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Abstract

The wealth dynamics of insurance companies strongly depends on the success of their investment strategies, but also on liquidity shocks which occur during unfavorable years, when indemnities to be paid to the clients exceed collected premia. An investment strategy that does not take liquidity shocks into account, exposes insurance companies to the risk of bankruptcy. This paper analyzes the behavior of insurance companies in an evolutionary framework. We show that an insurance company that merely satisfies regulatory constraints will eventually vanish from the market. We give a more restrictive no-bankruptcy condition for the investment strategies and we characterize trading strategies that are evolutionary stable, i.e. able to drive out any mutation. We study the existence of such strategies and the conditions under which financial and insurance markets are stable.

Keywords: insurance, portfolio theory, evolutionary finance.
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1 Introduction

Institutional investors, like pension plans or insurance companies, are usually active on asset markets that do not provide complete insurance against all possible risks. The performance of those institutions strongly depends on the success of their investment strategies. On the other hand, pension plans and insurance companies are also exposed to liquidity shocks, that occur when pensions or claims to be paid out to the clients exceed collected premia. In order to guarantee that insurance companies or pension plans are able to face their obligations with high probability, regulatory authorities impose solvency constraints, which are constraints on the investment strategies, such that a safely invested reserve capital exists. Apart from the regulatory constraints, institutional investors still face the problem of choosing the proportion of wealth to be invested prudently, in order to be able to cover future losses, but, on the other hand, also to exploit growth opportunities offered by financial markets.

In this paper we take a long-run perspective and we analyze the performance of insurance companies with an evolutionary model. According to this approach, investors’ trading strategies compete for the market capital and the endogenous price process is thus a market selection mechanism along which some strategies gain market capital while others lose. Analogously, also depending on their ability of facing liquidity shocks, insurance companies also compete for insurance contracts and the endogenously determined insurance premia represent a further selection mechanism. The successful investor is the one who better adapts to the market selection mechanisms. A successful investor will then increase her market share and, consequently, her impact on asset prices and insurance premia, thus, finally, on the market selection mechanisms. Financial and insurance markets are stable when a strategy which already dominates the market selection mechanisms is able to further increase its market share against any competitor and eventually be the sole survivor. Such strategies are called evolutionary stable (Hens and Schenk-Hoppé 2005).

In this paper we derive the conditions for the existence of evolutionary stable investment strategies under the assumption that insurance companies have the same risk profiles on the insurance market. Moreover, we characterize evolutionary stable strategies when they exist. In our analysis we do not specify investors’ objective functions, but we only look at strategies as the result of some (unknown) decision process. Indeed, any investment strategy, also if randomly chosen, can compete for capital: only the market selection mechanisms determine whether it may survive in the long-run or whether it will eventually disappear. Moreover, as shown in several studies, it is not clear whether investors derive their strategies from well-specified preferences and objective functions (see for example Mehra and Prescott 1985, Canner, Mankiw, and Weil 1997), or whether their investment strategies are driven by the large number of constraints (internal or external) the investors are forced to satisfy, or by behavioral anomalies, which also affect institutional investors. Finally, it is also not clear a priori if an investment strategy that is chosen randomly cannot be evolutionary stable. Consequently, in order to derive evolutionary stability under the less restrictive assumptions on investment strategies, we do not impose any objective function for the portfolio selection problem, but we take the investment strategies as given and focus on the market selection mechanisms.

The evolutionary model we present in this paper has one long-lived risky asset and cash.
Withdrawals and savings are the difference between collected premia and indemnities to be paid. We assume that insurance companies price insurance contracts according to a value-at-risk condition, set up by regulatory authorities to ensure that insurance companies can face their obligations with high probability. The value-at-risk condition also relates to investment strategies, i.e. the proportion of wealth safely invested, so that, finally, regulatory constraints represent minimal requirements on investors’ investment strategies. A bankruptcy occurs when the investment’s payoff and premia are not enough to pay the indemnities. Moreover, since borrowing and short-selling are not allowed, investors that go to bankrupt simply disappear from the market. This assumption is sustainable when focusing on the long-run evolution of simple strategies, as we do in this paper.

We first establish a no-bankruptcy condition for the investment strategies. The no-bankruptcy condition is the minimal sufficient condition on the trading strategies that ensures that, in the presence of any type of competitor or trading strategy, the investor is able to face liquidity shocks almost surely. In fact, since asset prices are endogenously determined, it happens that, also depending on other agents’ strategies and wealth shares, any strategy violating the no-bankruptcy condition has a strictly positive probability of going bankrupt. In particular, if an investor is the unique survivor at some point in time, the no-bankruptcy condition is sufficient but also necessary to avoid almost surely going bankrupt. Nevertheless, while investors with strategies satisfying the no-bankruptcy condition are almost sure not to go bankrupt, we also show that investors who use the simple strategy that corresponds to the no-bankruptcy boundary, will eventually disappear from the market. Finally, we show that the condition for the existence of evolutionary stable strategies is related to the growth rate of the trading strategies in a neighborhood of the strategy investing according to the no-bankruptcy boundary, when asset prices are dominated by the latter. If this growth rate is strictly increasing in a neighborhood above the no-bankruptcy boundary, then for any monomorphic population investing according to a strategy satisfying the no-bankruptcy condition, there exists a mutant strategy that is able to gain market share. Moreover, in this case, the sole strategy capable to drive out any mutation will eventually disappear from the market almost surely owing to liquidity shocks. Therefore, if the condition for the existence of evolutionary stable strategies is not satisfied, while it can happen that an aggressive investor who maximizes the growth rate is able to increase her market share against any mutant strategy, she will eventually disappear from the market. This means that markets remain unstable, so that new strategies can gain market share and eventually go bankrupt. Moreover, since the investors’ exposure to the insurance market corresponds to their market shares, the insurance market will also remain unstable over time. Finally, this result also suggests that in the presence of liquidity shocks, a two-dimensional criterion referred to both the growth rate and the probability of incurring bankruptcy should be considered.

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1 A portfolio rule is called a simple strategy if the proportion of wealth put on each asset is constant over time.

2 A monomorphic population of investors is a set of investors investing according to the same investment strategy.
This paper aims to contribute to the development of the evolutionary portfolio theory, which started with the seminal paper of Blume and Easley (1992), where an asset market model was first introduced to study the market selection mechanism and the long run evolution of investors’ wealth and assets’ prices. In their model, Blume and Easley (1992) consider diagonal securities, where there are no transaction costs and positive proportional saving rates are exogenously given. In the case of complete markets with diagonal securities, Blume and Easley (1992) show that there is a unique attractor of the market selection mechanism and prices do not matter. With simple strategies and constant, identical saving rates across investors, the sole survivor is the portfolio rule known as “betting your beliefs” (Breiman 1961), where the proportion of wealth placed on each asset is the probability of the corresponding state of nature. This strategy can also be generated by maximizing the expected logarithm of relative returns, which is known as the Kelly rule, studied in discrete time by Kelly (1956), Breiman (1961), Thorp (1971) and Hakansson and Ziemba (1995) (for an overview, see also Ziemba 2002) and, in continuous-time, by Pestien and Sudderth (1985), Heath, Orey, Pestien, and Sudderth (1987) and Karatzas and Shreve (1998), among others.

Hens and Schenk-Hoppé (2005) proposed a more general setting, with incomplete markets, general short-lived assets that are reborn each time, and constant, positive, proportional, and identical saving rates across investors. In their evolutionary model, the equilibrium notion refers to wealth distributions that are invariant under the market selection process. The authors show that invariant wealth distributions are generated by monomorphic populations. Moreover, they introduce the concept of evolutionary stable portfolio rules, also to be examined in this paper. The main result of Hens and Schenk-Hoppé (2005) is that, in the case of ergodic state processes, there is a unique evolutionary stable portfolio rule, which is the one that places on each asset the proportion of wealth corresponding to the expected relative payoff of the asset. In Evstigineev, Hens, and Schenk-Hoppé (2006) this result is extended to a model with long-lived assets, under the assumption of a Markov state of the world. Introducing long-lived assets allows us to take into account the capital gains and losses due to changes in asset prices. This will also be highly relevant in the presence of liquidity shocks, as we will discuss in this paper. Moreover, in Evstigineev, Hens, and Schenk-Hoppé (2002) it is also shown that, with independent and identically distributed state of the world processes, the strategy that invests according to relative dividends is the only simple portfolio rule that gathers total wealth asymptotically: a generalization of the results obtained by Blume and Easley (1992). Sandroni (2000), and Blume and Easley (2006) have also studied the case of long-lived assets. Their main result is that, with complete markets, among all infinite horizon expected utility maximizers, those that happen to have rational expectation will eventually dominate the market and this result holds independently of investors’ risk aversion. In his model, Sandroni (2000) also includes endogenously determined positive and proportional saving rates.

All these models assume that withdrawals and savings are a positive proportion of the current wealth, so that bankruptcy is excluded. Under this assumption, the only criterion that

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*A system of securities is called diagonal if for each state of nature there is exactly one asset which has a strictly positive payoff.*
matters for a trading strategy to be evolutionary stable is its exponential growth rate in the presence of a mutant strategy. This paper shows that with non-proportional and maybe negative withdrawal rates, a second criterion has to be considered. In fact, even if a strategy has the maximal exponential growth rate in the presence of any mutant, it can disappear in the event of an exogenously determined liquidity shock.

In the classical finance approach with exogenously given price dynamics, asset-liability management models already assume that investors maximize the investment’s expected payoff less penalties for bankruptcy or targets not met (see Carino, Myers, and Ziemba 1998, Carino and Ziemba 1998). Liu, Longstaff, and Pan (2003) consider a price dynamic for the risky assets with jumps (event risk) and assume utility functions identical to minus infinity for strictly negative terminal wealth, so that no portfolio rule with a strictly positive probability of going bankrupt will be optimal. They obtain lower (since they do not exclude short-selling) and upper bounds for the proportion of wealth to be put on the risky assets and they provide optimal portfolio weights. Alternatively, Browne (1997) distinguishes between survival problem and growth problem. He first looks at portfolio rules that maximize the probability of surviving in the so-called danger zone (where bankruptcy has strictly positive probability to occur) and, second, he considers portfolio rules that maximize the growth rate in the safe zone, where bankruptcy is almost surely excluded. Browne (1997) identifies wealth-level dependent strategies, but in his time-continuous setup, no optimal strategy is found for the danger-zone, and a weaker optimality criterion is introduced. The optimal strategy for the safe-zone corresponds to a generalization of the Kelly criterion previously discussed. Zhao and Ziemba (2000) propose a model with a reward function on minimum subsistence, i.e. the objective function to maximize equals the sum of the expected final wealth and a concave increasing function on the maximum over the wealth levels that are almost surely smaller than the final wealth. In this way, the optimal portfolio rule solves a trade-off between expected payoff and minimum subsistence.

The rest of this paper is organized as follows. In the next section we present the model setup. In Section 3 we derive the non-bankruptcy condition on investment strategies, which ensures that liquidity shocks do not cause bankruptcy. In Section 4 we present the main results of the paper. Section 5 concludes. Technical results and proofs are given in the Appendix.

2 The model

2.1 Preliminaries

Time is discrete and denoted by $t = 0, 1, 2, \ldots$. Uncertainty is modeled by a stochastic process $(S_t)_{t \in \mathbb{Z}}$ with values in some infinite space $S$, endowed with power $\sigma$-algebra $2^{|S|}$. $\mathcal{F}_t = \sigma(S_0, S_1, \ldots, S_t)$ denotes the $\sigma$-algebra giving all the information available at time $t$ and $\mathcal{F} = \sigma(\bigcup_{t \in \mathbb{Z}} \mathcal{F}_t)$. Let $\Omega = S^\mathbb{Z}$ be the space of sample paths $(s_t)_{t \in \mathbb{Z}}$, where $s_t$, $t \in \mathbb{Z}$ is the realization of $S_t$ on $S$. Finally, $\mathbb{P}$ denotes the unique probability measure on $(\Omega, \mathcal{F})$.
generated by \((S_t)_{t \in \mathbb{Z}}\).

There are \(i = 1, \ldots, I (I \geq 2)\) investors (insurance companies) with initial wealth \(w_i^0\), that invest their wealth in the financial market, which consists of one long-lived risky asset and cash. Cash is risk-free both in terms of its return \(R = 1 + r \geq 1\) and price, which is taken as numéraire. The risky asset pays a dividend \(D_t(s^t) \geq 0\) at time \(t\), depending on history \(s^t = (\ldots, s_{-1}, s_0, s_1, \ldots, s_t)\) up to time \(t\). The price \(q_t\) of the risky asset is determined by the market clearing condition.

At time \(t\) each investor \(i\) can also decide to sell \(\delta_i^t \geq 0\) insurance contracts on one single future stochastic claim \(X_{t+1}^{s_{t+1}} \geq 0\) (identical for all investors). The premium of each contract \(P_t\) is \(\mathcal{F}_t\)-measurable (it depends only on information available up to time \(t\)), determined by the market clearing condition on the insurance market at time \(t\), and is paid at time \(t+1\) by the buyer of the insurance contract, who is supposed to be external to the economy just defined, i.e. buyers of insurance contracts do not participate in the financial market. The amount collected or withdrawn by investor \(i\) at time \(t+1\) is then given by the difference

\[
C_{t+1}^i = \delta_i^t (X_{t+1} - P_t)
\]

between claims and premia.

Let \(w_i^t\) be the total wealth of investor \(i\) at time \(t\) after payment of claims and premia collection, \(m_i^t \geq 0\) and \(a_i^t \geq 0\) be the unit of cash and risky asset, respectively, held by investor \(i\) at time \(t\). The budget constraint is given by

\[
w_i^t = m_i^t + q_t a_i^t.
\]

The wealth of investor \(i\) evolves as follows:

\[
w_{t+1}^i = (1 + r) m_i^t + (D_{t+1} + q_{t+1}) a_i^t - C_{t+1}^i.
\]

We say that investor \(i\) goes bankrupt during period \(t\) if and only if \(w_i^t \leq 0\). In this case she uses all her wealth to pay the indemnities and vanishes from the market, i.e. we arbitrarily write \(m_s^i = a_s^i = 0\) for all \(s \geq t\) (and thus we also set \(w_s^i = 0\) for all \(s \geq t\)). Note that the investor’s wealth at time \(t\) also depends on the price \(q_t\) of the risky asset, which is determined at equilibrium by investors’ demand for the risky asset and supply, as will be discussed in Subsection 2.3. Thus, time \(t\) investors’ strategies may cause bankruptcy. Let \(\mathcal{I}_t = \{i | w_i^t > 0\}\) be the set of investors who survive period \(t\). Obviously, \(\mathcal{I}_t \subseteq \mathcal{I}_{t-1}\) and thus \(m_{s}^i = a_{s}^i = 0\) for all \(i \notin \mathcal{I}_{t-1}\). Investor \(j\) is said to be the sole survivor at time \(t\) if and only if \(\mathcal{I}_t = \{j\}\).

\[\text{To be formally correct, the wealth evolution of equation 2 should be replaced by}
\]

\[
w_{t+1}^i = [(1 + r) m_i^t + (D_{t+1} + q_{t+1}) a_i^t - C_{t+1}^i]^+,
\]

where for \(x \in \mathbb{R}\), \(x^+ = \max(0, x)\). We prefer to keep the notation simpler and since we are essentially looking at strategies that survive in the long run, the wealth evolution of those strategies is correctly given by equation 2.
The sequence of actions and realization during a typical period $t$ is as follows. First, the exogenous dividend process $D_t$ and the stochastic claim $X_t$ realize. Second, investors derive their investment strategies $m_i^t$ and $a_i^t$, and the price $q_t$ is simultaneously determined by the market clearing condition. Finally, depending on the wealth level $w_i^t$, investors derive the number of insurance contracts $\delta_i^t$ and the insurance premium $P^t$ is simultaneously determined by the market clearing condition for the insurance market. Similarly to Wenzelburger (2004) and Böhm and Chiarella (2005), the timing of actions and realizations is summarized in Figure 1.

![Figure 1 about here.]

### 2.2 Insurance market

We first focus on the insurance market. Insurance companies determine the number of contracts $\delta_i^t$ to be sold at time $t$ using the following quintile principle (see Schnieper 1993, Embrechts 2000):

$$
\mathbb{P}[\delta_i^t (X_{t+1} - P_t) > \alpha_i^t w_i^t | s^t] = \epsilon_i^t
$$

(3)

where the pair $(\alpha_i^t, \epsilon_i^t)$ is the loss acceptability of investor $i$, i.e. the proportion of current wealth $\alpha_i^t$ that company $i$ is ready to lose in period $t+1$ and the probability $\epsilon_i^t$ the company is ready to be hit by losses exceeding $\alpha_i^t w_i^t$. We do not specify in what way insurance companies determine their loss acceptability parameters, since this is not relevant in this paper, as discussed in the introduction. The amount $\alpha_i^t w_i^t$ serves as the technical reserve for the insurance exposure, and therefore has to be invested prudently by investor $i$. In our setting, this means that

$$
\alpha_i^t w_i^t \geq R m_i^t \quad \forall i \in I_t.
$$

(4)

The insurance market is regulated and solvency constraints $\overline{\alpha}$ and $\overline{\epsilon}$ are imposed by regulatory authorities, i.e. for all $i \in I_t$ we have

$$
\alpha_i^t \in (0, \overline{\alpha}) \quad \text{and} \quad \epsilon_i^t \leq \overline{\epsilon}.
$$

(5)

Finally, if $\delta_i^t (X_{t+1} - P_t)$ is strictly greater than the technical reserves, we say that investor $i$ faces a liquidity shock. From equation (3) it is clear that investor $i$ faces liquidity shocks with probability $\epsilon_i^t$ during period $t+1$.

From equation (3), we derive explicitly the number of contracts $\delta_i^t$ for any given premium $P_t$. Let $\mu_t = \mathbb{E}[X_t | \mathcal{F}^{t-1}]$ and $\sigma_t^2 = \text{Var}(X_t | \mathcal{F}^{t-1})$ be the conditional expectation and the conditional variance of $X_t$, given $\mathcal{F}^{t-1}$, respectively, and let $F_t$ be the conditional cumulative distribution function of $Y_t = \frac{X_t - \mu_t}{\sigma_t}$, i.e.

$$
F_t(y) = \mathbb{P}[Y_t \leq y | \mathcal{F}^{t-1}].
$$

Moreover, $F_t^{-1}$ denotes the generalized inverse of $F_t$.

To avoid premium $P_t$ fully covering the insurance risk, we impose the following restrictions:

...
Assumption 1 (Insurance market). For $t \in \mathbb{Z}$ and $i \in \mathcal{I}_t$, let $(\alpha^i_t, \epsilon^i_t)$ and $(\tilde{\alpha}^i_t, \tilde{\epsilon}^i_t)$ be two possible choices for the loss acceptability parameters of investor $i$. Let $\alpha^i_t = \tilde{\alpha}^i_t$. Then for all premia $P_t$

$$\delta^i_t > \tilde{\delta}^i_t \implies \epsilon^i_t > \tilde{\epsilon}^i_t.$$

This assumption says that for given technical reserves, the probability of liquidity occurring strictly increases with the number of insurance contracts sold. If this were not satisfied, then it would be possible to cover additional insurance risk only through collected premia, which is not a fair pricing rule. Since $\delta^i_t = 0$ solves equation (3) with $\alpha^i_t = 0$ and $\epsilon^i_t = 0$, Assumption 1 also implies that an insurance company without technical reserves, selling a strictly positive number of contracts, faces liquidity shocks with a strictly positive probability. Assumption 1 indirectly imposes restrictions on equilibrium premia, as shown in the following lemma.

Lemma 1. If Assumption 1 holds, then for all $t \in \mathbb{Z}$ and $i \in \mathcal{I}_t$: 

$$P_t < \mu_{t+1} + \sigma_{t+1} F^{-1}_{t+1}(1 - \epsilon^i_t).$$

The proof is given in Appendix 6.1. From equation (3) and Lemma 1, we obtain 

$$\delta^i_t \left[ \mu_{t+1} + \sigma_{t+1} F^{-1}_{t+1}(1 - \epsilon^i_t) - P_t \right] = \alpha^i_t \epsilon^i_t,$$

or

$$\delta^i_t = \frac{\alpha^i_t \epsilon^i_t}{\mu_{t+1} + \sigma_{t+1} F^{-1}_{t+1}(1 - \epsilon^i_t) - P_t}.$$  

(6)

(7)

Lemma 1 also ensures that $\delta^i_t \geq 0$. Equation (7) says that the supply for insurance contracts of each investor $i$ is proportional to her technical reserve and decreases with decreasing probability $\epsilon^i_t$.

We now derive the equilibrium premium $P_t$. Since we have a single claim $X_{t+1}$, the demand for insurance contracts is exogenously given and corresponds to 1 for all $t$. The market clearing condition for the insurance market is therefore given by 

$$\sum_i \delta^i_t = 1$$

for all $t$. Using equation (7) for $\delta^i_t$ and solving the market clearing condition with respect to $P_t$ we obtain 

$$P_t = \mu_{t+1} + \sigma_{t+1} \sum_i \delta^i_t F^{-1}_{t+1}(1 - \epsilon^i_t) - \sum_i \alpha^i_t \epsilon^i_t.$$  

(8)
\[
\sigma_{t+1} \sum_i \delta_i \cdot F_{t+1}^{-1}(1 - \epsilon_i) - \sum_i \alpha_i w_i
\]
is the so-called loading factor and is supposed to be strictly positive. From the last equation, we see that the premium of the insurance contract increases with increasing conditional variance, as one would expect, and decreases when the “average” reserve \(\sum_i \alpha_i w_i\) increases. Moreover, a safer investor, who has a smaller \(\alpha_i\) or a smaller \(\epsilon_i\) than a riskier investor, contributes to an increase in the premium, from which all investors benefit (see Ceccarelli 2002).

Equations (7) and (8) can be solved for \(\delta_i\) and \(P_t\): they provide a unique solution with a strictly positive premium (this will become clear for the special case considered below; however, we give a general proof of the existence and uniqueness of a solution in Appendix 6.2).

Since the goal of this paper is to analyze the long-run wealth evolution of investors with respect to their investment strategies on financial markets, we assume that their profiles on insurance markets are identical, meaning that they possess the same loss acceptability parameters. While this assumption is obviously not realistic, it allows to isolate the impact of investment strategies on long-run survival. In general, it is not clear whether an investor who has higher loss acceptability will growth faster. In fact, while it is true (as shown in equation (7)) that higher loss acceptability means greater liquidity shocks (for both the probability and the amount), it must also be said that investors who sell a larger number of contracts benefit from growth opportunities when premia are greater than claims. Moreover, fewer technical reserves means less exposure to liquidity shocks (as discussed above), but also less restrictive constraints for the investment strategies. In other word, those investors can put less money into the risk-free asset and profit from growth opportunities on the financial market. We address these issues in other papers. Here we make the following assumption (see also Leippold, Vanini, and Trojani 2003).

Assumption 2 (Loss acceptability).

Investors’ “loss acceptability” is constant over time and identical for all investors, i.e. \(\alpha_i = \alpha \in (0, \alpha]\) and \(\epsilon_i = \bar{\epsilon}\) for all \(t\) and \(i = 1, \ldots, I\).

Note that in equations (7) and (8), where all investors possess the same parameter \(\epsilon_i\), the number of contracts \(\delta_i\) does not depend on \(\epsilon_i\) anymore and therefore the magnitude of liquidity shocks is minimized for all investors if \(\epsilon_i = \bar{\epsilon}\). Moreover, by Assumption 2 and

\footnote{In fact, it is well known from the ruin theory, that if \(P_t \leq \mu_{t+1}\), i.e. if the premium at time \(t\) is smaller or equal to the conditional expectation of next period claims given all information available at time \(t\), then for any value for the initial wealth (without financial market) the probability of going bankrupt is equal one (see Feller 1971, page 396).}
equations (7) and (8) it follows

\[
P_t = \mu_{t+1} + \sigma_{t+1} F_{t+1}^{-1}(1 - \bar{\tau}) - \alpha \sum_i w_i^t,
\]

(9)

\[
\delta_i^t = \frac{w_i^t}{W_t},
\]

(10)

where \(W_t = \sum_{j=1}^I w_j^t\) is the total market wealth. Investor \(i\)'s supply for insurance contracts corresponds to her relative wealth.

We conclude the characterization of the insurance market by introducing a precise structure for claim \(X_{t+1}\). In particular, we assume that the total claim \(X_{t+1}\) is proportional to the aggregate wealth available at time \(t\), meaning that the amount of insured claims increases or decreases depending on the aggregate success of the investors (a similar assumption will also be made for the dividend process). This assumption also prevents a shock from destroying the economy. The proportional factor is supposed to be independent of the history up to time \(t\) and can be interpreted as the liquidity shock factor for the economy. Mathematically we have

\[
X_{t+1} = \eta_{t+1} W_t,
\]

(11)

where \(\eta_{t+1} \in [0, 1]\) is independent of \(\mathcal{F}^t\) and \(W_t = \sum_{i \in I} w_i^t = \sum_{i=1}^I w_i^t\) is the aggregate wealth available in the economy at time \(t\). From equation (11) it follows that \(\mu_{t+1} = W_t E[ \eta_{t+1} ]\) and \(\sigma_{t+1}^2 = W_t^2 \text{Var}(\eta_{t+1})\). Moreover, \(\eta_{t+1} \sim G_{t+1}\) where \(F_{t+1}(y) = G_{t+1}(\frac{y}{W_t}), \forall y\). Thus

\[
P_t = (\mu(\eta_{t+1}) + \sigma(\eta_{t+1}) G_{t+1}^{-1}(1 - \bar{\tau}) - \alpha) W_t.
\]

Therefore, premium \(P_t\) is strictly positive for all \(t\), if the loading factor \((\sigma(\eta_{t+1}) G_{t+1}^{-1}(1 - \bar{\tau}) - \alpha) W_t\) is greater than zero for all \(t\). Moreover, for the sake of simplicity, we make the following assumption:

**Assumption 3 (Liquidity shocks).**

Liquidity shocks \((\eta_t)_{t \geq 1}\) are independent and identically distributed, i.e. \(G_t = G\) for all \(t\),

\(\eta_t \sim \eta \sim G\), where \(G\) is a continuous cumulative distribution function.

Let \(\mu = E[\eta]\) and \(\sigma^2 = \text{Var}(\eta)\), then by Assumptions 2 and 3

\[
P_t = \mu W_t + \sigma G^{-1}(1 - \bar{\tau}) W_t - \alpha W_t = (\beta - \alpha) W_t,
\]

(12)

\[
C_{t+1}^i = (\eta_{t+1} - \beta + \alpha) w_i^t.
\]

(13)

where \(\beta = \mu + \sigma G^{-1}(1 - \bar{\tau})\). As discussed above for the general case, we impose that the loading factor \((\sigma G^{-1}(1 - \bar{\tau}) - \alpha) W_t\) is strictly positive, i.e. \(\alpha < \min\{\bar{\tau}, \sigma G^{-1}(1 - \bar{\tau})\}\). Then \(\beta - \alpha > \beta - \sigma G^{-1}(1 - \bar{\tau}) = \mu > 0\) and thus \(P_t > 0\) for all \(t\).
2.3 Financial market

We now turn our attention to the financial market. We suppose that the risky asset is in fixed
supply, normalized to one. Instead, the supply of cash is exogenously given by cumulated
dividends and collected premia less withdrawals. The market clearing conditions are

\[
\sum_{i=1}^{I} a_{i}^{t} = \sum_{i \in I_t} a_{i}^{t} = 1
\]

\[
M_t = R \sum_{i \in I_t} m_{i-1}^{t} + D_t \sum_{i \in I_t} a_{i-1}^{t} - q_t \left(1 - \sum_{i \in I_t} a_{i-1}^{t}\right) - C_t
\]

where \( M_t = \sum_{i \in I_t} m_i^t \) and \( C_t = \sum_{i \in I_t} C_i^t \). Note that \( \sum_{i \in I_t} m_{i-1}^{t} \leq M_{t-1} = \sum_{i \in I_{t-1}} m_i^{t-1} \) since \( I_t \subseteq I_{t-1} \).

To be consistent with Assumption 3 and in order to avoid that dividends shrink compared
to insurance shocks (and thus the risky asset becomes irrelevant), we make the following
assumption for the dividend process\(^6\)

**Assumption 4 (Dividend process).**

(i) For each \( t \),

\[ D_t = d_t W_{t-1} , \]

\(^6\)This assumption, together with Assumption 3, solves the difficulty encountered by Hens and Schenk-Hoppé (2006), where the rate of return on the long-lived asset eventually dominates that of the numéraire, so that the strategy that invests only in long-lived asset is able to drive out any other strategy. Hens and Schenk-Hoppé (2006) suggest basing evolutionary finance model on Lucas (1978), where assets’ payoffs are in terms of a single perishable consumption good. In this way, the consumption rate is at least as big as the growth rate of the total payoff of the market. In our model, also without relying on Lucas (1978), both the pricing rule for insurance contracts (which also determines \( C_t \)) and Assumptions 3 and 4 ensure that the rate of “consumption” increases proportionally to the growth rate of the total payoff. Moreover, as we will discuss later, if assets’ payoffs were in terms of perishable consumption goods, it would not be possible to find a trading strategy that preserves wealth (the reserve capital in the insurance business) and have positive growth rates.
for some process \((d_t)_{t>0}\), with \(d_t \sim d \sim H\) independently and identically distributed with cumulative distribution function \(H\) on \([0,1]\).

(ii) \[
P[d > 0] = 1 - H(0) \in (0,1),
\]
i.e. at each time dividends have strictly positive probability of being zero and of being strictly positive.

Let \(\lambda^i_t \in [0,1]\) be the proportion of wealth invested in the risky asset by investor \(i \in \mathcal{I}_t\) at time \(t\). We have \[
a^i_t = \frac{\lambda^i_t w^i_t}{q_t} \quad \text{and} \quad m^i_t = (1-\lambda^i_t) w^i_t.
\]
We call the sequence \((\lambda^i_t)_{t\geq 0 | i \in \mathcal{I}_t}\) the trading strategy of investor \(i\) and \(\lambda^i_t\) the strategy of investor \(i\) at time \(t\). We use the convention that \(\lambda^i_t = 0\) if \(i \notin \mathcal{I}_t\). Note that \(\lambda^i_t\) is a random variable, i.e. it depends on the state of the world up to time \(t\), \(s^t\). Other assumptions on the process defining the trading strategy \((\lambda^i_t)_{t \geq 0}\) will be introduced later. Here, we simply impose the following restriction on the strategies at time \(t\), \((\lambda^i_t)_{i \in \mathcal{I}_t}\), to prevent the price of the risky asset from becoming zero.

Assumption 5 (Investors’ strategies).

For each \(t\) such that \(|\mathcal{I}_t| > 1\), there exist \(i,j \in \mathcal{I}_t\) with \((1-\lambda^i_t) \lambda^j_t > 0\).

Assumption 5 essentially states that if more than one investor survives period \(t\), then there exists at least one survivor with a strictly positive proportion of her wealth invested in the risky asset and one survivor with a strictly positive proportion of her wealth invested in the risk-free asset. Naturally, when a survivor has a mixed strategy, \(\lambda^i_t \in (0,1)\), then Assumption 5 is obviously satisfied when \(i = j\). If \(|\mathcal{I}_t| = 1\), then it might occur that the sole survivor uses a strategy that invests all her wealth in the risk-free asset. The strategy \(\lambda^i_t = 1\) is excluded by the solvency constraint (1) that is equivalent to

\[
1 - \lambda^i_t \geq \frac{\alpha}{R} \iff \lambda^i_t \leq 1 - \frac{\alpha}{R} =: \overline{\lambda} \in (0,1),
\]
i.e., for each investor the proportion of wealth invested in the risky asset is bounded from above by \(\overline{\lambda}\). It seems to be a natural restriction for an insurance company (or a pension

\footnote{A strategy \((\lambda^i_t)\), is called a \textit{mixed strategy} if and only if it assigns a strictly positive percentage to every asset, for all \(t\). In our setting, a mixed strategy is characterized by \(\lambda^i_t \in (0,1)\) for all \(t\) (see Evstigneev, Hens, and Schenk-Hoppé 2002).}
fund), as shown for example in Davis (2001, Tables 5 and 6) for life insurances and pension funds of several countries.

Let \( \lambda_t = (\lambda_1^t, \ldots, \lambda_I^t) \)', then the market clearing condition for the risky asset implies

\[
q_t = \sum_{i=1}^{I} \lambda_i^t w_t.
\]

(17)

Note that for \( i \notin \mathcal{I}_t \), \( w_i^t = 0 \) by our convention and thus \( \lambda_i^t w_t = \sum_{i \in \mathcal{I}_t} \lambda_i^t w_i^t \). We rewrite equation (2) as follows

\[
w_{i+1}^t = \left[ R (1 - \lambda_i^t) + (d_{t+1} W_t + q_{t+1}) \frac{\lambda_i^t}{q_t} - (\eta_{t+1} - \beta + \alpha) \right] w_i^t.
\]

(18)

3 The no-bankruptcy condition

A necessary condition for long-term survival is not to go bankrupt. Consequently, strategies that do not eliminate bankruptcy with probability 1 are avoided by investors with long-term horizons.

In this section we characterize trading strategies that allow insurance companies to survive almost certainly any liquidity shock. In the next section we assume that investors adopt such strategies and we analyse the long-run wealth evolution and characterize evolutionary stable strategies.

We derive a no-bankruptcy condition that sets an upper bound for the proportion \( \lambda_i^t \) of wealth invested in the risky asset. We will show that an investor with a strategy that does not prevent bankruptcy at each period has a strictly positive probability of vanishing from the market, even if she is the sole survivor. Moreover, if an investor uses a simple strategy that does not prevent bankruptcy, she has probability 1 of vanishing from the market, even if at some point in time she is the sole survivor and thus dominates asset prices. In particular, an investor holding only the risky asset (i.e. \( \lambda_i^t = 1 \)) becomes extinct with probability 1. This result shows that Theorem 1 in Hens and Schenk-Hoppé (2006) does not hold in cases where bankruptcy is not excluded.

We first consider the case \( |\mathcal{I}_t| = 1 \) for some \( t > 0 \), i.e. \( \mathcal{I}_t = \{ j \} \) for some \( j \in \{1, \ldots, I\} \). We restrict ourselves to strategies \( \lambda_i^t > 0 \). If \( \lambda_i^t = 0 \), the bankruptcy is clearly excluded since \( R > \sup \text{supp}(\eta) - \beta + \alpha \). The price of the risky asset at time \( t \) is given by \( q_t = \lambda_i^t w_i^t \) and the aggregate wealth at time \( t \) is \( W_t = w_i^t \): from equation (18) it follows immediately that

\[
j \in \mathcal{I}_{t+1} \Leftrightarrow R (1 - \lambda_i^t) - (\eta_{t+1} - \beta + \alpha) + d_{t+1} > 0.
\]

Let \( \underline{\eta} = \inf \text{supp}(\eta), \overline{\eta} = \sup \text{supp}(\eta) \) and \( \underline{d} = \sup \text{supp}(d) \) and \( K \) be the continuous multivariate cumulative distribution of \( (\eta, d) \) on \( [\underline{\eta}, \overline{\eta}] \times [0, \underline{d}] \), i.e.

\[
K(x, y) = \mathbb{P}[\eta \leq x, d \leq y].
\]

\footnote{\text{supp}(\eta) denotes the support of \( \eta \).}
Moreover, let $\tilde{K}(z) = \mathbb{P}[d - \eta \leq z] = \int_{\mathbb{R}} \int_{0}^{x+z} dK(x,y)$ be the cumulative distribution function of $d - \eta$. Then

$$\mathbb{P}[j \in I_{t+1}] = \mathbb{P}[d_{t+1} - \eta_{t+1} > -R (1 - \lambda^j_t) - \beta + \alpha]$$

and thus

$$\mathbb{P}[j \in I_{t+1}] = 1 - \tilde{K}(-R (1 - \lambda^j_t) - \beta + \alpha)$$

where $\lambda = \inf \supp(\tilde{K})$. We call this latter equation the no-bankruptcy condition.

Since $\lambda = -\lambda + \frac{\beta + \alpha}{R}$, the solvency constraint (16) is a stronger condition on the strategies than the no-bankruptcy condition (19), if $\beta > -\lambda$ i.e. if higher shocks (greater than $\beta$) on the insurance market and small dividends (less than $\eta - \beta$) in the financial market do not occur simultaneously, which is not a realistic assumption. This is due to the fact that the solvency constraint disregards dividends, and thus does not take into consideration the (positive) correlation between shocks and dividends, such that higher shocks might have a smaller impact on investors' wealth since they are compensated by higher dividends. By contrast, if $\beta < -\lambda$ (which is the most common case, as for example when insurance shocks and dividends are considered independent), the no-bankruptcy condition (19) is stronger than the solvency constraint; and thus investors just care about the no-bankruptcy condition (19). In this case, the solvency constraint (16) does not eliminate bankruptcy. In the sequel we make the following assumption on the joint distribution of $(\eta,d)$:

Assumption 6 (Shocks and dividends joint distribution).

For all $\delta_1 > 0$ and $\delta_2 > 0$,

$$\mathbb{P}[\eta > \overline{\eta} - \delta_1, d \leq \delta_2] > 0,$$

i.e., big shocks and very small dividends have strictly positive probability to jointly occur.

Assumption 6 implies the following Lemma on the distribution of $d - \eta$.

Lemma 2. For all $\delta > 0$,

$$\mathbb{P}[d - \eta \leq -\overline{\eta} + \delta] > 0$$

and thus $\lambda = -\overline{\eta}$, i.e. maximal shocks and zero dividends have strictly positive probability to occur together.
The proof of Lemma 2 is given in Appendix 6.3. Under Assumption 6, the strategy \( \lambda \) corresponds to \( \frac{R-\eta+\beta-\alpha}{R} \) and is a stronger condition on the strategies than the solvency constraint, since obviously \( \beta < -\eta \). From now on, we assume
\[
\lambda = \frac{R-\eta+\beta-\alpha}{R}.
\]

Let us now consider a single survivor \( j \) with a simple strategy \( \lambda^j > \lambda \). Then, at each period, she will have a strictly positive probability of going bankrupt and therefore \( \mathbb{P}[j \in \cap_t \mathcal{I}_t] = 0 \), meaning that she will vanish almost surely from the market. We state these results in the following Lemma.

**Lemma 3.** Let \( \mathcal{I}_t = \{j\} \) for some \( t \) and \( j \in \{1, \ldots, I\} \), i.e. investor \( j \) is the single survivor at time \( t \). The following holds:

(i) If \( \lambda^j > \lambda \), then investor \( j \) has strictly positive probability of going bankrupt during period \( t+1 \).

(ii) If \( \lambda^s > \lambda \) for all \( s \geq t \), then investor \( j \) will almost surely eventually vanish from the market. In particular, if investor \( j \) uses a simple strategy \( \lambda^j > \lambda \), then she will eventually almost surely vanish from the market.

Let us now consider the case \( |\mathcal{I}_t| > 1 \). Without loss of generality we set \( \mathcal{I}_t = \{1,2\} \): if \( \mathcal{I}_t = \{i_1, \ldots, i_n\} \) with \( n = |\mathcal{I}_t| > 2 \), then we can still reduce the original setting to a 2-investor setting by defining a “new investor” with strategy \( \xi_s \in [0,1] \) at time \( s \in \{t, t+1\} \) and wealth \( w_s \), where
\[
\xi_s = \frac{\sum_{l=2}^{n} \lambda_{s}^i w_{s}^{i_l}}{\sum_{l=2}^{n} w_{s}^{i_l}}, \quad w_s = \sum_{l=2}^{n} w_{s}^{i_l}.
\]
The price of the risky asset at time \( s \in \{t, t+1\} \) is then given by \( q_s = \lambda_{s}^i w_{t+1}^i + \xi_s w_s \). Thus let us assume that \( \mathcal{I}_t = \{1,2\} \). Then, as shown in Appendix 6.4 from the wealth evolution (18) it follows immediately that for \( i = 1,2 \)
\[
i \in \mathcal{I}_{t+1} \Leftrightarrow R (1 - \lambda_{t+1}^i) + d_{t+1} W_t \frac{\lambda_{t+1}^j}{q_t} + w_{t+1}^j \lambda_{t+1}^j \frac{\lambda_{t+1}^j}{q_t} - (\eta_{t+1} - \beta + \alpha) > 0,
\]
where \( j \neq i \). This equation is a necessary and sufficient condition for avoiding bankruptcy for investor \( i \), which also depends on other investors’ wealths and strategies, through the term.
Speculating on other investors’ strategies, investor $i$ could essentially put less wealth on the risk-free asset than allowed under the no-bankruptcy condition (19). While this would imply a strictly positive probability of going bankrupt when investor $i$ dominates asset prices, the no-bankruptcy condition is not necessary to avoid almost surely bankruptcy in the presence of competitors, when they significantly invest in the risky asset. However, the no-bankruptcy condition is the minimal condition on investment strategies that almost surely eliminates bankruptcy in the presence of any type of competitor. In fact, an investor who systematically violates the no-bankruptcy condition (19) will eventually disappear from the market with probability 1, if her opponents are investing all their wealth in the risk-free asset, i.e. an investment strategy that systematically violates the no-bankruptcy condition is almost surely driven out by the risk-free strategy. Thus the no-bankruptcy condition is the minimal condition that ensures that no investor will go bankrupt with probability 1, regardless of other investors’ strategies. In the sequel, because of the long horizon perspective considered here, and following the approach of Liu, Longstaff, and Pan (2003), we use the no-bankruptcy condition to ensure that investors almost surely do not face bankruptcy.

4 The main results

In this section we characterize evolutionary stable strategies, as defined by Evstigneev, Hens, and Schenk-Hoppé (2002). An evolutionary stable strategy is a strategy that is able to drive out any new, distinct market participant when it dominates asset prices, i.e. when it possesses almost the entire market capital. A formal definition will be given later.

From the previous section, it is clear that an investor who uses a strategy that does almost certainly not eliminate bankruptcy, will eventually disappear from the market, also if at some point in time she is the only survivor, as shown in Lemma 3. Therefore, a strategy that does not prevent bankruptcy cannot be evolutionary stable. This motivates the following assumption:

**Assumption 7 (The no-bankruptcy condition).**

For all $i \in \mathcal{I}_t$ and all $t \in \mathbb{Z}$,

\[ \lambda^i_t \in [0, \lambda]. \]

---

*In their setting, bankruptcy is penalized with minus infinity utility, so that no optimal strategy will allow final negative wealth with strictly positive probability.*

17
Using Assumption 7 we rewrite the wealth dynamics (18) as follows:

\[
w_{t+1}^i = \frac{w_t^i}{\sum_j (1 - \lambda_t^{j+1}) \lambda_t^j w_t^j} \times \left[ d_{t+1} W_t \lambda_t^i + [R(\lambda - \lambda_t^i) + (\bar{\eta} - \eta_{t+1})] \left( \lambda_t^i w_t + \sum_{j \neq i} (\lambda_t^j - \lambda_t^i) \lambda_t^j w_t^j \right) \right].
\]

This result is explicitly derived in Appendix 6.5.

Let \( r_t^i = \frac{w_t^i}{W_t} \) be the wealth share of investor \( i \) at time \( t \), and \( \zeta_t^i = \frac{\lambda_t^i}{\sum_i \lambda_t^i} \) for \( i = 1, \ldots, I \) and \( t \in \mathbb{Z} \). Vector \( r_t = (r_t^1, \ldots, r_t^I)' \) is the vector of wealth shares, i.e. \( r_t \in \Delta^{I-1} = \{ r \in \mathbb{R}_+^I \mid \sum_i r_t^i = 1 \} \). By Assumption 7, \( \zeta_t^i \in [0, 1] \) and \( \zeta_t^i = 1 \) iff \( \lambda_t^i = \lambda \) and we obtain

\[
w_{t+1}^i = \frac{r_t^i W_t}{\sum_j (1 - \Delta \zeta_{t+1}^j) \zeta_t^j r_t^j} \times \left[ d_{t+1} \zeta_t^i + [R(1 - \zeta_t^i) + (\bar{\eta} - \eta_{t+1})] \left( \zeta_t^i r_t + \Delta \sum_{j \neq i} (\zeta_t^j - \zeta_t^i) \zeta_{t+1}^j r_t^j \right) \right].
\]

Let

\[
\theta_{t+1} = \zeta_t^i r_t d_{t+1} + \sum_k \left( R \Delta (1 - \zeta_t^k) + (\bar{\eta} - \eta_{t+1}) \right) r_t^k \left( \zeta_t^i r_t + \Delta \sum_j (\zeta_t^j - \zeta_t^i) \zeta_{t+1}^j r_t^j \right),
\]

then the total market wealth satisfies

\[
W_{t+1} = \frac{\theta_{t+1}}{\sum_j (1 - \Delta \zeta_{t+1}^j) \zeta_t^j r_t^j} W_t,
\]

where the ratio \( \frac{\theta_{t+1}}{\sum_j (1 - \Delta \zeta_{t+1}^j) \zeta_t^j r_t^j} \) is the growth rate of the economy. We reformulate wealth dynamic in terms of wealth shares and we obtain:

\[
r_{t+1}^i = \frac{r_t^i}{\theta_{t+1}} \left[ d_{t+1} \zeta_t^i + [R \Delta (1 - \zeta_t^i) + (\bar{\eta} - \eta_{t+1})] \left( \zeta_t^i r_t + \Delta \sum_j (\zeta_t^j - \zeta_t^i) \zeta_{t+1}^j r_t^j \right) \right], \quad (21)
\]

For \( i = 1, \ldots, I \) let

\[
f^i(r_t, t) = \frac{r_t^i}{\theta_{t+1}} \left[ d_{t+1} \zeta_t^i + \left( R \Delta (1 - \zeta_t^i) + (\bar{\eta} - \eta_{t+1}) \right) \left( \zeta_t^i r_t + \Delta \sum_j (\zeta_t^j - \zeta_t^i) \zeta_{t+1}^j r_t^j \right) \right], \quad (22)
\]

then

\[
r_{t+1}^i = f^i(r_t, t)
\]
or
\[ r_{t+1} = f(r_t, t), \]  
(23)

where \( f = (f^1, \ldots, f^I)' \). Although it does not appear explicitly in the definition of \( f^i_t \), function \( f_t \) also depends on the state of the world \( s_{t+1} \) up to time \( t + 1 \), through investors' strategies at time \( t + 1 \), the dividend \( d_{t+1} \) and the liquidity shock factor \( \eta_{t+1} \). The market selection process (23) generates a random dynamical system (see Arnold 1998) on the simplex \( \Delta^{I-1} \). Given a vector of initial wealth shares \( r \in \Delta^{I-1} \) and \( t > 0 \), the map
\[
\phi(t, \omega, r) = f(s^t, \cdot) \circ f(s^{t-1}, \cdot) \circ \cdots \circ f(s^1, r),
\]  
(24)
on \( \mathbb{N} \times \Omega \times \Delta^{I-1} \) gives the investors' wealth shares at time \( t \), if the state of the world is \( \omega = (s_t)_{t \in \mathbb{Z}} \).

We are interested in vectors of wealth shares that are invariant under market selection mechanism \( \phi \). We introduce the following definition:

**Definition 1 (Fixed point).** The vector of relative wealth shares \( r \in \Delta^{I-1} \) is called a deterministic fixed point of \( \phi \), if and only if
\[
\phi(t, r, \cdot) = r
\]
almost surely for all \( t \). If the distribution of market shares \( r \) is a deterministic fixed point, then \( r \) is said to be invariant under the market selection process (24).

One can easily check that the vectors of wealth shares \( r = e^i \) for \( i = 1, \ldots, I \) are deterministic fixed points of \( \phi \), where \( e^i \) is the Kronecker delta for the \( i \)-th component. The following Lemma shows the \( e^i \)'s are the unique fixed points of \( \phi \).

**Lemma 4.** Let \( r \) be a deterministic fixed point of \( \phi \). Then \( r = e^i \) for some \( i = 1, \ldots, I \).

The proof is given in Appendix 6.6. The Lemma implies that in order to analyze invariant wealth share distributions we can restrict ourselves to monomorphic populations (i.e. all investors with a strictly positive market share possess the same trading strategy). In fact, since invariant wealth share distributions correspond to monomorphic populations, the stability of investment strategies is related to the stability of the associated fixed points \( e^i \). Therefore, we study the perturbation of a monomorphic population of strategies, by considering two distinct trading strategies \( \lambda^i \) (the dominant strategy) and \( \lambda^j \) (the strategy of a new participants), with market shares \( r^i_t \) and \( r^j_t = 1 - r^i_t \), respectively.
**Definition 2** (Evolutionary stable strategies). A trading strategy $\lambda^i$ is called evolutionary stable if, for all strategies $\lambda^j$, there is a random variable $\epsilon > 0$ such that $\lim_{t \to \infty} \phi^i(t, \omega, r) = 1$ for all $r^i \geq 1 - \epsilon(\omega)$.

We start by considering trading strategies that corresponds to the no-bankruptcy bound $\overline{\lambda}$. We derive our first main result. We prove that an insurance company investing according to the no-bankruptcy bound $\overline{\lambda}$ will be driven out from the market by any insurance company using a strategy that is bounded away from $\overline{\lambda}$. From equation (21) it follows that for $i, k \in \mathcal{I}_t$,

$$\frac{r^i_{t+1}}{r^k_{t+1}} = \left(\frac{r^i_t}{r^k_t}\right) \times \frac{d_{t+1} \zeta^i_{t+1} + [R \Delta (1 - \zeta^i_t) + (\overline{\eta} - \eta_{t+1})](\zeta^i_{t+1} r_t + \Delta \sum_{j \neq i} (\zeta^j_t - \zeta^i_t) \zeta^j_{t+1} r^j_{t+1}]}{d_{t+1} \zeta^k_{t+1} + [R \Delta (1 - \zeta^k_t) + (\overline{\eta} - \eta_{t+1})](\zeta^k_{t+1} r_t + \Delta \sum_{j \neq k} (\zeta^k_t - \zeta^k_t) \zeta^j_{t+1} r^j_{t+1}).}$$

Let us now suppose that only two investors exist. The first investor is using a simple strategy corresponding to the no-bankruptcy bound, i.e. $\lambda^1_t = \overline{\lambda}$ for all $t$. The second investor is using a strategy which is bounded away from the no-bankruptcy condition, as well as from the strategy putting the wealth only on the risk-free asset, i.e. $\delta < \lambda^2_t < \overline{\lambda} - \delta$ for all $t > 0$ and for some $\delta > 0$. Using the notation introduced above, we have $\zeta_t^1 = 1$ for all $t$ and $\zeta_t^2 \in (\delta, 1 - \delta)$ for all $t$ and $\delta = \frac{\Delta}{2} > 0$. We obtain the following result.

**Theorem 1.** Under Assumptions and given an investor with $\zeta^1_t = 1$ for all $t > 0$ and an investor with $\zeta^2_t \in (\delta, 1 - \delta)$ for all $t > 0$ and some $\delta > 0$, the investor with the simple strategy corresponding to the no-bankruptcy boundary will almost surely vanish from the market.

Theorem states that, while being at the boundary of the no-bankruptcy condition means that bankruptcy is excluded with probability 1, the market selection mechanism still forces such an investor to vanish from the market, if other investors are using strategies that are bounded away from $\overline{\lambda}$. Therefore, the trading strategy $\lambda^1_t = \overline{\lambda}$ is not evolutionary stable. In fact, even if this strategy possesses almost the entire wealth, an investment strategy that is bounded away from $\overline{\lambda}$ is able to drastically perturb the distribution of wealth shares and to drive out $\overline{\lambda}$.

We now exclude investors using $\overline{\lambda}$ and we consider again a population of two investors, where the dominant investor 1 has market share $r^1_t$ and the new-comer investor 2 has market share $1 - r^1_t$. The wealth share dynamic for investor 1 is obtained from (21) by

$$\psi(r^1_t) = f^1(r^1_t, 1 - r^1_t).$$

The derivative of $\psi$ evaluated at $r^1_t = 1$ corresponds to

$$\left.\frac{\partial \psi(r^1_t)}{\partial r^1_t}\right|_{r^1_t=1} = \frac{\frac{\zeta^2_t d_{t+1} + [R \Delta (1 - \zeta^2_t) + (\overline{\eta} - \eta_{t+1})](\zeta^1_t + (\zeta^2_t - \zeta^1_t) \Delta \zeta^1_{t+1}]}{d_{t+1} + R \Delta (1 - \zeta^1_t) + \overline{\eta} - \eta_{t+1}}}{\zeta^2_t d_{t+1} + [R \Delta (1 - \zeta^2_t) + (\overline{\eta} - \eta_{t+1})](\zeta^1_t + (\zeta^2_t - \zeta^1_t) \Delta \zeta^1_{t+1}]}.$$
The right-hand side of this last equation corresponds to the exponential growth rate of the trading strategy \( \lambda^2 \), when investor 1 owns total market wealth (see equation (21)). Note that by Theorem 1, we can impose without loss of generality that \( \zeta_1 \neq 1 \). Thus, the derivative

\[
\frac{\partial \psi(r_1^t)}{\partial r_1^t} \bigg|_{r_1^t=1}
\]

is well defined and bounded for \( \zeta_2 \in [0, 1] \).

We restrict ourselves to the case where the process \((S_t)_{t \in \mathbb{Z}}\) determining the state of nature is i.i.d. and we denote by \( \mu \) the distribution of \( S_t \) on \( S \). Then, the growth rate of investor 2’s market share in a small neighborhood of \( r_1^t = 1 \) is equal to

\[
g_{\zeta_1}(\zeta_2) = \int_{S} \tilde{g}_{\zeta_1}(\zeta_2(s^0), s^0) \mu^N(ds^0)
\]

where

\[
\tilde{g}_{\zeta_1}(\zeta_2(s^0), s^0) = \int_{S} \mu(ds) \log \left\{ \zeta_1(s^0)^{-1} \left[ d(s) + R \Delta (1 - \zeta_1(s^0)) + \bar{\eta} - \eta(s) \right]^{-1} \left[ \zeta_2(s^0) d(s) + \left( R \Delta (1 - \zeta_2(s^0)) + (\bar{\eta} - \eta(s)) \right) \left( \zeta_1(s^0) + (\zeta_2(s^0) - \zeta_1(s^0)) \Delta \zeta_1(s^0, s) \right) \right] \right\}.
\]

The function \( \zeta_2(s^0) \mapsto \tilde{g}_{\zeta_1}(\zeta_2(s^0), s^0) \) is continuous, strictly concave on \([0, 1]\) for all \( s^0 \) and, obviously \( \tilde{g}_{\zeta_1}(\zeta_1(s^0), s^0) = 0 \) for all \( s^0 \). Moreover,

\[
\frac{\partial \tilde{g}_{\zeta_1}(\zeta_2(s^0), s^0)}{\partial \zeta_2(s^0)} \bigg|_{\zeta_2(s^0)=\zeta_1(s^0)} = \frac{1}{\zeta_1(s^0)} \times \\
\int_{S} \mu(ds) \frac{d(s) - R \Delta \zeta_1(s^0) + R \lambda^2 \zeta_1(s^0, s)(1 - \zeta_1(s^0)) + (\bar{\eta} - \eta(s)) \Delta \zeta_1(s^0, s)}{d(s) + \left( R \Delta (1 - \zeta_1(s^0)) + (\bar{\eta} - \eta(s)) \right)}.
\]

The following theorem holds:

**Theorem 2.** Let the state of nature \((s_t)_{t \in \mathbb{Z}}\) be determined by an i.i.d. process. Suppose that all investors use simple strategies \( \lambda^i(\omega) \equiv \lambda^i \in (0, 1) \).

(i) If

\[
\mathbb{E}\left[\frac{d}{d + \bar{\eta} - \eta}\right] + \lambda \mathbb{E}\left[\frac{\bar{\eta} - \eta}{d + \bar{\eta} - \eta}\right] \geq R \lambda \mathbb{E}\left[\frac{1}{d + \bar{\eta} - \eta}\right],
\]

then there is no evolutionary stable investment strategy.
(ii) If
\[ \mathbb{E}\left[ \frac{d}{d + \eta - \eta} \right] + \lambda \mathbb{E}\left[ \frac{\eta - \eta}{d + \eta - \eta} \right] < R \lambda \mathbb{E}\left[ \frac{1}{d + \eta - \eta} \right], \]

then there is a unique evolutionary stable investment strategy \( \lambda^* = \lambda \zeta^* \) where \( \zeta^* \) satisfies:
\[ \int_S \mu(ds) \frac{[d(s) - R \lambda \zeta + R \lambda^2 \zeta (1 - \zeta) + (\eta - \eta(s)) \lambda \zeta]}{d(s) + R \lambda (1 - \zeta) + \eta - \eta(s)} = 0. \quad (25) \]

The proof is given in Appendix 6.8. Theorem 2 provides a description of market stability, for both the financial and insurance markets. In fact, from Section 3 we learn that any monomorphic population of simple trading strategies \( \lambda > \lambda^* \) disappears almost surely owing to liquidity shocks. Therefore, both financial and insurance markets are highly unstable when dominated by investment strategies that do not fulfill the no-bankruptcy condition. Note that the percentage of contracts sold by any insurance company corresponds to its market share and thus financial and insurance markets, if such is the case, are both dominated by the same strategy. Moreover, from Theorem 1 the simple investment strategy at the boundary of the no-bankruptcy condition, while it obviously satisfies the no-bankruptcy condition, it is driven out of the market by any strategy that is bounded away from the no-bankruptcy boundary and the risk-free asset. Finally, the condition for the existence of evolutionary stable strategies given in Theorem 2 relates to the growth rate of a mutant investment strategy in a neighborhood of \( \zeta = 1 \), i.e. \( \lambda = \lambda^* \). More precisely, the derivative of the growth rate of a mutant strategy when asset prices are dominated by \( \lambda \) corresponds to \( \mathbb{E}\left[ \frac{d}{d + \eta - \eta} \right] + \lambda \mathbb{E}\left[ \frac{\eta - \eta}{d + \eta - \eta} \right] - R \lambda \mathbb{E}\left[ \frac{1}{d + \eta - \eta} \right] \). Therefore, if this last expression is positive, then for any strategy that satisfies the no-bankruptcy condition, i.e. for any \( \lambda = \zeta \lambda^* \) for some \( \zeta \in [0, 1] \), there exists a mutant strategy that is capable of conquering market share. In other words, if equation (i) in Theorem 2 holds, then both financial and insurance markets are unstable. In this case, there will be one single strategy able to drive out any mutation, but this strategy does not fulfill the no-bankruptcy condition and therefore will disappear almost certainly owing to liquidity shocks. Under case (i), an aggressive investment strategy that only maximizes the growth rate, might dominate the market, but eventually disappears!

Example

Let us suppose that \( d \) and \( \eta \) are independent, \( d \) is distributed on \([0, 0.1]\) with \( H(0) = \mathbb{P}[d = 0] = 0.01\), \( (d|d > 0) \sim \text{Beta}(2, 2) \) and, \( \eta \sim \text{Uniform}(0.05, 0.1) \). Moreover, we take \( \alpha = 0.04 \) and \( R = 1.025 \), then \( \lambda^* = 0.93796 \). According to Theorem 2 there exists a unique evolutionary
stable strategy that is $\zeta^* = 0.213528$, i.e only 20.03% of the wealth is invested on the risky asset. The evolution of market shares is shown in Figure 2.

If $(d|d > 0) \sim \text{Beta}(2, 3)$, then the unique evolutionary stable strategy is $\lambda^* = 17.66\%$; $\zeta^*$ decreases with decreasing $\mathbb{E}[d]$.

[Figure 2 about here.]

5 Conclusion

In this paper we propose an evolutionary portfolio model with bankruptcy. The investors are insurance companies and the amount of wealth that is withdrawn or collected at each period corresponds to the difference between indemnities, which must be paid out and premia paid in. If this difference is negative, insurance companies face liquidity shocks that could force investors to withdraw their entire wealth, thus forcing the company into bankruptcy. We introduce the no-bankruptcy condition on investment strategies, which ensures that bankruptcy is excluded with probability 1; and we analyse the set of simple strategies that are evolutionary stable if the state of nature is generated by an i.i.d. process. We prove that invariant wealth shares distribution only corresponds to monomorphic populations. Moreover, we show that, depending on the dividend and liquidity shock processes, either there exists a unique evolutionary stable strategy or, no evolutionary strategy exists. We give the condition that characterize existence of evolutionary stable strategies and, if this condition is satisfied, we also characterize those strategies.

The case where no evolutionary stable strategy exists is interesting, because it corresponds to the situation where markets remain unstable over time because no investment strategy dominates in the long-run. In fact, also if at some point in time prices are dominated by some strategy (like a momentum strategy in case of bubbles), either there exists an alternative strategy that is able to gain market share against the dominant strategy and perturb asset prices or, the dominant strategy is too risky and will disappear owing to liquidity shocks.

This work shows that when savings and withdrawals are not a positive percentage of the investor’s wealth, then one should also take into consideration a risk dimension, that is the probability of going bankrupt. This dimension is ignored in previous model adopting an evolutionary perspective, since it is usually assumed that investors only face positive withdrawals. Thus in this paper we introduce the risk dimension in the Evolutionary Portfolio Theory.
References


25


6 Appendix

6.1 Proof of Lemma 1

Proof. Let us suppose that

\[ P_t \geq \mu_{t+1} + \sigma_{t+1} F_{t+1}^{-1}(1 - \epsilon^i_t). \]

for some \( t \) and \( i \in I_t \). Then

\[ \frac{P_t - \mu_{t+1}}{\sigma_{t+1}} \geq F_{t+1}^{-1}(1 - \epsilon^i_t) \]

and thus for all \( \delta > 0 \)

\[ \mathbb{P}[\delta (X_{t+1} - P_t) > 0] = \mathbb{P}[X_{t+1} - P_{t+1} > 0] = \mathbb{P}[Y_{t+1} > \frac{P_t - \mu_{t+1}}{\sigma_{t+1}}] \leq \epsilon^i_t \]

independently from \( \delta \). This contradicts Assumption 1 since the last inequality shows that the probability of liquidity shocks is in fact independent of the number of contracts, with fixed technical reserves. \( \square \)

6.2 Existence and uniqueness of \( \delta^i_t \) and \( P_{t+1} \)

For the sake of simplicity we drop the index \( t \) from the notation of equations (7) and (8). Using the expression (7) for \( \delta^i \) in (3), we obtain

\[ P = \mu + \sum_i \left( \frac{\sigma \theta^i}{\mu + \sigma \theta^i - P} - 1 \right) \alpha^i w^i, \]

where \( \theta^i = F^{-1}(1 - \epsilon^i) \). Let \( f : [0, \min_i \{\mu + \sigma \theta^i\}] \to \mathbb{R} \) be defined by \( f(P) = \mu + \sum_i \left( \frac{\sigma \theta^i}{\mu + \sigma \theta^i - P} - 1 \right) \alpha^i w^i \). \( f \) is well defined on \([0, \min_i \{\mu + \sigma \theta^i\}]\) and continuous differentiable, with \( f'(P) = \sum_i \frac{\sigma \theta^i}{(\mu + \sigma \theta^i - P)^2} \alpha^i w^i > 0 \), \( f''(P) = \sum_i \frac{2 \sigma \theta^i}{(\mu + \sigma \theta^i - P)^3} \alpha^i w^i > 0 \), i.e. \( f \) is strictly
increasing and convex. Moreover, $f(P) \nearrow \infty$ as $P \nearrow \min_i \{\mu + \sigma \theta_i\}$ and

$$
\begin{align*}
    f(0) &= \mu + \sum_i \left( \frac{\sigma \theta_i}{\mu + \sigma \theta_i} - 1 \right) \alpha^i w^i = \mu - \mu \sum_i \frac{\alpha^i w^i}{\mu + \sigma \theta_i} \\
    &= \mu - \mu \sum_i \delta^i|_{P=0} = 0, \\
    f'(0) &= \sum_i \frac{\sigma \theta_i \alpha^i w^i}{(\mu + \sigma \theta_i)^2} \leq \max_i \left\{ \frac{\sigma \theta_i}{\mu + \sigma \theta_i} \right\} \sum_i \frac{\alpha^i w^i}{\mu + \sigma \theta_i} < 1,
\end{align*}
$$

since $\mu > 0$. Thus $f(P) \geq 0$ and it possesses exactly two fixed points: $P = 0$ and $P^* \in (0, \min_i \{\mu + \sigma \theta_i\})$. Therefore, there is a unique premium $P^* > 0$ which satisfies equations (7) and (8). Moreover, by equation (7), $\delta^i$ is also uniquely defined for all $i$.

[Figure 3 about here.]

6.3 Proof of Lemma 2

Proof.

$$
\begin{align*}
    \mathbb{P}[d - \eta \leq -\bar{\eta} + \delta] &= \mathbb{P}[d \leq \eta - \bar{\eta} + \delta] \\
    &= \int_{0<\delta_1<\delta} \mathbb{P}[d \leq \eta - \bar{\eta} + \delta | \eta - \bar{\eta} > -\delta_1] \ d\mathbb{P}[\eta - \bar{\eta} > -\delta_1] \\
    &\geq \int_{0<\delta_1<\delta} \mathbb{P}[d \leq -\delta_1 + \delta | \eta - \bar{\eta} > -\delta_1] \ d\mathbb{P}[\eta - \bar{\eta} > -\delta_1] \\
    &= \int_{0<\delta_1<\delta} \underbrace{\mathbb{P}[d \leq -\delta_1 + \delta, \eta > \bar{\eta} - \delta_1]}_{>0} \ d\mathbb{P}[\eta - \bar{\eta} > -\delta_1] \\
    &> 0
\end{align*}
$$

Thus, $\tilde{K}(-\bar{\eta} + \delta) > 0$ for all $\delta > 0$, i.e. $\tilde{k} = -\bar{\eta}$. \qed
6.4 Derivation of equation 20

Proof. (i) Suppose that \( i \in I_{t+1} \). Then \( w_{t+1}^i > 0 \) and by equation (18)

\[
\begin{align*}
    w_{t+1}^i &= \left( 1 - \lambda_{t+1}^i \right) w_t^i + d_{t+1} W_t \frac{\lambda^i_t w_t^i}{q_t} \\
               &\quad + (w_{t+1}^1 \lambda_{t+1}^1 + w_{t+1}^2 \lambda_{t+1}^2) \frac{\lambda^i_t w_t^i}{q_t} - (\eta_{t+1} - \beta + \alpha) w_t^i,
\end{align*}
\]

and thus

\[
\begin{align*}
    w_{t+1}^i \left( 1 - \lambda_{t+1}^i \frac{\lambda^i_t w_t^i}{q_t} \right) &= \left( 1 - \lambda_{t+1}^i \right) w_t^i + d_{t+1} W_t \frac{\lambda^i_t w_t^i}{q_t} \\
    &\quad + w_{t+1}^j \lambda_{t+1}^j \frac{\lambda^i_t w_t^i}{q_t} - (\eta_{t+1} - \beta + \alpha) w_t^i,
\end{align*}
\]

where \( j \neq i \). Since \( \lambda_{t+1}^i \neq 1 \) (solvent restriction), then \( \left( 1 - \lambda_{t+1}^i \frac{\lambda^i_t w_t^i}{q_t} \right) > 0 \), and thus from \( w_{t+1}^i > 0 \) it follows that

\[
R \left( 1 - \lambda_{t}^i \right) + d_{t+1} W_t \frac{\lambda^i_t}{q_t} + w_{t+1}^j \lambda_{t+1}^j \frac{\lambda^i_t}{q_t} - (\eta_{t+1} - \beta + \alpha) w_t^i > 0.
\]

Since \( i \in I_t \), then \( w_t^i > 0 \) and therefore dividing the last inequality by \( w_t^i \) we obtain

\[
R \left( 1 - \lambda_{t}^i \right) + d_{t+1} W_t \frac{\lambda^i_t}{q_t} + w_{t+1}^j \lambda_{t+1}^j \frac{\lambda^i_t}{q_t} - (\eta_{t+1} - \beta + \alpha) > 0.
\]

(ii) Suppose now that

\[
R \left( 1 - \lambda_{t}^i \right) + d_{t+1} W_t \frac{\lambda^i_t}{q_t} + w_{t+1}^j \lambda_{t+1}^j \frac{\lambda^i_t}{q_t} - (\eta_{t+1} - \beta + \alpha) > 0,
\]

where \( j \neq i \). Then for \( i \in I_t \),

\[
w_t^i \left[ R \left( 1 - \lambda_{t}^i \right) + d_{t+1} W_t \frac{\lambda^i_t}{q_t} + w_{t+1}^j \lambda_{t+1}^j \frac{\lambda^i_t}{q_t} - (\eta_{t+1} - \beta + \alpha) \right] > 0,
\]

and thus \( w_{t+1}^i > 0 \), since

\[
w_{t+1}^i = \left[ w_t^i \frac{R \left( 1 - \lambda_{t}^i \right) + d_{t+1} W_t \frac{\lambda^i_t}{q_t} + w_{t+1}^j \lambda_{t+1}^j \frac{\lambda^i_t}{q_t} - (\eta_{t+1} - \beta + \alpha)}{1 - \lambda_{t+1}^i \frac{\lambda^i_t w_t^i}{q_t}} \right]^+.\]
and $1 - \lambda_{t+1}^i \frac{\lambda_t^i w_t^i}{q_t} > 0$ by equation (16).

\[\square\]

6.5 Derivation of the wealth dynamics

Following Hens and Schenk-Hoppé (2006), we rewrite the wealth dynamics (18). We define

$$B_t^i = \frac{\lambda_t^i w_t^i}{\lambda_t^i w_t^i},$$

and

$$A_t^i = R (1 - \lambda_t^i) w_t^i + B_t^i d_{t+1} W_t - (\eta_{t+1} - \beta + \alpha) w_t^i.$$

By Assumption 7, we have

$$w_{t+1} = A_t + B_t \lambda_{t+1} \mathbf{w}_{t+1}$$

or

$$(I - B_t \lambda_{t+1}) w_{t+1} = A_t$$

where $A_t = (A_1^1, \ldots, A_I^1)'$, $B_t = (B_1^1, \ldots, B_I^1)'$ and $I$ is the identity on $\mathbb{R}^I$. Note that for $i \notin I_t$, $A_t^i = B_t^i = 0$. The inverse of $I - B_t \lambda_{t+1}$ is given by $I + (1 - \lambda_{t+1}^i B_t)^{-1} B_t \lambda_{t+1}^i$, provided that $\lambda_{t+1}^i B_t \neq 1$ (see Horn and Johnson 1985, Sec. 0.7.4). It can be easily checked that $\lambda_{t+1}^i B_t < 1$ if there exists an investor $i \in I_{t+1}$ with $\lambda_t^i < 1$ and $\lambda_{t+1}^i > 0$ and this is still the case when $|I_t| > 1$, by Assumptions 5 and 7. If $I_t = \{j\}$ for some $j$, then investor $j$ is already the unique survivor and the wealth evolution is easily obtained. Therefore, in the sequel we only consider the case $|I_t| > 1$. Under the assumption of no default during period $t+1$, the wealth evolution can then be written as

$$w_{t+1} = (I - B_t \lambda_{t+1})^{-1} A_t = \left( I + \frac{B_t \lambda_{t+1}^i}{1 - \lambda_{t+1}^i B_t} \right) A_t,$$

(26)

The $i$-th component follows:

$$w_{t+1}^i = A_{t+1}^i + \frac{B_t^i}{1 - \sum_j \lambda_{t+1}^j B_t^j} \sum_j \lambda_{t+1}^j A_t^j$$

$$= A_{t+1}^i + \frac{\lambda_t^i w_t^i}{\sum_j \lambda_t^j w_t^j - \sum_j \lambda_{t+1}^j \lambda_t^j w_t^j} \sum_j \lambda_{t+1}^j A_t^j$$

$$= w_t^i \left[ R (1 - \lambda_t^i) + \frac{\lambda_t^i}{\lambda_t^i w_t^i} d_{t+1} W_t - (\eta_{t+1} - \beta + \alpha) \right.$$  

$$+ \lambda_t^i \frac{\sum_j \lambda_{t+1}^j \left( R (1 - \lambda_t^i) + \frac{\lambda_t^j}{\lambda_t^j w_t^j} d_{t+1} W_t - (\eta_{t+1} - \beta + \alpha) \right)}{\sum_j (1 - \lambda_{t+1}^j) \lambda_t^j w_t^j} \right]$$
Thus
\[
\begin{align*}
\lambda^i_{t+1} &= \frac{w^i_t}{\lambda^i_t w_t} \left[ R \left( 1 - \lambda^i_t \right) \lambda^T_t w_t + \lambda^i_t d_{t+1} W_t - (\eta_{t+1} - \beta + \alpha) \lambda^T_t w_t \right. \\
&\quad + \lambda^i_t \sum_j w^j_{t+1} \left( R \left( 1 - \lambda^j_{t+1} \right) \lambda^T_{t+1} w_t + \lambda^j_{t+1} d_{t+1} W_t - (\eta_{t+1} - \beta + \alpha) \lambda^T_{t+1} w_t \right) \\
&\quad \left. \right] \\
&= \frac{w^i_t}{\lambda^i_t w_t} \sum_j (1 - \lambda^j_{t+1}) \lambda^j_t w^j_t \\
&\quad \left[ \sum_j (1 - \lambda^j_{t+1}) \lambda^j_t w^j_t \left( R \left( 1 - \lambda^j_t \right) \lambda^T_j w_t + \lambda^j_t d_{t+1} W_t - (\eta_{t+1} - \beta + \alpha) \lambda^T_t w_t \right) \right. \\
&\quad + \lambda^j_t \left( \sum_j w^j_{t+1} \left( R \left( 1 - \lambda^j_{t+1} \right) \lambda^T_{t+1} w_t + \lambda^j_{t+1} d_{t+1} W_t - (\eta_{t+1} - \beta + \alpha) \lambda^T_{t+1} w_t \right) \right) \\
&\quad \left. \right] \\
&= \frac{w^i_t}{\sum_j (1 - \lambda^j_{t+1}) \lambda^j_t w^j_t} \\
&\quad \left[ d_{t+1} W_t \lambda^i_t + \left( R \left( 1 - \lambda^i_t \right) - (\eta_{t+1} - \beta + \alpha) \right) \left( \lambda^T_t w_t + \sum_j \lambda^j_t \lambda^i_{t+1} w^j_t \right) \right]
\end{align*}
\]

and therefore
\[
\lambda_{t+1} = \frac{w^i_t}{\sum_j (1 - \lambda^j_{t+1}) \lambda^j_t w^j_t} \\
\left[ d_{t+1} W_t \lambda^i_t + \left( R \left( 1 - \lambda^i_t \right) + (\eta - \eta_{t+1}) \right) \left( \lambda^T_t w_t + \sum_{j \neq i} \lambda^j_t \lambda^i_{t+1} w^j_t \right) \right].
\]

We use that
\[
\lambda = \frac{R - \eta + \beta - \alpha}{R} \iff R + \beta - \alpha = R \lambda + \eta.
\]

### 6.6 Proof of Lemma 4

**Proof.** Let assume that \( r^i = r^i_{t+1} = r^i_t \in (0, 1) \). Then
\[
\theta_{t+1} = d_{t+1} \zeta^i_t + \left( R \lambda (1 - \zeta^i_t) + (\eta - \eta_{t+1}) \right) \left( \zeta^i_t r_t + \lambda \sum_j (\zeta^j_t - \zeta^i_t) \zeta^j_{t+1} r^j_t \right).
\]
or equivalently

\[ dt_{t+1} \sum_{k \neq i} \zeta^k_i r^k_t + \]

\[ + \sum_{k \neq i} \left( R \Delta (1 - \zeta^k_i) + (\rho - \eta_{t+1}) \right) r^k_t \left( \zeta^i_t r_t + \Delta \sum_j (\zeta^j_i - \zeta^j_t) \zeta^j_{t+1} r^j_t \right) = dt_{t+1} \zeta^i_t (1 - r^i_t) + \]

\[ + (1 - r^i_t) \left( R \Delta (1 - \zeta^i_i) + (\rho - \eta_{t+1}) \right) \left( \zeta^i_t r_t + \Delta \sum_j (\zeta^j_i - \zeta^j_t) \zeta^j_{t+1} r^j_t \right). \]  

(27)

Since \( 1 - r^i_t = \sum_{k \neq i} r^k_t \), the right-hand side of equation (27) corresponds to

\[ dt_{t+1} \sum_{k \neq i} \zeta^k_i r^k_t + \sum_{k \neq i} \left( R \Delta (1 - \zeta^k_i) + (\rho - \eta_{t+1}) \right) r^k_t \left( \zeta^i_t r_t + \Delta \sum_j (\zeta^j_i - \zeta^j_t) \zeta^j_{t+1} r^j_t \right) \]

and thus equation (27) is equivalent to

\[ 0 = dt_{t+1} \sum_{k \neq i} (\zeta^i_i - \zeta^i_t) r^k_t + \zeta^i_t r_t R \Delta \sum_{k \neq i} (\zeta^k_i - \zeta^i_t) r^k_t \]

\[ + \sum_{k \neq i} \left( R \Delta (1 - \zeta^i_i) + (\rho - \eta_{t+1}) \right) r^k_t \Delta \sum_j (\zeta^j_i - \zeta^j_t) \zeta^j_{t+1} r^j_t \]

\[ + \sum_{k \neq i} \left( R \Delta (1 - \zeta^k_i) + (\rho - \eta_{t+1}) \right) r^k_t \Delta \sum_j (\zeta^j_i - \zeta^j_t) \zeta^j_{t+1} r^j_t \]

\[ = dt_{t+1} \sum_{k} (\zeta^i_i - \zeta^i_t) r^k_t + \Delta \sum_{k} (\zeta^i_i - \zeta^i_t) r^k_t \sum_j \zeta^j_{t+1} r^j_t \]

\[ + R \Delta^2 \sum_{k} (\zeta^i_i - \zeta^i_t) r^k_t \sum_j (1 + \zeta^j_i) \zeta^j_{t+1} r^j_t \]

\[ - R \Delta^2 \sum_{k} (\zeta^i_i - \zeta^i_t)(\zeta^i_i + \zeta^i_t)r^k_t \sum_j \zeta^j_{t+1}r^j_t - R \Delta \sum_{k} (\zeta^i_i - \zeta^i_t)r^k_t \sum_j \zeta^j_t r^j_t. \]

Let us first suppose that \( \sum_k (\zeta^i_i - \zeta^i_t) r^k_t = 0 \). Then \( \zeta_i = \xi_t \), where \( \xi_t = \frac{\sum_{k \neq i} \zeta^k_t r^k_t}{1 - r^i_t} \). Moreover, the last equation is equivalent to

\[ \sum_{k} (\zeta^i_i - \zeta^i_t) \zeta^k_t r^k_t = 0 \]
and thus
\[ \xi_t^2 = \sum_{k \neq i} (\zeta_t^k)^2 r_t^k. \]
This last equation implies \( \zeta_t^k = 0 \) for all \( k \), or \( r_t^k = 0 \) for \( k \neq i \). In the first case we have a contradiction to Assumption 5. In the second case we have a contradiction to \( r_t^i \in (0, 1) \).

Let us now suppose that \( \sum_k (\zeta_t^i - \zeta_t^k) r_t^k \neq 0 \). Without loss of generality, we take \( \sum_k (\zeta_t^i - \zeta_t^k) r_t^k > 0 \) (the same argument can also be used for the case \( \sum_k (\zeta_t^i - \zeta_t^k) r_t^k < 0 \)). Then

\[
0 = d_{t+1} + \lambda (\eta - \eta_{t+1}) \sum_j \zeta_{t+1}^j r_t^j - R \lambda \sum_j (1 - \lambda \zeta_{t+1}^j) \zeta_t^j r_t^j \\
- \frac{R \lambda^2}{\sum_l (\zeta_t^i - \zeta_t^l) r_t^l} \sum_k \left( (\zeta_t^i)^2 - (\zeta_t^k)^2 - \sum_l (\zeta_t^i - \zeta_t^l) r_t^l \right) r_t^k \sum_l \zeta_{t+1}^l r_t^l \\
= d_{t+1} + \lambda (\eta - \eta_{t+1}) \sum_j \zeta_{t+1}^j r_t^j - R \lambda \sum_j (1 - \lambda \zeta_{t+1}^j) \zeta_t^j r_t^j \\
- \frac{R \lambda^2}{\sum_l (\zeta_t^i - \zeta_t^l) r_t^l} \sum_{k \neq i} \zeta_t^k (1 - \zeta_t^k) r_t^k \sum_j \zeta_{t+1}^j r_t^j.
\]

Since \( r_{t+1}^i = r_t^i \in (0, 1) \), the set \( \{ \sum_j \zeta_{t+1}^j r_t^j = 0 \} \) has probability zero by Assumption 5.

Thus \( \sum_j \zeta_{t+1}^j r_t^j > 0 \) almost surely. Let \( \delta > 0 \), then by Assumptions 4 and 6, the set \( \{ s_{t+1} | d_{t+1} = 0, \eta - \eta_{t+1} < \delta \} \) has strictly positive probability independently from \( s^t \). Thus

\[
(1 - \lambda) \sum_j \zeta_t^j r_t^j + \frac{\lambda}{\sum_l (\zeta_t^i - \zeta_t^l) r_t^l} \sum_{k \neq i} \zeta_t^k (1 - \zeta_t^k) r_t^k < \delta.
\]

Since this is true for all \( \delta > 0 \), \( \zeta_t^j = 0 \) for all investors with strictly positive wealth share at time \( t \), a contradiction to Assumption 5. Therefore, \( r_t^i = 0 \) or \( r_t^i = 1 \). \( \square \)
6.7 Proof of Theorem 1

Proof. On \( \{d_{t+1} - \eta_{t+1} > -\bar{\eta}\} \) (by Assumptions [1] and Assumption [2] this set has probability 1) we have

\[
\frac{r^2_{t+1}}{r^1_{t+1}} = \left( \frac{r^2_t}{r^1_t} \right) \frac{d_{t+1} \Delta^2 + [R \Delta (1 - \zeta^2_t) + (\bar{\eta} - \eta_{t+1})]}{d_{t+1} + (\bar{\eta} - \eta_{t+1})} \left( \zeta^T_t \mathbf{r}_t - (1 - \zeta^2_t) \Delta r^1_t \right)
\]

\[
= \left( \frac{r^1_t}{r^2_t} \right) \frac{d_{t+1} \zeta^2_t + [R \Delta (1 - \zeta^2_t) + (\bar{\eta} - \eta_{t+1})]}{d_{t+1} + (\bar{\eta} - \eta_{t+1})} \left( \zeta^T_t \mathbf{r}_t + (1 - \zeta^2_t) \Delta \zeta^2_{t+1} r^2_t \right)
\]

\[
\geq \left( \frac{r^1_t}{r^2_t} \right) \min \left( \frac{d_{t+1} \zeta^2_t + [1 - (1 - \zeta^2_t) \Delta]}{d_{t+1} + (\bar{\eta} - \eta_{t+1})} \left[ R \Delta (1 - \zeta^2_t) + (\bar{\eta} - \eta_{t+1}) \right],
\right.

\[
\left. \frac{d_{t+1} + [R \Delta (1 - \zeta^2_t) + (\bar{\eta} - \eta_{t+1})]}{d_{t+1} + (\bar{\eta} - \eta_{t+1})} \left( \zeta^2_t (1 - \Delta \zeta^2_{t+1}) + \Delta \zeta^2_{t+1} \right) \right)
\]

\[
\geq \left( \frac{r^1_t}{r^2_t} \right) \min \left( \frac{d_{t+1} \zeta^2_t + [1 - (1 - \zeta^2_t) \Delta]}{d_{t+1} + (\bar{\eta} - \eta_{t+1})} \left[ R \Delta (1 - \zeta^2_t) + (\bar{\eta} - \eta_{t+1}) \right],
\right.

\[
\left. \frac{d_{t+1} + [R \Delta (1 - \zeta^2_t) + (\bar{\eta} - \eta_{t+1})]}{d_{t+1} + (\bar{\eta} - \eta_{t+1})} \left( \zeta^2_t (1 - \Delta \zeta^2_{t+1}) + \Delta \zeta^2_{t+1} \right) \right)
\]

By the first inequality, we use that

\[
f(r; d, \eta, \zeta, \tilde{\zeta}, \Delta, R) = \frac{d \xi + [R \Delta (1 - \zeta) + (\bar{\eta} - \eta)] \left[ 1 - (1 - \zeta) \Delta - (1 - \zeta) (1 - \lambda) r \right]}{d + (\bar{\eta} - \eta) \left[ 1 - (1 - \zeta) (1 - \Delta \tilde{\zeta}) r \right]}
\]

is strictly increasing, strictly decreasing or flat as function of \( r \), depending on the parameters \( d, \eta, \zeta, \tilde{\zeta}, \Delta \) and \( R \). Thus, the minimum of the function is attained for \( r = 1 \) or \( r = 0 \).

By the second inequality, we use that

\[
[\zeta^2_t (1 - \Delta \zeta^2_{t+1}) + \Delta \zeta^2_{t+1}] < 1,
\]

\[34\]
for all $\zeta_t^2, \zeta_{t+1}^2 \in [0, 1]$.

Iteratively, we obtain

$$\log \frac{r_{t+1}^2}{r_t^2} \geq \sum_{\tau=1}^{t+1} \log \left( \frac{R(1-\bar{\eta})}{d_r + \bar{\eta} - \eta_r} \right) + \log \frac{r_0^2}{r_0^2}. $$

Let $\epsilon < R(1-\delta)\delta \lambda$, then

$$\log \frac{r_{t+1}^2}{r_t^2} \geq C \sum_{\tau=1}^{t+1} \log 1_{(d_r+\bar{\eta} - \eta_r \leq \epsilon)} + \log \frac{r_0^2}{r_0^2},$$

where $C = \log \frac{R(1-\delta)\delta \lambda}{\epsilon} > 0$ by definition of $\epsilon$. By the Theorem,

$$\lim_{t \to \infty} \frac{1}{t^2} \log \frac{r_{t+1}^2}{r_t^2} \geq C \lim_{t \to \infty} \frac{1}{t+1} \left( \sum_{\tau=1}^{t+1} 1_{(d_r+\bar{\eta} - \eta_r \leq \epsilon)} + \log \frac{r_0^2}{r_0^2} \right) = C \tilde{K}(d-\eta \leq -\bar{\eta} + \epsilon) = \gamma > 0.$$

by Assumption 6. Thus $\frac{r_t^2}{1-r_t^2} = \frac{r_t^2}{r_0^2} \approx \exp(t \gamma) \to \infty$ as $t \to \infty$, i.e $r_t^2 \to 1$ almost surely. \qed

### 6.8 Proof of Theorem 2

**Proof.** An investment strategy $\zeta^*$ is evolutionary stable if $g_{\zeta^*}(\zeta) < 0$ for all $\zeta \in [0, 1]$. From Theorem\ref{thm:1} we have $\zeta^* \neq 1$. Moreover, if investors strategies are simple, we have

$$g_{\zeta^*}(\zeta^2) = \tilde{g}_{\zeta^*}(\zeta^2).$$

Thus $\zeta^2 \mapsto g_{\zeta^*}(\zeta^2)$ is continuous, strictly concave and $g_{\zeta^*}(\zeta^1) = 0$. Therefore, $g_{\zeta^*}(\zeta) < 0$ for all $\zeta \in [0, 1]$ if and only if

$$\frac{d g_{\zeta^*}(\zeta^2)}{d \zeta^2} \bigg|_{\zeta^2 = \zeta^1} = 0.$$

Hence, $\zeta^*$ is an evolutionary stable investment strategy if it solves

$$0 = \int_S \mu(ds) \frac{[d(s) - R\lambda \zeta + R\lambda^2 \zeta(1-\zeta) + (\bar{\eta} - \eta(s))\lambda \zeta]}{d(s) + R\lambda(1-\zeta) + \bar{\eta} - \eta(s)}.$$
Let

\[ h(\zeta; d, \eta) = \frac{[d - R \Delta \zeta + R \Delta^2 \zeta (1 - \zeta) + (\eta - \eta) \Delta \zeta]}{d + R \Delta (1 - \zeta) + \eta - \eta} \]

Then \( \frac{\partial h(\zeta, d, \eta)}{\partial \zeta} < 0 \) for \( \zeta \in [0, 1] \), thus \( h \) is strictly decreasing on \([0, 1]\) for all \( d \) and \( \eta \) and,

\[ h(0; d, \eta) = \frac{d}{d + R \Delta (\eta - \eta)} \geq 0 \text{ for all } d \text{ and } \eta, \text{ and strictly positive for } d > 0. \]

Thus

\[ \int_S \mu(ds) h(0; d(s), \eta(s)) > 0 \]

since by Assumption [4] the set \( \{ s \in S | d(s) > 0 \} \) has strictly positive probability. Moreover,

\[ h(1; d, \eta) = \frac{d - R \Delta + (\eta - \eta) \Delta}{d + \eta - \eta} \text{ and} \]

\[ \int_S \mu(ds) h(1; d(s), \eta(s)) = \int_S \mu(ds) \frac{d(s) - R \Delta + (\eta - \eta) \Delta}{d(s) + \eta - \eta(s)} \]

\[ = \int_S \mu(ds) \frac{d(s)}{d(s) + \eta - \eta(s)} + \lambda \int_S \mu(ds) \frac{\eta - \eta(s)}{d(s) + \eta - \eta(s)} \]

\[ = \mathbb{E} \left[ \frac{d}{d + \eta - \eta} \right] + \lambda \mathbb{E} \left[ \frac{\eta - \eta}{d + \eta - \eta} \right] - R \lambda \mathbb{E} \left[ \frac{1}{d + \eta - \eta} \right]. \]

Therefore

\[ \int_S \mu(ds) h(1; d(s), \eta(s)) < 0 \]

if and only if

\[ \mathbb{E} \left[ \frac{d}{d + \eta - \eta} \right] + \lambda \mathbb{E} \left[ \frac{\eta - \eta}{d + \eta - \eta} \right] - R \lambda \mathbb{E} \left[ \frac{1}{d + \eta - \eta} \right] < 0. \]

Thus, if the condition \( \mathbb{E} \left[ \frac{d}{d + \eta - \eta} \right] + \lambda \mathbb{E} \left[ \frac{\eta - \eta}{d + \eta - \eta} \right] - R \lambda \mathbb{E} \left[ \frac{1}{d + \eta - \eta} \right] \geq 0 \) holds, then the function

\[ \int_S \mu(ds) h(\zeta; d(s), \eta(s)) \]

is strictly positive on \([0, 1]\) and therefore no evolutionary stable strategy exists. Note that since \( g_{\zeta_1} \) is continuous and strictly concave, if no evolutionary stable strategy exists, then no local evolutionary stable strategy can exist either. In fact, in a small neighborhood of \( \zeta_1 \) there exists an investment strategy \( \zeta_2 > \zeta_1 \) with \( g_{\zeta_1}(\zeta_2) > 0 \). This proves 36
(i).

If \( \mathbb{E} \left[ \frac{d}{d + \eta - \eta} \right] + \lambda \mathbb{E} \left[ \frac{\eta}{d + \eta - \eta} \right] - R \lambda \mathbb{E} \left[ \frac{1}{d + \eta - \eta} \right] < 0 \), then there exists exactly one \( \zeta^* \in (0, 1) \) such that \( \int_S \mu(ds) h(\zeta^*; d(s), \eta(s)) = 0 \) and therefore \( g_{\zeta^*}(\zeta) < 0 \) for \( \zeta \in (0, 1) \), \( \zeta \neq \zeta^* \). Obviously \( \zeta^* \) solves

\[
\int_S \mu(ds) h(\zeta; d(s), \eta(s)) = 0,
\]

which proves (ii). \( \square \)
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Figure 1: Timing of actions and realizations.
Figure 2: Evolution of market shares, for the $\lambda^*$ strategy of Theorem 2 (full line), the strategy $\lambda$ (dotted line), the risk free strategy (dashed-dotted line), and a randomly chosen strategy in $(0, \lambda)$ (dashed line). In figure (a) all investors have the same initial wealth, while in figure (b) the strategy $\lambda^*$ initially possesses only the 2% of the market capital.
Figure 3: Proof of the existence and uniqueness of equilibrium insurance premium. The demand function $f(P)$ has one strictly positive fixed point $P^*$. 