finally, before using these expectations in the fundamental equation (14), it is useful to define \( H_{t+i} = e_{t+i}e'_{t+i} \). We have that

\[
E_t(H_{t+i}) = E_t(\mathbf{e}_{t+i}\mathbf{e}'_{t+i}) = \\
\begin{bmatrix}
E_t(H_{t+i}) & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix}
\]

\[ = \mathbf{q} \text{vech}^{-1} \{ \mathbf{B}' \mathbf{h}_t \} \mathbf{q}' \]  

(23)

where \( \mathbf{q}' = \begin{bmatrix} \mathbf{I}_{2+p} & 0_{(2+p) \times (2+p)(j-1)} \end{bmatrix} \).

2.5 Fundamental PD ratio

Once the expectations have been defined, it is possible to derive the fundamental log PD ratio as a function of the estimated VAR-GARCH and of the observable variables. For that, let us rewrite equation (14) with the new notation:

\[
pd_t = \sum_{i=1}^{\infty} \rho^{i-1} \mathbf{g}' E_t(\mathbf{y}_{t+i}) + \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho^{i+j-2} \mathbf{g}' E_t(\mathbf{A}_{t+i,t+j}) \mathbf{g} + c  
\]

(24)

where \( \mathbf{g}' = \begin{bmatrix} \mathbf{g} & 0_{(2+p)(j-1)+p} \end{bmatrix} \) and \( \mathbf{A}_{t+i,t+j} = \mathbf{e}_{t,i+t}\mathbf{e}'_{t,t+j} \) with \( \mathbf{e}_{t,t+i} = \begin{bmatrix} \varepsilon_{t,t+i} & 0 & \ldots \end{bmatrix} \).

The vector \( \mathbf{g}' \) is a row vector which selects \( x_t \) in \( \mathbf{y}_t \) and multiplies it by \( g \).\(^{15}\)

Let us first consider the second part of the right hand side of equation (24). Using the fact that \( \mathbf{y}_{t+1} = \mathbf{A}_t \mathbf{y}_t + \mathbf{e}_{t+1} \), we can express the error made at time \( t \) about the

\(^{15}\)Note that, given assumption 2, expected third and higher moments are constant. Therefore, the remainder \( R_t \) of the Taylor expansion, and thus \( c_t \), are also constant.
vector $y_{t+i}$ as a function of one-period shocks

$$e_{t,t+i} = \sum_{k=1}^{i} A^{i-k} e_{t+k}$$ \hfill (25)

Plugging that into $\Lambda_{t+i,t+j}$ yields

$$\Lambda_{t+i,t+j} = \sum_{k=1}^{i} \sum_{l=1}^{j} A_{t}^{i-k} E_t (e_{t+k} e_{t+l}) (A_{i}^l)^{j-l}$$ \hfill (26)

Using this expression and the fact that the error terms are not autocorrelated,\footnote{Cf. equation (16).} we can rewrite equation (24) as

$$p_d = \sum_{i=1}^{\infty} \rho^{i-1} g' E_t (y_{t+i}) + \frac{1}{2} \sum_{i=1}^{\infty} \rho^{i-1} g' (I - \rho A)^{-1} E_t (H_{t+i}) (I - \rho A')^{-1} g + c$$ \hfill (27)

We can use the expectation derived in equations (21) and (23) into our fundamental equation to get

$$p_d = \sum_{i=1}^{\infty} \rho^{i-1} g' A^i y_t + \frac{1}{2} \sum_{i=1}^{\infty} \rho^{i-1} g' (I - \rho A)^{-1} q \text{vech}^{-1} \{ B' h_t \} q' (I - \rho A')^{-1} g + c$$ \hfill (28)

Finally, using the fact that the $\text{vech}^{-1}$ operator is has the following properties

$$\text{avech}^{-1} (x) = \text{vech}^{-1} (ax)$$ \hfill (29)

$$\text{vech}^{-1} (x) + \text{vech}^{-1} (y) = \text{vech}^{-1} (x + y)$$ \hfill (30)
we get the last fundamental equation

\[ pd_t = g' (I - \rho A)^{-1} A y_t + \frac{1}{2} g' (I - \rho A)^{-1} \text{vech}^{-1} \{ (I - \rho B)^{-1} B h_t \} q' (I - \rho A')^{-1} g + c \]

Equation (31) expresses the fundamental log PD ratio as a function of 1) the observable variables in \( y_t \), 2) the estimated covariance matrix in \( h_t \), 3) the estimated VAR-GARCH dynamic in \( A \) and \( B \), 4) the coefficient of relative risk aversion \( \gamma \) in \( g \) and 5) the linearization parameter \( \rho \). The next sections present the empirical estimation of this fundamental equation.

### 3 Data

The next assumption concerns the set of variables \( y_t \) that is used to forecast future fundamentals (i.e. future consumption and dividends).

**Assumption 3** The representative agent uses past and present fundamentals (i.e. consumption and dividends) to forecasts future fundamentals.

Following assumption 3, the data set contains stock prices, dividends, consumption and consumer price data for four countries: United States, United Kingdom, Japan and Switzerland. All series are monthly. Stock prices are measured by the S&P 500 index for United States, the FTSE All shares index for Untied Kingdom and the MSCI indexes for Japan and Switzerland. Monthly data are obtained by taking the monthly average of daily prices. Dividends are computed with dividend yield data given with each stock price indexes. Consumption is measured by personal consumption expenditures in the United States and by the households’ consumption expenditure for the other countries. Consumption data for the United Kingdom, Japan and Switzerland are quarterly. They have been converted to monthly data by using Eviews cubic spline
conversion method. Whenever necessary, real values are obtained by dividing nominal data by the Consumer Price Index of each country. The samples go from January 1965 through to January 2007 for the United States and the United Kingdom, from January 1970 through to January 2007 for Japan and from March 1970 through to January 2007 for Switzerland. All data stem from Datastream, except consumption data for Japan and Switzerland which stem from the IMF database and the Swiss National Bank database respectively. Figure 1 displays the log nominal stock price indexes and the log PD ratios. Figure 2 shows the log nominal consumption and the log nominal dividends. Finally, Figure 3 gives the log consumer price levels.

4 Long-term evolution of the SDF

The fundamental model states that the SDF is a linear function of real consumption growth rates. Before moving on to the empirical estimation of the fundamental price, I investigate in more details the long term evolution of the SDF and its consequences on PD ratios.

4.1 Real vs. nominal long term consumption growth rates

A first glance at Figures 2 and 3 suggests that nominal consumption growth rate and inflation have decreased between 1965 and 2007 in all countries. Figure 4 gives the 10-year moving average of nominal and real consumption growth rates. In each country, we observe an important and continuous decrease of nominal consumption growth rates after 1980. Real growth rates in contrary seem to remain stable. Structural break tests confirm this observation. Table 1 gives the results of the sup$F$ structural break test developed by Bai and Perron (1998 and 2003a) for nominal and real consumption growth rates in the different countries. This test assesses the probability of structural

---

17This method assigns each value in the low frequency series to the last high frequency observation associated with the low frequency period, then places all intermediate points on a natural cubic spline connecting all the points.
Price indexes: S&P 500 for the United States, FTSE All shares for the United Kingdom, MSCI for Japan and Switzerland. Dividends are computed with the dividend yield associated with each of these indexes. Prices are nominal. Both variables are expressed in logarithm. Source: Datastream.
Consumption: personal consumption expenditures for the United States and household’s final consumption for the United Kingdom, Japan and Switzerland. Dividends are computed with the dividend yield associated with each stock price index. Both variables are nominal and expressed in logarithm. Source: Datastream, IMF and Swiss national bank.
Figure 3: Consumer Price Indexes

Consumer price indexes are expressed in logarithm. Source: Datastream.

breaks in the average growth rate and estimate their most probable date. The Bai and Perron’s sup$F$ statistics clearly detects some structural breaks in nominal consumption growth rates for each country. After each break, the average nominal rates decrease. By contrast, no break is detected in the real consumption growth rate. This difference seems to be the result of the structural breaks observed in inflation for all countries (cf. Table 1). All these results point toward a decrease of long term nominal consumption growth rates during the observed period and a stability of real consumption growth rates.

\[18\] The only exception is Japan for which one break is detected in 1979.08. The complete results of Bai-Perron tests (in particular the estimated jumps for average consumption growth rates) are available on request.
4.2 Long term evolution of PD ratios

Given Definition 1, the representative agent should maximize the utility of its present and future real consumption. In the set-up of this paper, this implies that the marginal rate of intertemporal substitution for consumption (i.e. the SDF), and thus the PD ratio, are a linear function of the real consumption growth rate. As shown in section 4.1, the real consumption growth rate is stable in the long term, which implies that the fundamental PD ratio should also be stable in the long term. However, Bai and Perron’s tests clearly detect at least one structural break in the average PD ratio (cf. Table 1).\textsuperscript{19} This is theoretically not compatible with a stable real consumption growth

\textsuperscript{19}The complete results of the structural break tests for PD ratios are not presented here. They are available on request. They show that a structural break in nominal consumption growth is followed by several breaks in the PD ratio average. The delay between the break in nominal consumption growth and those in the PD ratio average is rather long (e.g. up to 10 years after the 1985.09 break
Table 1: Structural Breaks

<table>
<thead>
<tr>
<th>Country</th>
<th>1 break vs. 0 break</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>United States</td>
<td>PD ratio 375.0639**</td>
<td>1995.10</td>
</tr>
<tr>
<td></td>
<td>Nom. cons. growth rate 30.6263**</td>
<td>1985.10</td>
</tr>
<tr>
<td></td>
<td>Inflation 108.2467**</td>
<td>1982.08</td>
</tr>
<tr>
<td></td>
<td>Real cons. growth rate 4.2978</td>
<td></td>
</tr>
<tr>
<td>United Kingdom</td>
<td>PD ratio 174.3398**</td>
<td>1993.09</td>
</tr>
<tr>
<td></td>
<td>Nom. cons. growth rate 112.8094**</td>
<td>1990.07</td>
</tr>
<tr>
<td></td>
<td>Inflation 82.5351**</td>
<td>1982.06</td>
</tr>
<tr>
<td></td>
<td>Real cons. growth rate 5.0535</td>
<td></td>
</tr>
<tr>
<td>Japan</td>
<td>PD ratio 341.9726**</td>
<td>1983.08</td>
</tr>
<tr>
<td></td>
<td>Nom. cons. growth rate 210.4146**</td>
<td>1979.12</td>
</tr>
<tr>
<td></td>
<td>Inflation 79.7999**</td>
<td>1977.05</td>
</tr>
<tr>
<td></td>
<td>Real cons. growth rate 13.2779**</td>
<td>1979.08</td>
</tr>
<tr>
<td>Switzerland</td>
<td>PD ratio 311.9437**</td>
<td>1993.01</td>
</tr>
<tr>
<td></td>
<td>Nom. cons. growth rate 74.3461**</td>
<td>1975.11</td>
</tr>
<tr>
<td></td>
<td>Inflation 58.5082**</td>
<td>1975.11</td>
</tr>
<tr>
<td></td>
<td>Real cons. growth rate 7.8739</td>
<td></td>
</tr>
</tbody>
</table>

* (**) denotes the rejection of the null hypothesis of 0 breaks for the alternative hypothesis of 1 break at a 5% (1%) confidence level. The critical value for the test are given by Bai and Perron (2003b) for a trimming parameter of 0.15. The estimated break date is given in the last column.

rate. Furthermore, in Figure 5, we observe an odd correspondence between the (inverse) nominal average consumption growth rate and the average PD ratio. It clearly shows that the 10-year moving average of the PD ratio has increased in each country during the observation period, soon after a decrease in the nominal consumption growth rate. This is in line with the predictions of a model in which the representative agent maximizes nominal instead of real consumption. Indeed, in that case, a permanent decrease in investors’ nominal consumption growth rates will induce a permanent increase in their marginal rate of intertemporal substitution for consumption. This makes future dividends more valuable to them and yields an appreciation of asset prices. The next section suggests one theoretical explanation which could justify why

for the United States). However, a Bai-Perron test with a rolling window shows that the hypothesis of no break in consumption growth would be rejected for the first time in 1992.02 at a 10% confidence level and in 1995.01 at a 1% confidence level. These results suggest that it takes time for the agents to notice a structural break and that they might take it gradually into account depending on the confidence they have in their estimations. This is in line with the succession of structural breaks observed in the PD ratio and the delay between the breaks in consumption growth and those in the PD ratio. Note that Lettau and van Nieuwerburg (2005) find results similar to mine for breaks in PD ratio mean for the United States.
Figure 5: Link Between PD Ratios and (minus) Consumption Growth Rates

The black line is the 10–year moving average of the PD ratio expressed in logarithm (left axis). The dashed line is (minus) the 10–year moving average of the consumption growth rate, expressed in annualized terms (right axis).

agents use nominal instead of real consumption.

4.3 The money illusion hypothesis

Money illusion is a possible explanation for the negative link observed between nominal consumption growth and PD ratios (or, equivalently, for the discrepancy between a stable real consumption growth rate and increasing PD ratios). Fisher (1928) gives the following definition for money illusion:"20

"[Money illusion] is the failure to perceive that the dollar, or any other unit of money, expands or shrinks in value"

20Citation taken from Brunnermeier and Julliard (2007).
If an investor suffers from money illusion, then she does not (or at least falsely) take inflation into account. This can induce her to measure consumption in nominal rather than in real terms. Since the preliminary analysis of the data suggests that asset prices might be linked to nominal rather than to real consumption growth rates rather (cf. section 4.2), I will construct an alternative fundamental price, based on nominal fundamentals, and compare it to the results obtained with real fundamental. The difference between them can be (at least partially) attributed to money illusion.

As seen in section 4.1, nominal consumption growth rates are characterized by structural breaks and a general decline over time. A rational agent (even if she suffers from money illusion) should realize (immediately or after some delay) that the long term consumption growth rate is changing and thus adapt her asset valuation to the new growth rate. To take this phenomenon into account, I use the following hypothesis

**Assumption 4** *The representative agent bases her forecasts about future consumption growth rates on the average nominal consumption growth rate observed in the last five years.*

By taking the average nominal growth rate over the last five years instead of the average over the whole period, the agent gets an estimate of the long term growth rate which is currently driving consumption. Assumption 4 does not mean that the agent abandons the use the VAR-MGARCH model, but rather that she uses it to forecasts deviations from the long term average. Note that I do not use Assumption 4 to compute the fundamental price without money illusion since the real consumption growth rate is stable.

## 5 Econometric methodology

To be able to compute the fundamental price given by equation (31), we must know $A$, $B$, $\rho$ and $c$. The next two sections explain how to estimate these coefficients. Note that
each estimation is made with the sample observable at time $t$, such that the sample used reflects exactly the information set of the representative agent at that time.

### 5.1 Estimation of the VAR-GARCH model

The first step of the computation of the fundamental stock price is to estimate the matrices $A$ and $B$ of the VAR-GARCH model. This requires the simultaneous estimation of equations (15) and (17). For the estimation of the GARCH dynamic to remain computationally feasible, it is necessary to restrict the number of coefficients in $A$ and $B$. For that, I make the following assumption:

**Assumption 5** *Each element of the conditional covariance matrix depends only on its own last value and last residuals.*

This assumption means that the GARCH dynamic is a BEKK model (Engle and Kroner 1995) with one lag and no exogenous variable. The BEKK model can be estimated by the traditional maximization of the log likelihood function as explained in Hamilton (1994, p. 670).

I estimated a VAR-GARCH model for each period using only the data available at that time. With this procedure, I placed myself exactly in the same situation as investor living at time $t$. This yields a truly "out-of-sample" estimation of the fundamental price, i.e. based only on ex-ante data. In each period I used 6 lags in the VAR part of the model. In addition to the matrices $A$ and $B$, the estimation of the VAR-GARCH also gives an estimation of the conditional variance-covariance vector $h_t$.

### 5.2 Estimation of the relative risk aversion coefficient

After the estimation of the matrices $A$ and $B$, the only unknown remaining in the fundamental equation (31) is the relative risk aversion coefficient $\gamma$ appearing in $g$. 

26
Note that, without using the matrix notation, equation (31) is equivalent to

\[ pd_t = -\gamma q_{1,t} + q_{2,t} + \frac{1}{2} \gamma^2 q_{3,t} - \gamma q_{4,t} + \frac{1}{2} q_{5,t} + c \]  

(32)

where \( q_{1,t} \) and \( q_{2,t} \) are the first and second elements of \((I - \rho A)^{-1} Ay_t\) for \( A \) and \( \rho \) estimated with the sample available at time \( t \), and \( q_{3,t} \), \( q_{4,t} \) and \( q_{5,t} \) are the \((1,1)\)-th, \((1,2)\)-th and \((2,2)\)-th element of

\[(I - \rho A)^{-1} \text{vech}^{-1} \left\{ (I - \rho B)^{-1} Bh_t \right\} q' (I - \rho A')^{-1} \]  

(33)

for \( A, B, \rho \) and \( h_t \) estimated with the sample available at time \( t \). The parameters \( \gamma \) and \( c \) can then be estimated, for each period, by estimating equation (32) with OLS and a sample containing only the variables \( q_{t-k} \) (for all \( k \) between 0 and \( t-1 \)) available at this time.

6 Estimated fundamental stock prices

The fundamental log PD ratio can be computed with equation (31) for parameters estimated as explained in section 5. As mentioned in section 4.3, I use real data for the model without money illusion and nominal data for the models with money illusion. The fundamental price can then be recovered from the fundamental log PD ratio by adding \( d_t \) (in real terms for the model without money illusion and in nominal terms for the model with money illusion). Figure 6 shows the estimated fundamental stock prices (with and without money illusion) and the observed prices (left panels) and the estimated fundamental PD ratios (with and without money illusion) and observed PD ratios (right panels). The gap between the observed and the fundamental prices is presented in Figure 7.

In each country, we observe that the price can diverge significantly and for long
The black line is the observed stock price. The grey line is the estimated fundamental stock price. The dashed line is the estimated fundamental stock price with money illusion. All variables are expressed in logarithm.
Figure 7: Gap Between the Observed and the Fundamental Prices

Each panel presents the difference between the (log) observed nominal stock price and the estimated (log) fundamental stock price without money illusion (grey line) and with money illusion (dashed line).

periods from its fundamental value, with both models (with and without money illusion). The American stock market is characterized by a huge gap at the end of the nineties, which is associated with the Internet bubble. According to the money illusion model, this bubble has disappeared in the last years, whereas the model without money illusion still estimates that the stock are overvalued today. Before the bubble, and according to both models, the stock prices were most of the time undervalued, with some short episodes of overvaluation (e.g. in 1974 or in 1987). The stock market in the United Kingdom follows a similar pattern, with a smaller overvaluation at the end of the nineties (and a bigger undervaluation after the crash in 1974). The Japanese

\[^{21}\text{Zhong et al. (2003) and Black, Fraser and Groenewold (2003) get a similar period of undervaluation between the second half of the 70s and the first half of the 90s. The former uses ex-post data for sample going from 1871 through to 1997 (as in Shiller 1981); the latter use a general equilibrium framework with samples going from 1947 through to 2002 for quarterly data and from 1929 through to 2001 for annual data.}\]
market has known a long period of relatively important overvaluation at the end of the eighties and in the very beginning of the nineties. After that, the models with and without money illusion differ significantly. The model with money illusion indicates that stocks were undervalued whereas the model indicates that stock were overvalued until 2004. Finally, the Swiss market displays a pattern similar to the United Kingdom with the exception that the main overvaluation period begins earlier in the nineties and decrease mostly in 1998. Note that the Swiss market has suffered more than other western countries of the 1998 crisis (Russian/ LTCM crisis). July 1998 has remained the historical maximum until the beginning of 2006 in Switzerland, whereas stock prices recovered much more rapidly in other western countries.22

We can note that, for the United States, the United Kingdom and Switzerland, the largest difference between the model with and without money illusion are observed during the nineties bubble and in its aftermath. In each countries, the model without money illusion indicates a larger bubble, which has not totally disappeared yet. This is specially obvious in the United States. In these three markets, the model with money illusion seems closer to the observed price than the model without money illusion (this question is studied in more details in section 8.1). The case of Japan is different from the three other countries, both models gives very different results. The model without money illusion indicates an overvaluation in the last 20 years whereas the model with money illusion display a period of undervaluation in the last 10 years. At the first glance, no model seems to describe the observed data better than the other.

Finally, note that the volatility of the observed price is significantly greater than the volatility of the fundamental price without money illusion in all countries. This is in-line with the stock price volatility puzzle (Shiller 1981, LeRoy and Porter 1981). The volatility of the stock price is also significantly greater than the fundamental stock

22More precisely, for the Swiss market, the record high of the 21 July 1998 (1931.49) has been beaten for one day (23 August 2000) before plunging until the beginning of 2003 with other western markets. On the other hand, the stock price index returned at its level of July 1998 only five months later in the United States and ten months later in the United Kingdom.
price with money illusion in the United States and in Switzerland. The volatility of the stock price in the United Kingdom is greater than the one of its fundamental value with money illusion but the difference is not significant. Finally, in Japan, the volatility of the stock price is smaller than the one of the fundamental price with money illusion.

7 Do stock prices go back to their fundamental value?

As it is shown in the previous section, prices can significantly diverge from their fundamental value for long periods. This conclusion naturally raises the question of the existence (or absence) of a link between observed and fundamental prices. I study this question from two points of view. Firstly, I examine the dynamic of the gaps and check if they tend to disappear after some time. Secondly, I check if the fundamental price can help to forecast future prices out-of-sample. If the fundamental price is able to give good out-of-sample forecasts, then it is a sign that there is a link between them. The first approach is in-sample and the second is out-of-sample.

7.1 Gap dynamic analysis

Firstly, I test for the presence of a unit root in the gap between the observed and the fundamental prices. If the unit root is rejected, then the gap is mean reverting, which means that it tends to disappear after some time and that the observed price eventually goes back to its fundamental value.

Table 2 presents the results of the Augmented Dickey-Fuller test and the Phillips-Perron test for unit root. For the United States, a unit root cannot be rejected at a 5% confidence level for the model without money illusion. The results are less clear for the

---

23 This approach is equivalent to testing for cointegration between the fundamental and the observed prices. In our case, the cointegrating vector is know a priori and equal to $[1 \ -1]$. 

31
model with money illusion. In the United Kingdom, the unit root is rejected for both models. In Japan, the unit root cannot be rejected for the model with money illusion. Results are less clear for the model without money illusion. Finally, in Switzerland, the unit root is strongly rejected for the model with money illusion. These results suggest that, firstly, the gap for the model without money illusion and for the model with money illusion are not mean reverting in the US and Japan respectively. This is a sign that the fundamental price give by these models are probably not linked with the observed price. Secondly, the gap for both models in the UK and for the money illusion model in Switzerland are mean reverting, indicating a link between these fundamental prices and the observed prices.

In complement to the unit root tests, I estimate the impulse-response function of the different gaps. The results are presented in Figure 8. Each impulse-response function indicates how the gap evolves after a shock. All functions show that the gap generated by a random shock tends to disappear after some time. However, the gap vanishes more quickly for the model with money illusion in the United States, in the United Kingdom and in Switzerland. In Japan, we observe the opposite situation. Note that the gap computed without money illusion for the United States decays only very slowly. The half-life\(^24\) of the gap is one year and two months in the United Kingdom and one year and 4 months in Switzerland against about 3.5 years in Japan\(^25\) and more that 4 years

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\(^{24}\)The half-life corresponds to the number of months needed for the gap to wipe out half of the initial shock.

\(^{25}\)The half-life for Japan without money illusion is of 2.5 years.

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<table>
<thead>
<tr>
<th></th>
<th>Without money illusion</th>
<th>With money illusion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ADF test</td>
<td>PP test</td>
</tr>
<tr>
<td>United States</td>
<td>-0.8149</td>
<td>-0.8814</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>-2.9596**</td>
<td>-2.2296*</td>
</tr>
<tr>
<td>Japan</td>
<td>-1.3898</td>
<td>-1.9448*</td>
</tr>
<tr>
<td>Switzerland</td>
<td>-1.9100</td>
<td>-2.0673*</td>
</tr>
</tbody>
</table>

\* (**) denotes the rejection of the null hypothesis of unit root at a 5% (1%) confidence level. The test is performed with an equation without trend or constant. ADF means Augmented Dickey-Fuller and PP means Phillips-Perron.
in the United States for the model with money illusion. This is a sign that there is a link between the observed and the fundamental prices, but that this link is weak and that it takes time for the gap to die out after a shock. This suggests that the C-CAPM fundamental price does play a role in the long term evolution of the stock price, but that it performs rather poorly in explaining the short term dynamic of the stock price.

### 7.2 Forecast accuracy analysis

This section analyses the ability of the information contained in the fundamental price to forecast future stock prices. Apart from its obvious practical applications, the forecast ability of the fundamental price is another way to study the link between the price and its fundamental value. If the fundamental price is able to give good out-of-sample forecasts, then it is a sign that there is a link between them. I compare four forecasting models:

1. **Random walk with drift (benchmark model)**: for this model, I first estimate the growth rate of the observed price on the observable sample at time $t$. Then, I use the estimated growth rate to forecast the future price for different horizon starting at the price observed at time $t$.

2. **Fundamental random walk with drift**: similarly to the benchmark model, I first estimate the growth rate of the fundamental price on the observable sample at time $t$. Then, I use the estimated growth rate to forecast the future fundamental price for different horizon starting at the fundamental price at time $t$. I set the forecast for the future price equal to the forecast for the fundamental price.

3. **Error correction model**: in this model, I check if the gap observed at time $t$ helps to predict the forecasting error at time $t + h$. For that, I estimate the following equation:

$$p_t - E_{t-h} (p_t) = \beta_0 + \beta_1 g_{t-h}$$  \hfill (34)
Figure 8: Impulse-Response Function for the Gaps

Impulse-response functions for the gaps between the fundamental and the observed price (expressed in months, horizontal axis). Upper panel: without money illusion. Lower panel: with money illusion.
on the sample observable at time $t$, where $E_{t-h}(p_t)$ are the forecasting errors made by the random walk with a drift model.\textsuperscript{26} I then correct the random walk with a drift forecasts with the estimated error.

4. **Fundamental price and gap dynamic**: this model combined the fundamental random walk dynamic with drift described above with the dynamic of the gap. The latter is estimated with the following equation:

\[
\Delta g_t = \beta_0 + \beta_1 g_{t-1} + \beta_2 \Delta g_{t-1} + \epsilon_t. \tag{35}
\]

where $g_t$ is the gap observed at time $t$. I first forecast the dynamic of the gap with the previous equation and then add the forecasted gap to the forecasted fundamental price obtained with the fundamental random walk with drift.

The three last models have been tested with the fundamental price given by the models with and without money illusion. Note that, in addition to these models, I have also estimated models in which the observed price or the fundamental price is a random walk or a AR(1) process. However, their forecasts are constantly outperformed by those of the random walk with drift models.\textsuperscript{27} Their results are therefore not presented here. Finally, remember that all forecasts are made out-of-sample. The forecast accuracy is measured by the mean absolute error.\textsuperscript{28}

Figure 9 presents relative forecast accuracy of the different fundamental forecasting models with the benchmark for different forecast horizons (horizontal axis). A value below one indicates that the model is more accurate than the benchmark.

\textsuperscript{26}To estimate equation (34), I first make out-of-sample forecasts for a subsample $[t-k+1,t]$ of the entire sample $[0,t]$ available at time $t$ by using the dynamic observed on $[0,t-k]$. I then use the forecast errors in equation (34).

\textsuperscript{27}The only exception is Japan for which the random walk and the AR(1) forecasts were slightly better than the random walk with a drift for both the observed and the fundamental price. However, using them as benchmark does not change the results presented in the next sections.

\textsuperscript{28}Using the root mean square error to measure the forecast accuracy does not change the results.
Each panel compares the forecast accuracy of different models with the forecast accuracy of the random walk with drift model (benchmark). The black line corresponds to the fundamental random walk with drift model, the dotted line to the error correction model and the dashed line to the fundamental price and gap dynamic model. Each panel gives the ratio of the model’s average absolute errors over the benchmark’s average absolute error. A value under 1 indicates that the model is more accurate than the benchmark. The comparison is made for different forecasting horizons (in months, horizontal axis).
The results are different for each country. In the United States, the fundamental price *with money illusion* significantly improves forecast accuracy, whereas the model without money illusion does not. With money illusion, the fundamental random walk with a drift forecasts outperforms the benchmark for forecast horizons longer than 2 years. For an horizon of 6 years, the improvement is of 40%. For the United Kingdom, both models do improve forecasts accuracy. The model without money illusion gives the better forecasts for horizons longer than 3 years with an improvements of about 40% in comparison with the benchmark. The model with money illusion give better forecasts for horizons shorter than 3 years. Note that in both countries, the error correction model constantly outperforms the benchmark. The main findings for the United States and the United Kingdom is that, by using the information contained in the fundamental price, it is possible to improve the forecasts of the benchmark *for all forecast horizons*. The performance of the fundamental price with money illusion is particularly striking in the United Kingdom: the benchmark is outperformed by at least 20% for all horizons longer than 3 months!

For Japan, the model *without money illusion* also improves forecasts for horizons longer than 3 years by using the fundamental price and gap dynamic approach. Its performance is however, less impressive than for the United States and the United Kingdom with a maximum improvement of about 20% for an 6-year forecast horizon. For Switzerland, the forecasting accuracy of the fundamental model can be considered as worse than the one of the benchmark, although the fundamental random walk with a drift gives slightly more accurate forecasts than the benchmark for horizons longer than 5 years.

These results tend to attest the presence of a relatively strong link between the fundamental and the observed stock prices in the United States (for the model with money illusion), in the United Kingdom (for both models) and in Japan (for the model without money illusion). This link is inexistent or at most weak in Switzerland.
8 Is it really worth computing the CAPM fundamental price?

The goal of this section is to check if the efforts made in constructing the C-CAPM fundamental price are really worth or if computationally less intensive fundamental models give similar results. For this, I first decompose the fundamental price to see which component is has a significant importance on the fundamental price. Secondly, I compare the forecasts of simpler models to see if they outperform the C-CAPM in terms of forecast accuracy. Finally, I analyze the fundamental forecasts in different point in time to see if the C-CAPM fundamental price is more relevant in certain periods than in others.

8.1 Fundamental price decomposition

In this section, I decompose the fundamental price into the main innovations that have been added by this paper to the original method of Campbell and Shiller (1988a). The goal is to see which components really change the results. For that, I first estimate a fundamental price without money illusion, without second moments and over the all sample, as in Campbell and Shiller (1988a). Secondly, I estimate the same model but by using, at each point in time, only the information that was available at that time (out-of-sample estimation). Thirdly, I add the estimated conditional second moments and finally, I introduce money illusion. Results are presented in Figure 10. Note that the difference between the fundamental model with and without second moments is not visually perceptible. The fundamental price with second moments and without money illusion is therefore not represented in Figure 10.\textsuperscript{29,30}

The first observation is that, in each country, the in-sample estimated fundamental

\textsuperscript{29}The fundamental PD ratio with second moments and without money illusion si represented in Figure 6 (right panel).

\textsuperscript{30}The only exception is Japan, for which the first order moments and second order moments models are slightly different in the beginning of the sample.
PD ratio without money illusion does not differ much from the average PD ratio (i.e. it is almost a flat line). Introducing out-of-sample estimation is a significant first step in the direction of the observed dynamic, especially in the first half of the sample. Money illusion also introduces significant modifications to the PD ratio fundamental dynamic, in particular in the second half of the sample. Table 3 studies in more details how close the different fundamental PD ratios are from the observed ones. It gives the percentage of the total variance that is explained by the different components (i.e. the coefficient of determination).\(^{31}\)

Table 3 shows that neither the out-of-sample approach nor the second moments

\(^{31}\)The coefficient of determination is computed by 

\[
R^2 = 1 - \frac{\sum (pd_t - pd_{t,\text{avg}})^2}{\sum (pd_t - \overline{pd_t})^2}
\]

where \(pd_t\) is the observed log PD ratio, \(\overline{pd_t}\) its average and \(pd_{t,\text{avg}}\) the estimated fundamental PD ratio. A \(R^2\) smaller than one indicates that the sum of square deviations between observed and fundamental PD ratios is greater than the observed variance.
do significantly change the fit between the fundamental model and the observed data. The only exception is Japan, for which the second moment model have a slightly better fit than the first moment model. Only the introduction of money illusion significantly improve the fit of all models. Note however that having a better fit does not necessarily imply better forecasts. For example, the model with money illusion for Japan has a better fit that the one for United Kingdom. The former performs however very poorly compared to the latter in terms of forecasts.

8.2 Forecasts with alternative fundamental models

This section compares the forecast accuracy of the C-CAPM fundamental price developed in this paper with simpler had-hoc fundamental models. I compare the C-CAPM model with three simpler models:

1. **Trend model**: in this model, the fundamental price is determined by fitting a linear trend with the observed prices.

2. **Hodrick-Prescott filter model**: in this model the fundamental price is determined by estimating a Hodrick-Prescott filter with the observed price.32

3. **Moving average PD ratio**: in this model, the fundamental price is determined by the 10-year moving average of the PD ratio.

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32The smoothing parameter is set to 230400, which corresponds to the hypothesis of an average gap of 20% between the fundamental and the observed price and an average annual change of 4% in the stock price long term trend.
At every period, each fundamental price is estimated with the observed data only. I chose to compare the forecasts of the fundamental random walk with drift version of each model (with money illusion for the US and Switzerland and without money illusion for the UK and Japan, cf. section 7.2). The alternative fundamental models correspond to measures that are commonly used by practitioners to estimate imbalances on stock markets. For example, Borio and Lowe (2002) use the Hodrick-Prescott filter to identify bubble in different stock market. A constant PD ratio corresponds to the Gordon (1962) model. The moving average PD ratio can thus be considered as a Gordon model based on the only on the recent observed dynamic.
Figure 11 shows the forecast accuracy of the alternative fundamental models. None of the alternative model can simultaneously outperform the benchmark or the C-CAPM fundamental model. This means that when the C-CAPM is more accurate than the benchmark, then it is also more accurate than simpler fundamental models. Thus the C-CAPM fundamental model developed in this paper adds some value to simpler ad-hoc fundamental models.

### 8.3 When are the fundamental forecasts the most accurate?

The previous sections examine the forecasts accuracy of C-CAPM for the whole sample. However, it might happen that this model performs better in given periods than in others. This section addresses this question and studies more particularly if there is a link between the wideness of the gap between the fundamental and the observed prices and the precision of the forecasts.

Table 4 gives the relation between the wideness of the gap and the forecast error. It presents the Spearman correlation\(^{33}\) between the absolute value of the gap and the forecast accuracy improvement made by using the fundamental model.\(^{34}\) A positive correlation implies that a (positive or negative) wider gap increases the accuracy of fundamental forecasts in comparison to forecast based on the observed price.

The correlation analysis shows that, in the United Stated, in the United Kingdom and in Japan for horizon longer than 2 years, one year and 3 years respectively, a wider gap implies more accurate forecast for the fundamental model (with money illusion for the US, with and without money illusion for the UK and without money illusion for Japan). Since more accurate forecasts are a sign of a stronger link between the fundamental and the observed price, the positive correlation suggests that the further

\(^{33}\)Since the form of the function linking the (absolute) gap with the forecast error is a priori not known, I prefer to use the Spearman rank-order correlation coefficient rather than the traditional (linear) correlation (Pearson coefficient). Spearman correlation coefficient is independent of the form taken by the function. Spearman rank-order correlation coefficient measures the linear correlation between the ranks of each observation.

\(^{34}\)This variable takes a positive value when the fundamental price forecasts are better than the observed price forecasts and a negative value when they are worse than them.
Table 4: Correlation Between Gaps and Relative Forecast Errors

<table>
<thead>
<tr>
<th>Forecast horizon</th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
<th>4 years</th>
<th>5 years</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Without money illusion</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>United States</td>
<td>-0.7352*</td>
<td>-0.5862*</td>
<td>-0.4318*</td>
<td>-0.2822*</td>
<td>-0.1667*</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>0.1486*</td>
<td>0.1830*</td>
<td>0.4653*</td>
<td>0.5793*</td>
<td>0.6162*</td>
</tr>
<tr>
<td>Japan</td>
<td>0.6401*</td>
<td>0.2375*</td>
<td>0.0222</td>
<td>0.3088*</td>
<td>0.4460*</td>
</tr>
<tr>
<td>Switzerland</td>
<td>-0.5769*</td>
<td>-0.4409*</td>
<td>-0.3417*</td>
<td>-0.2597*</td>
<td>-0.1221</td>
</tr>
<tr>
<td><strong>With money illusion</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>United States</td>
<td>-0.3114*</td>
<td>0.0080</td>
<td>0.1655*</td>
<td>0.2666*</td>
<td>0.4623*</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>0.0373</td>
<td>0.3810*</td>
<td>0.5692*</td>
<td>0.5725*</td>
<td>0.5939*</td>
</tr>
<tr>
<td>Japan</td>
<td>-0.6479*</td>
<td>-0.5247*</td>
<td>-0.2999*</td>
<td>-0.2029*</td>
<td>-0.1706*</td>
</tr>
<tr>
<td>Switzerland</td>
<td>-0.5221*</td>
<td>-0.3641*</td>
<td>-0.3565*</td>
<td>-0.3137*</td>
<td>-0.1870*</td>
</tr>
</tbody>
</table>

* denotes the rejection of the null hypothesis of a coefficient equal to 0 at a 1% confidence level. The correlation is measured with the Spearman coefficient. A positive value means that a (positive or negative) wider gap implies an increase of the accuracy of the fundamental forecasts in comparison to the forecasts based on the observed price.

Apart the two prices are, the more they tends to return towards each other in the long term. In other terms, the "attraction force" exerted by the fundamental price on the observed price get stronger with the distance between them. We can also see that the strength of the link increases with the forecast horizon.

Building on the conclusions made for the United States, the United Kingdom and Japan, one could adopt the following strategy: if the gap is large, then one should use the fundamental price for the forecasts, otherwise one should use the observed price. Figure 12 compares the forecast accuracy of the fundamental random walk with drift for any value of the gap with those made when the gap is large. It shows that, for horizon longer than 5 years, the forecast accuracy improves of up to 15 and 20 percentage points in the United States and in the United Kingdom, respectively, if the forecast are made when the gap is large. In the United Kingdom, this strategy improves any forecasts with an horizon longer than 3 years. I find no improvement for Japan and Switzerland.
9 Conclusion

The results presented in this paper can be interpreted and exploited at different levels. The most obvious practical result is the ability of the C-CAPM fundamental model to forecast future prices for short horizon. As mentioned in the paper, the predictability of stock prices is not new. Several studies have shown that simple ratios such as dividend-price ratios or price-earnings ratios are useful to forecast stock prices in the long term. Typically, these ratios are able to forecasts price for horizons longer than five years. What is new here is that the C-CAPM fundamental model significantly shortens the horizons for which the forecast accuracy can be improved. Of course, the
improvement depends on the market and on the period in which the forecast is made, but the results presented in this paper suggests that is generally possible to find a way to combine the observed and the fundamental prices to improve forecasts for horizons shorter than five years. While long term forecasts are of little interest for traders or portfolio managers, whose performance are evaluated on shorter intervals, forecast horizons in the range of those proposed in this paper might find their place in longer term tactical portfolio allocation strategies. With that in mind, it would be useful to better study how significant and reliable the improvement made by the fundamental model is and if a portfolio based on fundamental forecasts would have generated excess returns in the past.

The C-CAPM fundamental model is also of interest for central banks and international organizations such as the IMF or the BIS. Indeed, such institutions show a growing interest in identifying imbalances on asset markets. Their fear is that such imbalances will eventually unwind and that their correction might have a significant impact on the economy. Thus, by detecting imbalances early enough, central banks hope to identify factors that could help them predicting the evolution of the economy and choosing an adequate policy. Two results indicate that the imbalances measured by the C-CAPM fundamental model would be relevant in this framework: firstly, there are strong empirical evidence of a link between the C-CAPM fundamental price and the observed stock price and secondly, the forecast horizon for which the fundamental model is helpful corresponds to the horizons, which is normally considered as pertinent for central banks policy (up to 3 or 5 years). In this context, an improvement of the model would be to relax assumption 5 and extend the set of variables used to forecast the fundamentals. As mentioned by Campbell (1999), consumption is not well forecasted by its own history. It is therefore unlikely that agents do only rely on its dynamic to form their expectations. Expanding the information set to other variables would refine the estimation of the gap between the price and its fundamental value and
give a better measure of potential imbalances.

Central bankers may also be interested by the evidence of money illusion find in this paper. In the presence of money illusion, inflation induce mispricing on stock markets. Therefore, as suggested by Campbell and Vuolteenaho (2004), a by-product of inflation-stabilizing monetary policies is to reduce the volatility of mispricing and foster stock market efficiency.

Finally, from an academic point of view, the results presented in this paper seem to rehabilitate a bit the empirical relevance of the C-CAPM with power utility. Indeed, since the seminal paper of Mehra and Prescott (1986) documenting the equity premium puzzle raised by using the C-CAPM with power utility, this model has been the target of multiple critiques. Numerous articles have contested or defended its use in the context of stock prices. In recent years, the trend seems to move toward modified utility functions such as habit formation or loss aversion. Most of these studies are based on the observation of stock returns. This paper shed a new light on the C-CAPM with power utility by looking at stock prices instead of returns. From this point of view, the empirical evidence is kinder with the power utility C-CAPM, at least for the long term evolution of stock prices. Note that, the methodology developed in this paper can be easily extended to any other linear SDF model. It would be interesting to check if utility functions that perform better than the power utility for stock returns give also better results for the long term dynamic of stock prices.
A Proofs

A.1 Second-order Taylor expansion of the log PD ratio

We start from the definition of the fundamental PD ratio in equation (5)

\[ PD_t = E_t \sum_{i=1}^{\infty} \prod_{j=1}^i M_{t+j} \gamma_{t+j} = E_t (PD_t^*) = \sum_{s \in S} \lambda_s PD_{s,t}^* \quad (36) \]

where \( S = \{ s_1, \ldots, s_N \} \) is the set of every possible states of nature \( s \). Taking the log of this equation yields

\[ \ln E_t (PD_t^*) = \ln \sum_{s \in S} \lambda_s PD_{s,t}^* = \ln \sum_{s \in S} \lambda_s e^{pd_{s,t}^*} = f (pd_{1,t}^*, \ldots, pd_{N,t}^*) \quad (37) \]

The second-order Taylor expansion of \( f (pd_{1,t}^*, \ldots, pd_{N,t}^*) \) around \( pd_t^* \) is

\[
f (pd_t^*, \ldots, pd_t^*) + \sum_{s \in S} \frac{\partial f}{\partial pd_{s,t}^*} \bigg|_{pd_{s,t}^* = pd_t^*} (pd_{s,t}^* - pd_t^*) + \]

\[ + \frac{1}{2} \sum_{s_1 \in S} \sum_{s_2 \in S} \frac{\partial^2 f}{\partial pd_{s_1,t}^* \partial pd_{s_2,t}^*} \bigg|_{pd_{s_1,t}^* = pd_t^*, \; pd_{s_2,t}^* = pd_t^*} (pd_{s_1,t}^* - pd_t^*) (pd_{s_2,t}^* - pd_t^*) + \]

\[ + R_t \quad (38) \]

where \( pd_t^* = \sum_{s} \lambda_s e^{pd_t^s} \) and \( R_t \) is the remainder of the Taylor expansion.

We have that

\[ f (pd_t^*, \ldots, pd_t^*) = \ln \sum_{s} \lambda_s e^{pd_t^s} = pd_t^* \quad (39) \]
The first order term is
\[
\sum_{s \in S} \frac{\partial f}{\partial p^{*}_{s,t}} \bigg|_{p^{*}_{s,t} = \bar{p}^{*}_{t}} (p^{*}_{s,t} - \bar{p}^{*}_{t}) = \sum_{s \in S} \lambda_s e^{p^{*}_{s,t}} \bigg|_{p^{*}_{s,t} = \bar{p}^{*}_{t}} (p^{*}_{s,t} - \bar{p}^{*}_{t}) = \sum_{s \in S} \lambda_s (p^{*}_{s,t} - \bar{p}^{*}_{t})
\]
(40)

The second derivative in the second order term is
\[
\frac{\partial^2 f}{\partial p^{*}_{s_1,t} \partial p^{*}_{s_2,t}} = -\lambda_{s_1} e^{p^{*}_{s_1,t}} \lambda_{s_2} e^{p^{*}_{s_2,t}} \left( \sum_{s} \lambda_s e^{p^{*}_{s,t}} \right)^2 \text{ if } s_1 \neq s_2
\]
(41)
\[
\frac{\partial^2 f}{\partial p^{*}_{s_1,t} \partial p^{*}_{s_2,t}} = \lambda_{s_1} e^{p^{*}_{s_1,t}} \sum_{s} \lambda_s e^{p^{*}_{s,t}} - \left( \lambda_{s_1} e^{p^{*}_{s_1,t}} \right)^2 \left( \sum_{s} \lambda_s e^{p^{*}_{s,t}} \right)^2 \text{ if } s_1 = s_2
\]
(42)

Evaluated at \(\bar{p}^{*}_{t}\), the second derivative is
\[
\frac{\partial^2 f}{\partial p^{*}_{s_1,t} \partial p^{*}_{s_2,t}} = -\lambda_{s_1} \lambda_{s_2} \text{ if } s_1 \neq s_2
\]
(43)
\[
\frac{\partial^2 f}{\partial p^{*}_{s_1,t} \partial p^{*}_{s_2,t}} = \lambda_{s_1} (1 - \lambda_{s_1}) \text{ if } s_1 = s_2
\]
(44)

Given that, the second order term is
\[
\sum_{s_1 \in S} \sum_{s_2 \in S} \frac{\partial^2 f}{\partial p^{*}_{s_1,t} \partial p^{*}_{s_2,t}} \bigg|_{p^{*}_{s,t} = \bar{p}^{*}_{t}} (p^{*}_{s_1,t} - \bar{p}^{*}_{t}) (p^{*}_{s_2,t} - \bar{p}^{*}_{t}) = \sum_{s \in S} \lambda_s (p^{*}_{s,t} - \bar{p}^{*}_{t})^2
\]
(45)

Recollecting all these results, we get that the Taylor expansion is
\[
f(p^{*}_{1,t}, \ldots, p^{*}_{N,t}) = \bar{p}^{*}_{t} + \sum_{s \in S} \lambda_s (p^{*}_{s,t} - \bar{p}^{*}_{t}) + \frac{1}{2} \sum_{s \in S} \lambda_s (p^{*}_{s,t} - \bar{p}^{*}_{t})^2 + R_t
\]
(46)
and thus
\[
\ln E_t (PD_t^*) = E_t (pd_t^*) + \frac{1}{2} V_t (pd_t^*) + R_t
\] (47)

A.2 First order approximation of the logarithm of a sum

Campbell, Lo and McKinlay (1997) show that it possible to approximate the logarithm of a sum by the sum of logarithm. First consider

\[
\ln (1 + PD^*) = \ln (1 + e^{pd^*})
\] (48)

where \(pd^* = \ln PD^*\). The second part of this equation can be approximated by a standard Taylor approximation around its mean. Define \(f(x) = \ln (1 + e^{pd^*})\) and take the Taylor approximation of it around its mean:

\[
f(pd^*) \approx f\left(\bar{pd}^*\right) + f'\left(\bar{pd}^*\right) \left(pd^* - \bar{pd}^*\right)
\] (49)

with \(f'\left(\bar{pd}^*\right) = e^{\bar{pd}^*} / \left(1 + e^{\bar{pd}^*}\right)\). Define \(\rho = 1 / \left(1 + e^{-\bar{pd}^*}\right)\) and plug it into the previous equation to get the final result

\[
\ln (1 + PD^*) \simeq \kappa + \rho pd^*
\] (50)

with \(\kappa = - \ln \rho - (1 - \rho) \ln (1/\rho - 1)\).

References


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