Partial Information and Default Hazard Process

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First version: December 2003
Current version: March 2004

This research has been carried out within the NCCR FINRISK project on “Conceptual Issues in Financial Risk Management”.

Abstract

This paper studies in some examples the role of information in a default-risk framework. In a first-passage model, we assume that investors obtain two types of information about the firm’s unlevered asset value at a discrete sequence of dates. The effects of information on the distributional properties of default time and on the valuation of default-risky bonds are analyzed. The discrete information arrivals induce jump-discontinuities in both the conditional default probability and the default hazard process. In such cases, it is better avoiding the intensity approach for valuation of default-sensitive contingent claims since the hazard process approach is more efficient.

Key words: credit risk, information, hazard process of default time, default probability

1 Introduction

In the first firm-value based models for valuation of risky debt (Black and Scholes, 1973; Merton, 1974), the default can take place only at the maturity of the debt. This is in contrast to the actual times of defaults on corporate debt to the extent that firms frequently default prior to the maturities of their debts. To obtain a more realistic model for the default time, Black and Cox (1976) reformulated the valuation problem as a first passage time problem of asset value to a deterministic boundary.

*We thank Geneviève Gauthier for comments on the first version of this paper. The second author acknowledges support by the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK). NCCR FINRISK is a research programme supported by the Swiss National Science Foundation.

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The first passage approach accounts for the safety covenants and debt subordination in corporate indentures. Safety covenants provide bondholders with the rights to enforce default, reorganization and liquidation of the firm prior to default when its financial performance is poor. In the first passage approach, poor performance is captured by a low value of the firm’s productive assets relative to the stipulated debt payments. The default occurs when asset value falls to some exogenously specified barrier, in which case the bondholders take over the firm and reorganize or liquidate it. This is why the default-triggering barrier is also referred to as the lower reorganization boundary.

The standard first-passage models are based on the assumption of perfect and continuous observation of the firm’s asset value. While there have been many generalizations and extensions of the basic first passage approach, such as to stochastic interest rates (Longstaff and Schwartz, 1995; Rutkowski, 2002) and endogenous bankruptcy (Leland, 1994; Leland and Toft, 1996; Leland, 1998), few studies investigated the effects of the information. In the standard first passage models, the time of default is a predictable stopping time in the filtration generated by asset value and its properties are not interesting from a mathematical point of view. The most interesting cases are when the information available to the market is some restriction of the full information.

The first systematic study of the consequences of the incomplete information in the structural credit risk models is Duffie and Lando (2001). The authors noticed that frequently the standard assumption of continuous observation of asset value in the first-passage models is not satisfied in practice. While bond investors in a publicly traded firm can estimate the level of its assets by observing the prices of its equity and debt, the bondholders in a privately held firm have to rely on periodic accounting reports received at discrete dates. Therefore, the information flows to bondholders in a privately held firm can be more realistically modeled by filtrations generated by discrete observations of asset value.

As noted by Duffie and Lando, for bond investors having access to this discrete information, the default time is a totally inaccessible time and admits intensity. In this way, the authors reconcile the structural and reduced form approaches to valuation of credit-risky bonds. Valuation with default intensity in the first-passage models with incomplete information leads to higher short-term credit spreads than those provided by the standard first passage models with continuous observations. Moreover, it is demonstrated that, for realistic parameter values, the credit spreads generated by the model are close to the actual spreads of defaultable bonds.

Kusuoka (1999) and Nakagawa (2001) investigate the case of continuous but imperfect observation of asset value. These authors assume observation of a smooth function of asset value and
solve the related filtering problem. It is shown that, with such information, default time admits default arrival intensity of the same form as in Duffie and Lando (2001).

Another study of credit risk with continuous but incomplete information is Cetin et al. (2003). In this study, the default occurs when the firm’s cash balances are negative for some pre-specified period and hit a default threshold level after that. The authors assume that the managers observe continuously the cash balances of the firm, while the market has a continuous subfiltration of the continuous manager’s information filtration. More specifically, the market has information on whether the firm is in financial distress, the duration of the distress and whether the default threshold has been reached. A solution for the price of a default-risky bond with zero recovery with this type of information is obtained.

Giesecke (2003) approaches the incomplete information problem from another perspective. In his model, investors have incomplete information about the default-triggering barrier for asset value. A solution for the conditional joint default probability and default correlation are provided. The author obtains explicit formulas in the case when asset value follows a geometric Brownian motion.

While the above studies focused on different types of incomplete information and on their consequences for valuation of default-risky bonds, still little is known about the dynamics of the default probability. This paper describes the effects of incomplete information about asset value on the distribution of default time. It also compares and evaluates the efficiency of the different reduced form approaches to valuation of risky bonds. The analysis is based on two specific types of information, obtained at a sequence of dates. First, we assume that at this discrete sequence of dates, the bondholders observe all the past asset values. Second, as in Duffie and Lando (2001), we assume that at the dates in the sequence, bondholders observe only the contemporaneous asset values. It is obvious that the second type of information is much more restrictive than the first one.

It turns out that, the discrete arrivals induce jumps in the probability of default and in the default hazard process at the observation dates. In such situations when the hypothesis (H) and condition (G) (see, e.g., Jeanblanc and Rutkowski, 2000b) are not satisfied. Our results suggest that, with discrete observations of asset value, the intensity approach is not as efficient for valuation of default-risky bonds as the hazard process approach. While the intensity valuation formula involves an expectation of complicated product terms involving the conditional default probability at all the observation dates, the hazard process approach requires information only about the conditional default probability at the current time and maturity date. The models described
here could serve as a basis for a study of the two valuation approaches with more complicated information structures (e.g., combining discrete with continuous but noisy observations).

The rest of the paper is organized as follows. In Section 2, we specify a standard first passage model for the default event. Section 3 introduces the first type of incomplete information. With a discrete sequence of observation times, which may correspond to the dates of releasing of periodic accounting reports, the investors obtain all the information about the past asset values. Section 4 investigates a second type of incomplete information, namely, the case when the investors only observe the contemporaneous asset value at these discrete dates. In Section 5, we compare the valuations of defaultable bonds by using the hazard process approach and the intensity approach with incomplete information. Section 6 contains concluding remarks.

2 Full information: continuous observations of asset value

In the first-passage models, it is usually assumed that the unlevered asset value $V$ of a firm follows the geometric Brownian motion process

$$dV_t = V_t((\mu - \delta)dt + \sigma dW^*_t),$$  \hspace{1cm} (1)

where $\mu$ and $\sigma$, (with $\sigma > 0$) are the drift rate and the percentage volatility of asset value, $\delta$ is the rate of payments to the holders of claims on the firm’s assets and $W^*$ is a Brownian motion under the historical probability $P^*$. The information flows of bondholders are modeled by filtrations generated by observations of asset value. In particular, we denote by $F^V$ the complete filtration generated by $V$, i.e., $F^V = (F^V_t, t \geq 0)$, where $F^V_t = \sigma(V_s, s \leq t)$.

Default occurs when asset value falls below some threshold level $\alpha$, which is assumed to be lower than the initial firm value $V_0 = v$. Then, the time of default $\tau$ is

$$\tau = \inf\{t : V_t \leq \alpha\} = \inf\{t : X_t \leq a\}$$  \hspace{1cm} (2)

with $X_t = \frac{1}{\sigma} \ln(V_t/v)$ and $a = \frac{1}{\sigma} \ln(\alpha/v)$.

Suppose that it exists a savings account paying a constant short term interest rate $r$. Hence, if one assumes that the firm’s asset value is a tradable asset, then the market is complete and arbitrage-free. In what follows, the computations are carried under the risk neutral probability measure $P$. We recall that under $P$

$$dV_t = V_t((r - \delta)dt + \sigma dW_t),$$

where $(W_t, t \geq 0)$ is a $(P,F^V)$-Brownian motion.
The solution of (3) is
\[ V_t = ve^{\sigma(W_t + \nu t)}, \]
where \( \nu = \frac{1}{\sigma}(r - \delta - \frac{\sigma^2}{2}) \). We rewrite the solution in a more convenient form:
\[ V_t = ve^{\sigma X_t}, \]
where \( X_t = \nu t + W_t \) is a \((P, \mathcal{F}^V_t)\)-Brownian motion with drift. It is important to notice that \( \mathcal{F}^V_t \) is equal to the filtrations \( \mathcal{F}^X_t \) and \( \mathcal{F}^W_t \) generated by \( X_t \) and \( W_t \), respectively.

Figure 1 displays \((X_t, 0 \leq t \leq T)\) and the default boundary with the base values of the parameters that we shall use in this study
\[ \sigma = 0.30; \ r = 0.04; \ \delta = 0.03; \ v = 100; \ \alpha = 80; \ T = 2. \] (4)

We choose a very high default boundary for illustrative purposes, since, in the simulations, we want to obtain paths close to default.

We denote the probability that the process \((X_s = W_s + \nu s, s \geq 0)\) does not hit the barrier \( z \) before time \( t \) by
\[ \Phi(\nu, t, z) = P\left(\inf_{s \leq t} X_s > z\right). \] (5)

It is important to notice that this probability depends on the drift rate \( \nu \). The reflection principle and elementary considerations lead to
\[ \Phi(\nu, t, z) = \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} \mathcal{N}\left(\frac{z + \nu t}{\sqrt{t}}\right), \quad \text{for } z < 0, \ t > 0, \]
\[ \Phi(\nu, 0, z) = 0, \quad \text{for } z \geq 0, \ t \geq 0, \]
\[ \Phi(\nu, 0, z) = 1, \quad \text{for } z < 0. \]

Consider the definition of default time in this setup. The event that the firm does not default up to the maturity \( T \) of the bond can be expressed as
\[ \{\tau > T\} = \left\{\inf_{s \leq T} V_s > \alpha\right\} = \left\{\inf_{s \leq t} V_s > \alpha\right\} \cap \left\{\inf_{t < s \leq T} V_s > \alpha\right\} \]
\[ = \left\{\inf_{s \leq t} X_s > \alpha\right\} \cap \left\{\inf_{t < s \leq T} X_s > a - X_t\right\} \]
\[ = \left\{\inf_{s \leq t} X_s > a\right\} \cap \left\{\inf_{s \leq T-t} \tilde{X}_{s-t} > a - X_t\right\}, \]
where \( \tilde{X} = (\tilde{X}_u = X_{t+u} - X_t, u \geq 0) \) is independent of \( \mathcal{F}^V_t \). Because of the stationarity and the independence of the increments of the Brownian motion, the process \((\tilde{X}_u, u \geq 0)\) is a \((P, \mathcal{F}^V_{t+u})\)-Brownian motion with drift.
Suppose that the firm has issued defaultable bonds with unit face values that pay their face values at maturity if there is no default or the recovery rate otherwise. We assume that there are no bankruptcy costs, so that bondholders receive all asset value at default, i.e. $\alpha$. Let the total face value of the firm’s outstanding bonds be $D$. Then, in the case of default, the residual firm value is distributed to all the bondholders and each bond recovers $\frac{\alpha}{D}$. A defaultable $T$-maturity bond can be evaluated by taking a conditional expectation under the risk neutral probability $P$ of the terminal payoff.

To facilitate the subsequent analysis, we introduce two simplifying approximations to the Black and Cox (1976) valuation problem. First, we assume that the default-triggering barrier is constant and equals $\alpha$. In fact, the case of an exponential barrier $f(t) = \alpha e^{\eta t}$, $\tau = \inf\{t: V_t \leq f(t)\}$ reduces to the constant case up to a change of parameters $\tau = \inf\{t: xe^{\sigma X_t} \leq \alpha e^{\eta t}\}$ with $Y_t = X_t - \frac{\eta}{\sigma} t$, $y = x$.

Second, in the formulation of Black and Cox, the barrier would be $\alpha$ up to $T$ and $D$ at $T$. Following this approach, one would define the default time (see also Bielecki and Rutkowski, 2002) as $\tau^* = \tau \wedge \tau^m$, where $\tau^m$ is the Merton’s default time, i.e., $\tau^m = T$ if $V_T < D$ and $\tau^m = \infty$ otherwise. We work with the default time $\tau$ rather than $\tau^*$. So our next valuation formula for a default-risky bond is an approximation to the Black and Cox formula. If the default triggering barrier is equal to the face value of the debt, (i.e., $\alpha = D$), the two formulas are equivalent. The approximation works well if $\alpha$ is close to $D$. We use this approximation, because we introduce more complex information structures in the next two sections, which complicate the valuation problem considerably.

With the above assumptions, when the information available to the market at time $t$ is the $\sigma$-algebra $\mathcal{F}^V_t$, the default-risky bond price $B_d(t, T)$ is given by

$$B_d(t, T) = E\left(e^{-r(T-t)}1_{\tau > T} + \frac{\alpha}{D} e^{-r(\tau-t)}1_{\tau \leq T}\mid \mathcal{F}^V_t\right) = e^{-r(T-t)}P(\tau > T\mid \mathcal{F}^V_t) + \frac{\alpha}{D} E\left(e^{-r(\tau-t)}1_{\tau \leq T}\mid \mathcal{F}^V_t\right).$$

The price of the defaultable bond is a sum of two components. We write

$$Z(t, T) := e^{-r(T-t)} P(\tau > T\mid \mathcal{F}^V_t)$$

for the first term, which is the price of a defaultable bond with zero recovery. It pays the face value of the bond at maturity if there is no default and zero otherwise. The second term is the value of
the recovery
\[ R(t, T) := \frac{\alpha}{D} E \left( e^{-r(T-t)} \mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V \right) \]
in the case of default.\(^1\)

The conditional survival probability \( P(\tau > T | \mathcal{F}_t^V) \) plays a key role in the term \( Z(t, T) \). It is expressed in terms of the probability of hitting the default barrier
\[
P(\tau > T | \mathcal{F}_t^V) = \mathbb{1}_{t < \tau} \Phi(\nu, T - t, a - X_t),
\]

hence
\[
Z(t, T) = e^{-r(T-t)} \mathbb{1}_{t < \tau} \Phi(\nu, T - t, a - X_t).
\]

The value of the recovery can also be easily computed. Indeed, using absolute continuity relationship, we can write
\[
R(t, T) = \frac{\alpha}{D} e^{rt} E \left( e^{-r\tau} \mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V \right) = \frac{\alpha}{D} e^{rt} E^{(\gamma)} \left( L_{\tau} e^{-r\tau} \mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V \right) = \frac{\alpha}{D} e^{rt} E^{(\gamma)} \left( e^{-r\tau} e^{(\nu - \gamma)(X_T - X_t)} \cdot e^{\frac{\nu^2 - \gamma^2}{2} (\tau - t)} \mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V \right),
\]
where \( E^{(\gamma)} \) denotes expectation under the probability measure \( P^{(\gamma)} \), defined by
\[
L_t = \frac{dP}{dP^{(\gamma)}} \bigg|_{\mathcal{F}_t^V} = \exp \left( (\nu - \gamma)X_t - \frac{\nu^2 - \gamma^2}{2} t \right).
\]

Under \( P^{(\gamma)} \), the process \( X \) is a Brownian motion with drift \( \gamma \). If we choose \( \gamma \) such that \( \gamma = \sqrt{2r + \nu^2} \), we obtain
\[
R(t, T) = \frac{\alpha}{D} e^{(\nu - \gamma)(a - X_t)} E^{(\gamma)} \left( \mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V \right) = \frac{\alpha}{D} e^{(\nu - \gamma)(a - X_t)} \left( 1 - E^{(\gamma)} \left( \mathbb{1}_{t < \tau} | \mathcal{F}_t^V \right) \right) = \frac{\alpha}{D} e^{(\nu - \gamma)(a - X_t)} \left( 1 - \mathbb{1}_{t < \tau} \Phi(\gamma, T - t, a - X_t) \right), \tag{7}
\]

\(^1\)If the recovery is paid at maturity, the computation is simpler.
Finally, the price of a defaultable bond is
\[
B_d(t, T) = e^{-r(T-t)\mathbb{1}_{t<\tau}}\Phi(\nu, T-t, a-X_t) + \frac{\alpha}{D}e^{(\nu-\gamma)(a-X_t)}\left(1 - \mathbb{1}_{t<\tau}\Phi(\gamma, T-t, a-X_t)\right).
\]
As a check, for \(r = 0\), the term on the right-hand-side reduces to \(\mathbb{1}_{t<\tau}\Phi(\nu, T-t, a-X_t)(1 - \frac{\alpha}{D}) + \frac{\alpha}{D}\) as expected from the definition.

One of the failures of the structural approach is that it produces counterfactually low spreads for the short-maturity default-risky bonds. This is a consequence of the fact that \(\tau\) is a stopping time in the filtration generated by the asset prices. If the asset value is substantially larger than the default boundary and the time to the maturity of the debt is short, the probability that the asset value falls to the boundary is small. As a result, the short-maturity credit spreads are quite low, which is in contrast to the empirically observed spreads of default-risky bonds. In the subsequent sections, we focus mainly on the evaluation of the first component of the price of a defaultable bond (i.e., the zero-recovery defaultable bond) under incomplete information.

3 First type of incomplete information: full observation at discrete times

In this section, supposing that the bondholders have access only to a subfiltration of the full-information filtration \(\mathcal{F}^V\), we compute the conditional default probability under restricted information. More specifically, we assume that the bondholders obtain all the information about the past values of the firm’s assets at the dates
\[
\mathbb{T} = \{t_1, t_2, t_3, \ldots\},
\]
where \((t_i, 1 \leq i)\) is an increasing sequence of observation times. We denote by \(\mathcal{H} = (\mathcal{H}_t, t \geq 0)\) the filtration generated by the observations of the past \(V\) at times \(t_1, \ldots, t_n\) with \(t_n \leq t < t_{n+1}\), that is,
\[
\mathcal{H}_t = \{\emptyset, \Omega\} \quad \text{for } t < t_1,
\]
\[
\mathcal{H}_t = \mathcal{F}^V_{t_1} = \sigma(V_s, s \leq t_1) \quad \text{for } t_1 \leq t < t_2,
\]
\[
\mathcal{H}_t = \mathcal{F}^V_{t_n} = \sigma(V_s, s \leq t_n) \quad \text{for } t_n \leq t < t_{n+1}.
\]

The filtration \(\mathcal{H}\) is a subfiltration of \(\mathcal{F}^V\) (i.e., \(\mathcal{H} \subset \mathcal{F}^V\)). It is constant between the observation dates. At each observation date \(t_i\), the \(\sigma\)-algebra \(\mathcal{H}_t\) is enlarged by \(\sigma(V_s, t_{i-1} \leq s < t_i)\). Clearly, \(\mathcal{H}_{t_i} = \mathcal{H}_{t_{i-1}} \lor \sigma(V_s, t_{i-1} \leq s < t_i)\).
Our aim is to characterize the distributional properties of $\tau$ when the available information is $H$. We write $F^1_t := P(\tau \leq t | H_t)$ for the $H$-conditional default probability and $\Gamma^1_t = -\ln(1 - F^1_t)$ for the $H$-hazard process of $\tau$. In this and the following section, we shall write $\Phi(t, a)$ for $\Phi(\nu, t, a)$.

3.1 First interval

In the case $t < t_1$, the conditional default probability $F^1_t$ is deterministic and is equal to the distribution function of $\tau$. Using the fact that $a < 0$, we obtain

$$F^1_t = P(\tau \leq t) = P\left(\inf_{s \leq t} X_s \leq a\right) = 1 - \Phi(t, a)$$

and $F^1$ is continuous on $[0, t_1[$.

3.2 Second interval

In the case $t_1 \leq t < t_2$, we have

$$F^1_t = P(\tau \leq t | H_{t_1}) = 1 - P(\tau > t | H_{t_1}) = 1 - \Phi(\inf_{s < t_1} X_s > a) P\left(\inf_{t_1 \leq s < t} X_s > a \bigg| F^V_{t_1}\right).$$

(9)

By using (6) and (9), we obtain

$$F^1_t = 1 - \Phi(t - t_1, a - X_{t_1}).$$

We denote by $\Delta F^1_{t_1}$ the jump of $F^1$ at $s$, i.e., $\Delta F^1_s = F^1_s - F^1_{s-}$. The process $F^1$ is continuous and increasing on $[t_1, t_2]$; the jump of $F^1$ at time $t_1$ on the set $\{\tau \leq t_1\}$ is a positive jump with size

$$\Delta F^1_{t_1} = 1 - (1 - \Phi(t_1, a)) = \Phi(t_1, a),$$

while on the set $\{\tau > t_1\}$ it is a negative jump equal to

$$\Delta F^1_{t_1} = 1 - \Phi(0, a - X_{t_1}) - (1 - \Phi(t_1, a)) = \Phi(t_1, a) - 1.$$

In the above computation, we have used that, on $\{\tau > t_1\}$, the inequality $a - X_{t_1} < 0$ holds, hence $\Phi(0, a - X_{t_1}) = 1$. Note that if $F^1_t = 1$, then $F^1_{t+s} = 1 \ \forall s > 0$.

3.3 General case

We easily extend the previous result to general observation times. For $t_i \leq t < t_{i+1}$, we can write

$$F^1_t = P(\tau \leq t | H_t) = 1 - P(\tau > t | F^V_{t_i})$$

$$= 1 - P\left(\inf_{s < t} X_s > a \bigg| F^V_{t_i}\right) = 1 - \Phi(\inf_{t_i \leq s < t} X_s > a \bigg| F^V_{t_i})$$

$$= 1 - \Phi(t - t_i, a - X_{t_i}).$$
The pure jump process $F^1$ is continuous and increasing on $[t_i, t_{i+1}]$. Its jump at time $t_i$ on the set $\{ \tau \leq t_i \}$ is a positive jump

$$\Delta F^1_{t_i} = 1 - (1 - \mathbb{1}_{\tau > t_{i-1}} \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}))$$

$$= \mathbb{1}_{\tau > t_{i-1}} \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}),$$

while on the set $\{ \tau > t_i \}$ it is a negative jump equal to

$$\Delta F^1_{t_i} = 1 - \Phi(0, a - X_{t_i}) - (1 - \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}))$$

$$= \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}) - 1.$$

Apparently, the process $F^1$ jumps to either one or zero at each observation date and this is illustrated in the following two figures. Figure 2 displays the conditional default probability $(F^1_t, 0 \leq t \leq T)$ and the hazard process $(\Gamma^1_t, 0 \leq t \leq T)$ for the following sequence of observation dates: $t_1 = 0.5$, $t_2 = 1$ and $t_3 = 1.5$. We have simulated the probability with the typical values of the parameters. The simulations resulted in a path with no default. The results show that there are jumps in $F^1$ and $\Gamma^1$ to zero at the times of observation of the firm’s assets. The jumps are with stochastic sizes depending on $V_{t_1}, V_{t_2}$ and $V_{t_3}$ for $t_2$, $t_3$ and $t_4$, respectively.

Figure 3 plots other simulation results for the same processes. In this example, the default occurs at time $\tau = 0.624$. At the next observation date, $t_2 = 1$, the conditional default probability jumps to one and the hazard process jumps to $+\infty$. The simulation results make it clear that, if there is a default between two observations, the process $F^1$ jumps to one at the second observation date.

**Lemma 1** The pure jump process $\zeta^1$ defined by $\zeta^1 = \sum_{i, t_i \leq \tau} \Delta F^1_{t_i}$ is an $H$-martingale.

**Proof:** Consider first the times $t_i \leq s < t \leq t_{i+1}$. In this case, it is obvious that $E(\zeta^1_s | H_s) = \zeta^1_s$ since $\zeta^1_t = \zeta^1_s = \zeta^1_t$, which is $H_s$-measurable.

It suffices to show that $E(\zeta^1_s | H_s) = \zeta^1_s$ for $t_i \leq s < t_{i+1} \leq t < t_{i+2}$. In this case $\zeta^1_s = \zeta^1_t + \Delta F^1_{t_{i+1}}$ and $\zeta^1_s = \zeta^1_t$. We have

$$E(\Delta F^1_{t_{i+1}} | H_s) = E\left( \mathbb{1}_{\tau > t_i} \mathbb{1}_{\tau \leq t_{i+1}} \Phi(t_{i+1} - t_i, a - X_{t_i}) + \mathbb{1}_{\tau > t_{i+1}} (\Phi(t_{i+1} - t_i, a - X_{t_i}) - 1) | H_s \right)$$

$$= \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i}) - E(\mathbb{1}_{\tau > t_{i+1}} | H_s)$$

$$= \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i}) - E(\mathbb{1}_{\tau > t_{i+1}} | F^1_{t_i})$$

$$= \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i}) - \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i})$$

$$= 0.$$

$\triangle$
3.4 Hazard process and default arrival intensity

Let $\mathbf{F}^* = (\mathcal{F}^*_t, t \geq 0)$ be some reference filtration, $F^*_t = P(\tau \leq t | \mathcal{F}^*_t)$ and $\mathcal{G}_t = \mathcal{F}^*_t \vee \sigma(\tau \wedge t)$. Write $\Gamma^*_t = - \ln(1 - F^*_t)$ for the $\mathbf{F}^*$-hazard process. The hazard process approach (Elliott et al. 2000, Jeanblanc and Rutkowski, 2000b) states that if $X$ is some integrable, $\mathcal{G}$-measurable random variable, then

$$E[X \mathbb{1}_{T < \tau} | \mathcal{G}_t] = \mathbb{1}_{t < \tau} E \left[ X e^{\Gamma_t - \Gamma_T} | \mathcal{F}^*_t \right].$$

The case of a default-risky bond with unit face value and zero recovery corresponds to $X = 1$. In what follows, we will write $D_t := \mathbb{1}_{t < \tau}$.

In the case where $\mathbf{F}^*$ is increasing and continuous, the process

$$D_t - \Gamma^*(t \wedge \tau) = D_t - \int_0^{t \wedge \tau} \frac{dF^*_s}{1 - F^*_s}$$

is a martingale.

If $\mathbf{F}^*$ is increasing (this is called condition (G) in Jeanblanc and Rutkowski, 2000b), the process

$$D_t - \int_0^{t \wedge \tau} \frac{dF^*_s}{1 - F^*_s}$$

is a martingale and $\Lambda^*_t \wedge \tau = \int_0^{t \wedge \tau} \frac{dF^*_s}{1 - F^*_s}$ is an increasing process. Intensity $\lambda^*$ is defined as

$$\lambda^*_t dt = \frac{dF^*_t}{1 - F^*_t}.$$

In the general case, the submartingale $F^*$ admits the decomposition $F^* = Z^* + A^*$ where $A^*$ is a predictable increasing process and $Z^*$ is an $\mathbf{F}^*$-martingale, we have that

$$D_t - \int_0^{t \wedge \tau} \frac{dA^*_s}{1 - F^*_s}$$

is a martingale.

Accordingly, the Doob-Meyer decomposition of the submartingale $(F^*_t, t \geq 0)$ is

$$F^*_t = \zeta^*_t + (F^*_t - \zeta^*_t),$$

where $\zeta^*$ is an $\mathbf{H}$-martingale; $F^{1,c} := F^1 - \zeta^1$ is a continuous, hence predictable increasing process. Since $F^1$ is not increasing, the hypothesis (H) does not hold.

From (10), the intensity of the default time is the $\mathbf{H}$-adapted process $\lambda^1$ defined as

$$\lambda^1_t dt = \frac{d(F^1_t - \zeta^1_t)}{1 - F^1_{t-}}.$$
Figure 1: Drifted Brownian motion path and the default boundary

\[ X_t = \nu t + W_t \]
\[ a = \frac{1}{\sigma} \log \left( \frac{\alpha}{V_0} \right) \]

Figure 2: Default probability and hazard process with information of the first type

Conditional default probability

Hazard process
Figure 3: Default probability and hazard process with information of the first type (path with default)

Figure 4: $T$-forward default probability with information of the first type
3.5 Implications for bond prices

When bondholders have access only to the filtration $\mathcal{H}$, the defaultable bond price is

$$E \left( e^{-r(T-t)} \mathbb{1}_{\tau > T} + \frac{\alpha}{D} e^{-r(T-t)} \mathbb{1}_{\tau \leq T} | \mathcal{H}_t \right) = e^{-r(T-t)} P(\tau > T | \mathcal{H}_t) + \frac{\alpha}{D} E \left( e^{-r(T-t)} \mathbb{1}_{\tau \leq T} | \mathcal{H}_t \right).$$

As in the full information case, the first term on the right-hand side is the value of a zero-recovery defaultable bond, while the second term is the value of the rebate. The value of the zero-recovery defaultable bond depends mainly on the conditional probability of survival to maturity. Indeed, following the same computations as in the previous section

$$P(\tau > T | \mathcal{H}_t) = \Phi(T - t_i, a - X_{t_i})$$

for $t_i \leq t < t_{i+1}$. (12)

Hence, the conditional probability of default up to maturity is

$$P(\tau < T | \mathcal{H}_t) = 1 - \Phi(T - t_i, a - X_{t_i})$$

for $t_i \leq t < t_{i+1}$.

Figure 4 illustrates the conditional probability of default up to the maturity of the bond. Clearly, the conditional default probability is piecewise constant. It jumps at the observation dates with stochastic sizes depending on asset values. The jumps induce jumps in the price of a zero-recovery defaultable bond. This is in contrast to the complete observation case, where the defaultable bond price jumps to the recovery value only at the default time.

In the literature, it is usually assumed that the default is announced and the default time is observed. The following theorem provides an analytical valuation formula for a zero recovery bond.

**Theorem 1** When the information available to bondholders is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$, the price of a zero-recovery defaultable bond is

$$\hat{Z}^1(t, T) = e^{-r(T-t)} \mathbb{1}_{t \leq \tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})}. \tag{13}$$

**Proof:** The price of a zero-recovery defaultable bond is given by the discounted value of the conditional survival probability, i.e.,

$$\hat{Z}^1(t, T) = e^{-r(T-t)} P(T < \tau | \mathcal{H}_t \vee \sigma(\tau \wedge t)) = e^{-r(T-t)} \mathbb{1}_{t \leq \tau} \frac{P(T < \tau | \mathcal{H}_t)}{P(t < \tau | \mathcal{H}_t)}. \tag{14}$$

To compute the fraction on the right-hand-side, we use the conditional survival probability in (12) and the fact that

$$P(\tau > s | \mathcal{H}_t) = \mathbb{1}_{\tau > t_i, \Phi(s - t_i, a - X_{t_i})} \quad \text{for} \ s \geq t,$$
hence,

\[ P(T < \tau | \mathcal{H}_t \vee \sigma(\tau \wedge t)) = \mathbb{I}_{t<\tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})}. \]

A substitution of this expression in (14) yields the result.

The price of the rebate part when the information is \( \mathcal{H}_t \vee \sigma(\tau \wedge t) \) equals

\[ \hat{R}^1(t, T) = \frac{\alpha}{D} E \left( e^{-r(\tau - t)} \mathbb{I}_{t<\tau} | \mathcal{H}_t \vee \sigma(\tau \wedge t) \right) = \frac{\alpha}{D} \frac{\mathbb{I}_{t<\tau}}{P(t < \tau | \mathcal{H}_t)} E \left( e^{-r(\tau - t)} \mathbb{I}_{t<\tau} | \mathcal{H}_t \right). \]

Assume that \( t_i < t < t_{i+1} \). From the Markov property,

\[ E \left( e^{-r(\tau - t_i)} \mathbb{I}_{t<\tau} | \mathcal{H}_t \right) = E \left( e^{-r(\tau - t_i)} \mathbb{I}_{t<\tau} | \mathcal{F}^V_{t_i} \right). \]

One can write

\[ E \left( e^{-r(\tau - t_i)} \mathbb{I}_{t<\tau} | \mathcal{H}_t \right) = E \left( e^{-r(\tau - t)} \mathbb{I}_{t<\tau} | \mathcal{H}_t \right). \]

where \( \Psi(x, u) = E(e^{-r\tau} \mathbb{I}_{\tau < u}) \) with \( \tau = \inf\{s : \nu s + W_s \leq x\} \). It follows that

\[ \mathbb{I}_{t<\tau} E \left( e^{-r(\tau - t)} \mathbb{I}_{t<\tau} | \mathcal{H}_t \right) = \mathbb{I}_{t<\tau} E \left( e^{-r(\tau - t)} \mathbb{I}_{t<\tau} | \mathcal{F}^V_{t_i} \right) = \mathbb{I}_{t<\tau} e^{r(t - t_i)} \left( E \left( e^{-r(\tau - t)} \mathbb{I}_{t<\tau} | \mathcal{F}^V_{t_i} \right) \right) = \mathbb{I}_{t<\tau} \left( \Psi(a - X_{t_i} - T - t) - \Psi(a - X_{t_i}, t - t_i) \right). \]

The computation of \( \Psi(\cdot, \cdot) \) can be found in Jeanblanc et al. (2003). For \( x = a - X_{t_i} < 0 \),

\[ \Psi(x, t) = e^{(\nu - \gamma)x} N \left( \frac{-\gamma t + x}{\sqrt{t}} \right) + e^{(\nu + \gamma)x} N \left( \frac{\gamma t + x}{\sqrt{t}} \right) = e^{(\nu - \gamma)x} \left[ 1 - N\left( \frac{\gamma t - x}{\sqrt{t}} \right) + e^{2\gamma x} N\left( \frac{\gamma t + x}{\sqrt{t}} \right) \right] = e^{(\nu - \gamma)x} \left[ 1 - \Phi(\gamma, t, x) \right] \]

with \( \gamma = \sqrt{2r + \nu^2} \). Therefore,

\[ \Psi(a - X_{t_i}, T - t) - \Psi(a - X_{t_i}, t - t_i) = e^{(\nu - \gamma)(a - X_{t_i})} \left\{ \Phi(\gamma, t - t_i, a - X_{t_i}) - \Phi(\gamma, T - t_i, a - X_{t_i}) \right\}. \]

Consequently, the value of the rebate is given by

\[ \hat{R}^1(t, T) = \frac{\alpha}{D} \frac{\mathbb{I}_{t<\tau} e^{r(t - t_i)} e^{(\nu - \gamma)(a - X_{t_i})} \left\{ \Phi(\gamma, t - t_i, a - X_{t_i}) - \Phi(\gamma, T - t_i, a - X_{t_i}) \right\}}{\Phi(t - t_i, a - X_{t_i})}. \]
Therefore, for $t_i < t < t_{i+1}$, the price of a defaultable bond when the information available on the market is $\mathcal{H}_t \lor \sigma(t \land t)$ is given by

\[
\hat{B}_1^d(t, T) = e^{-r(T-t)} \mathbb{I}_{t < \tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})} + \frac{\alpha}{D} e^{r(T-t)} e^{(\nu - \gamma)(a - X_{t_i})} \Phi(t - t_i, a - X_{t_i}) \{ \Phi(\gamma, t - t_i, a - X_{t_i}) - \Phi(\gamma, T - t_i, a - X_{t_i}) \}.
\]

A comparison between $B_1^d(t, T)$ and $\hat{B}_1^d(t, T)$ shows that the former depends on contemporaneous asset value (as indicated by the presence of $X_t$ in (8)), while the latter depends on asset value at the last observation date (through $X_{t_i}$ in (17)). Of course, both bond prices depend on whether the default-triggering barrier has been reached. However, it is clear that when the bondholders have access to $\mathbb{F}_V$, they use more up-to-date information to value default-risky bonds.

### 4 Second type of information

In this section, we introduce a second type of incomplete information, which is a further restriction of that of the first type. As in Duffie and Lando (2001), suppose that the bondholders observe asset value at the discrete times

\[ T = \{t_1, t_2, \ldots, t_n\}. \]

These dates may correspond to the dates of release of periodic accounting reports, such as balance sheets, profit and loss statements and cash flow statements by the firm. We assume that bondholders observe only the contemporaneous asset value rather than the contemporaneous and all the past asset values as was the case in the previous section. We denote by $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ the filtration generated by $V$ at dates $t_1, \ldots, t_n$. It follows that $\mathcal{F}_t$ is trivial for $t < t_1$, i.e.,

\[
\mathcal{F}_t = \{ \emptyset, \Omega \} \quad \text{for } t < t_1,
\]

\[
\mathcal{F}_t = \mathcal{F}_{t_1} = \sigma(V_{t_1}) = \sigma(X_{t_1}) \quad \text{for } t_1 \leq t < t_2,
\]

\[
\mathcal{F}_t = \mathcal{F}_{t_2} = \sigma(V_{t_1}, V_{t_2}) = \sigma(X_{t_1}, X_{t_2}) \quad \text{for } t_2 \leq t < t_3,
\]

and so on.

At each observation date $t_i$, the filtration $\mathbb{F}$ is enlarged with the observation of asset value at that date. Therefore, we have $\mathcal{F}_{t_i} = \mathcal{F}_{t_{i-1}} \lor \sigma(V_{t_i})$. Therefore, bondholders have access to a subfiltration of the filtration $\mathcal{H}$ (i.e., $\mathbb{F} \subset \mathcal{H} \subset \mathbb{F}_V$).

In what follows, we compute the conditional default probability with respect to this information $F_t^2 := P(\tau \leq t|\mathcal{F}_t)$. 

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4.1 On $t < t_1$

As with the first type of information, before $t_1$ bondholders do not observe anything, and we have

$$F_t^2 = 1 - \Phi(t, a).$$  \hspace{1cm} (18)

The default probability is deterministic.

4.2 On $t_1 < t < t_2$

In this interval, we have

$$F_t^2 = P(\tau \leq t|X_{t_1}) = 1 - P(\tau > t|X_{t_1}),$$  \hspace{1cm} (19)

where

$$P(\tau > t|X_{t_1}) = P\left(\inf_{s < t_1} X_s > a|X_{t_1}\right) = P\left(\inf_{s < t_1} X_s > a, \inf_{t_1 \leq u < t} X_u > a|X_{t_1}\right) = E\left(\mathbb{1}_{\inf_{s < t_1} X_s > a} P\left(\inf_{t_1 \leq s < t} X_s > a|\mathcal{F}_{t_1}^V\right)|X_{t_1}\right).$$

Using (6), we can evaluate the probability inside the expectation

$$P\left(\inf_{t_1 \leq s < t} X_s > a|\mathcal{F}_{t_1}^V\right) = \Phi\left(t - t_1, a - X_{t_1}\right).$$

Substituting above, we obtain

$$F_t^2 = 1 - \Phi(t - t_1, a - X_{t_1})P\left(\inf_{s < t_1} X_s > a|X_{t_1}\right).$$  \hspace{1cm} (20)

The term $P(\inf_{s < t_1} X_s > a|X_{t_1})$ corresponds to a drifted Brownian bridge. In the following lemma, this term is computed by using the joint law of the infimum and the current position of a Brownian motion with drift.

**Lemma 2** Let $X_t = W_t + \nu t$ and $m_t^X = \inf_{s \leq t} X_s$. Then, for $y < 0$, $x < x$

$$P(m_t^X \leq y|X_t = x) = \exp\left(-\frac{2}{\nu} y(y - x)\right).$$  \hspace{1cm} (21)

**Proof:** This is a classic result, we provide a demonstration to facilitate understanding. We know that, for $y \leq 0$, $y \leq x$,

$$P(X_t \geq x, m_t^X \leq y) = e^{2\nu y} \mathcal{N}\left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right).$$  \hspace{1cm} (22)
Taking a derivative with respect to $x$, leads to
\[ P(X_t \in dx, m^X_t \leq y) = \frac{1}{\sqrt{2\pi t}} e^{2\nu y} \exp\left(-\frac{1}{2} \left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right)^2\right) dx. \]

By the definition of $X$, we have
\[ P(X_t \in dx) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2} \left(\frac{x - \nu t}{\sqrt{t}}\right)^2\right) dx. \]

Since
\[ P(m^X_t \leq y|X_t = x) = \frac{P(X_t \in dx, m^X_t \leq y)}{P(X_t \in dx)}, \]
from
\[
2\nu y - \frac{1}{2} \left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right)^2 + \frac{1}{2} \left(\frac{x - \nu t}{\sqrt{t}}\right)^2 = 2\nu y - \frac{1}{2t} \left(4y^2 + 4y(-x + \nu t)\right) = \frac{2}{t} y(y - x),
\]
the equality follows.

\[ \triangle \]

Note that (21) does not depend on $\nu$. This is, in fact, a consequence of Girsanov’s theorem.

Since
\[ P(\inf_{s\leq t} X_s > a|X_t = x) = \Phi\left(t - t_1, a - X_{t_1}\right) \left[1 - \exp\left(-\frac{2}{t_1} a(a - X_{t_1})\right)\right], \]
the process $F^2$ is continuous and increasing in $[t_1, t_2]$. When $t$ approaches $t_1$ from above, for $X_{t_1} > a$, $F^2_{t_1} = \exp\left(-\frac{2}{t_1} a(a - X_{t_1})\right)$, because $\lim_{t\to t_1^+} \Phi(t - t_1, a - X_{t_1}) = 1$.

For $X_{t_1} > a$, the jump of $F^2$ at $t_1$ is
\[ \Delta F^2_{t_1} = \exp\left(-\frac{2}{t_1} a(a - X_{t_1})\right) - 1 + \Phi(t_1, a). \]

For $X_{t_1} \leq a$, $\Phi(t - t_1, a - X_{t_1}) = 0$ by the definition of $\Phi(\cdot)$ and
\[ \Delta F^2_{t_1} = \Phi(t_1, a). \]

### 4.3 General observation times $t_i < t < t_{i+1} < T$, $i \geq 2$

For $t_i < t < t_{i+1}$,
\[
P(\tau > t|X_{t_i}, \ldots, X_{t_i}) = P\left(\inf_{s \leq t} X_s > a \left| P(\inf_{t_i \leq s < t} X_s > a|\mathcal{F}_t)|X_{t_1}, \ldots, X_{t_i}\right)\right)
= \Phi(t - t_i, a - X_{t_i}) P\left(\inf_{s \leq t} X_s > a | X_{t_1}, \ldots, X_{t_i}\right).
\]
Write $K_i$ for the second term on the right-hand-side. It can be rewritten as

$$K_i = P\left( \inf_{s \leq t_i} X_s > a \left| X_{t_1}, \ldots, X_{t_i} \right. \right)$$

$$= P\left( \inf_{s \leq t_{i-1}} X_s > a \left| \inf_{t_{i-1} \leq s < t_i} X_s > a \left| X_s : s \leq t_{i-1}, X_{t_i} \right. \right. \right)_{X_{t_1}, \ldots, X_{t_i}}.$$

An analytical expression for the inside member can be obtained as follows:

$$P\left( \inf_{t_{i-1} \leq s < t_i} X_s > a \left| X_{t_1}, \ldots, X_{t_i} \right. \right) = P\left( \inf_{t_{i-1} \leq s < t_i} X_s > a \left| \inf_{t_i - t_{i-1} \leq s < t_i} X_s > a \left| X_{t_1}, X_{t_i} \right. \right. \right)_{X_{t_1}, \ldots, X_{t_i}}.$$

Therefore,

$$K_i = K_{i-1} \exp \left( -\frac{2}{t_i - t_{i-1}}(a - X_{t_{i-1}})(a - X_{t_i}) \right), \quad (24)$$

which can be solved recursively by using the initial condition $K_1$ computed in Section 4.1.

Hence, the general formula for the conditional default probability $F^2_t$ reads

$$P(\tau \leq t \left| F_t \right. ) = 1 \quad \text{if } X_{t_j} < a \text{ for at least one } t_j, t_j < t$$

$$= 1 - \Phi(t - t_i, a - X_{t_i})K_i,$$

where

$$K_i = 1 - \exp \left( -\frac{2}{t_1}a(a - X_{t_1}) \right) \left( 1 - \exp \left( -\frac{2}{t_2 - t_1}(a - X_{t_1})(a - X_{t_2}) \right) \right) \quad \ldots$$

$$\left( 1 - \exp \left( -\frac{2}{t_i - t_{i-1}}(a - X_{t_{i-1}})(a - X_{t_i}) \right) \right).$$

Let $\zeta^2$ be the pure jump process defined by

$$\zeta^2_t = \sum_{i: t_i \leq t} \Delta F^2_{t_i}.$$

**Lemma 3** The process $\zeta^2$ is an $F$-martingale.

**Proof:** Consider first the times $t_i \leq s < t \leq t_{i+1}$. In this case, it is obvious that $E(\zeta^2_s | H_s) = \zeta^2_s$ since $\zeta^2_t = \zeta^2_s = \zeta^2_{t_i}$, which is $H_s$-measurable.

It suffices to show that $E(\zeta^2_s | F_s) = \zeta^2_s$ for $t_i \leq s < t_{i+1} \leq t < t_{i+2}$. In this case, $\zeta^2_s = \zeta^2_{t_i}$ and $\zeta^2_t = \zeta^2_{t_i} + \Delta F^2_{t_{i+1}}$. Therefore,

$$E(\zeta^2_s | F_s) = E(\zeta^2_{t_i} + \Delta F^2_{t_{i+1}} | F_s)$$

$$= \zeta^2_{t_i} + E(\Delta F^2_{t_{i+1}} | F_s),$$

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which shows that it is necessary to prove that \( E(\Delta F_{t_{i+1}}^2 | \mathcal{F}_s) = 0 \).

Let \( s < u < t_{i+1} < v < t \). Then,

\[
E(F_v^2 - F_u^2 | \mathcal{F}_s) = E(\mathbb{1}_{u < \tau \leq v} | \mathcal{F}_s).
\]

When

\[
v \to t_{i+1}, \quad v > t_{i+1} \quad \text{and} \quad u \to t_{i+1}, \quad u < t_{i+1}, \quad F_v^2 - F_u^2 \to \Delta F_{t_{i+1}}^2.
\]

It follows that

\[
E(\Delta F_{t_{i+1}}^2 | \mathcal{F}_s) \to \lim_{u \to t_{i+1}} \lim_{v \to t_{i+1}} E(\mathbb{1}_{u < \tau \leq v} | \mathcal{F}_s) = E(\mathbb{1}_{\tau = t_{i+1}} | \mathcal{F}_s) = 0.
\]

The Doob-Meyer decomposition of \( F^2 \) is

\[
F_t^2 = \zeta_t^2 + (F_t^2 - \zeta_t^2),
\]

where \( \zeta_t^2 \) is an \( \mathcal{F} \)-martingale and \( F^2.c := F^2 - \zeta^2 \) is a continuous, hence predictable, increasing process. Duffie and Lando (2001) established that when the information on the market is \( \mathcal{F} \lor \sigma(t \land \tau) \), the time of default is totally inaccessible in this filtration and admits intensity. From the results of Elliott et al. (2000), the \( \mathcal{F} \)-intensity of \( \tau \) is the process \( \lambda^2 \) defined as

\[
\lambda_t^2 dt = \frac{d(F_t^2 - \zeta_t^2)}{1 - F_t^2 - \zeta_t^2}.
\]

The process \( (F_t^2 - \zeta_t^2, t \geq 0) \) is absolutely continuous with respect to the Lebesgue measure.

5 Comparison with intensity methods

In this section, we evaluate the efficiency and compare the intensity and the hazard process approaches to valuation of default-risky bonds. The martingale hazard process of \( \tau \) is the predictable increasing process \( \Lambda \), such that \( \mathbb{1}_{\tau \leq t} - \Lambda(t \land \tau) \) is a martingale in the reference filtration. Let us consider the intensity-based approach when the information is of the first type. Then, using the results of Elliott et al. (2000), or, more generally, some results from enlargement of filtrations, the \( \mathcal{H} \)-martingale hazard process of \( \tau \) is the \( \mathcal{H} \)-adapted process

\[
\Lambda_t^1 = \frac{d(F_t^1.c)}{1 - F_t^1}.
\]
Moreover, for \( t_n < t < t_{n+1} \), from the continuity of \( F^{1,c} \)

\[
\Lambda^1_t = \int_{[0,t]} \frac{dF^{1,c}_s}{1 - F^{1,c}_s} = \sum_{i=0}^{n-1} \int_{[t_i,t_{i+1}]} \frac{dF^{1,c}_s}{1 - F^{1,c}_s} + \int_{[t_n,t]} \frac{dF^{1,c}_s}{1 - F^{1,c}_s}
\]

and on \([t_i,t_{i+1}]\), \( F^{1,c} \) is differentiable. Therefore, we can write

\[
\Lambda^1_t = \sum_{i=0}^{n-1} \int_{[t_i,t_{i+1}]} \frac{dF^{1,c}_s}{1 - F^{1,c}_s - \zeta^1_{t_i}} + \int_{[t_n,t]} \frac{dF^{1,c}_s}{1 - F^{1,c}_s - \zeta^1_{t_n}}
\]

\[
= \sum_{i=0}^{n-1} - \ln(1 - F^{1,c}_s - \zeta^1_{t_i})|_{t_i}^{t_{i+1}} - \ln(1 - F^{1,c}_s - \zeta^1_{t_n})|_{t_n}
\]

\[
= - \sum_{i=0}^{n-1} \ln \frac{(1 - F^{1,c}_{t_{i+1}} - \zeta^1_{t_i})}{(1 - F^{1,c}_{t_i} - \zeta^1_{t_i})} - \ln \frac{(1 - F^{1,c}_{t_{n}} - \zeta^1_{t_n})}{(1 - F^{1,c}_{t_n} - \zeta^1_{t_n})}.
\]

(25)

The process \( F^{1,c}_s \) is differentiable and \( dF^{1,c}_s = f^{1,c}_s \, ds \). Hence, the \( H \)-intensity of \( \tau \) is \( \lambda^1_s = \frac{f^{1,c}}{1 - F^{1,c}_s - \zeta^1_s} \) on the interval \([t_i,t_{i+1}]\). Therefore, the \( H \)-martingale hazard process \( \Lambda^1 \) admits the following integral representation:

\[
\Lambda^1_t = \int_0^t \lambda^1_s \, ds.
\]

(26)

One can use this result for valuation of a zero-recovery default-risky bond with maturity \( T \).

However, working with the hazard process \( \Gamma^1 \) leads to a price of a defaultable bond with zero recovery as

\[
Z^1(t,T) = \mathbb{1}_{t<\tau} e^{-r(T-t)} E \{ \exp(\Gamma^1_t - \Gamma^1_T) | \mathcal{H}_t \}
\]

\[
= \mathbb{1}_{t<\tau} e^{-r(T-t)} \frac{1}{1 - F^1_t} E \{ 1 - F^1_T | \mathcal{H}_t \}
\]

\[
= \mathbb{1}_{t<\tau} e^{-r(T-t)} \frac{1}{1 - F^1_t} \{ E(1 - F^1_T - \zeta^1_T) | \mathcal{H}_t) \}
\]

\[
= \mathbb{1}_{t<\tau} e^{-r(T-t)} \frac{1}{1 - F^1_t - \zeta^1_t} \{ 1 - E(F^1_T | \mathcal{H}_t) - \zeta^1_t \}.
\]

(27)

where the fourth equality follows because the process \( \zeta^1 \) is an \( H \)-martingale.

It is rather obvious that the hazard process approach is more efficient. Using the intensity approach leads to a complicated valuation formula with expectation of product terms including jumps in the conditional default probability at the observation dates.

Likewise, when the information available on the market is of the second type, the \( F \)-martingale hazard process is

\[
\Lambda^2_t = - \sum_{i=0}^{n-1} \ln \frac{(1 - F^{2,c}_{t_{i+1}} - \zeta^2_{t_i})}{(1 - F^{2,c}_{t_i} - \zeta^2_{t_i})} - \ln \frac{(1 - F^{2,c}_{t_{n}} - \zeta^2_{t_n})}{(1 - F^{2,c}_{t_n} - \zeta^2_{t_n})}.
\]

(28)

The hazard process approach yields

\[
Z^2(t,T) = \mathbb{1}_{t<\tau} e^{-r(T-t)} \frac{1}{1 - F^{2,c}_t - \zeta^2_t} \{ 1 - E(F^{2,c}_T | \mathcal{F}_t) - \zeta^2_t \}.
\]
We investigated two cases, in which the intensity approach is not very efficient for valuation of default-contingent claims. Computation of integrals of intensity are non trivial. Moreover, when the information includes also observations of $\tau$, the process
\[ E \left[ e^{\Lambda^1_t - \Lambda^1_T} \middle| \mathcal{H}_t \vee \sigma(\tau \wedge t) \right] \]
has a jump at time $\tau$ and the no-jump condition of Duffie (1998a, 1998b) is not satisfied. One has to compute this jump in order to give the price of a contingent claim. These computations are long while obtaining the price $\hat{Z}^1(t, T)$ by the hazard process approach is much simpler.

6 Conclusion

This research investigated in some details the distributional properties of default time with incomplete information about asset value. As in Duffie and Lando (2001) we considered public information obtained on a discrete sequence of dates, but we introduced also another type of such information. In contrast to the full information case (in which the information is represented by a Brownian filtration generated by continuous observations of asset value), the two information filtrations are piecewise-constant and jump-discontinuous, incorporating surprises at the observation dates.

The discrete information arrivals induced jumps in both the default probabilities and the hazard processes. While the time of default is totally inaccessible in the asset filtrations and admits intensity, since the hypothesis (H) is not satisfied, it is better avoiding the intensity-based approach to valuation. Working with the intensity approach, the information about the contemporaneous jump sizes in the hazard process and the conditional default probability, at the observation dates, is lost. The hazard process approach to the valuation of credit-risky bonds should be preferred, since it uses all the information about the cumulative jump sizes and leads to much simpler valuation formulas.
References


