American Options with Stopping Time Constraints to Executive Stock Options

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American Options with Stopping Time Constraints *

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American Options with Stopping Time Constraints

Abstract

This paper concerns the pricing of American options with stochastic stopping time constraints expressed in terms of the states of a Markov process. Following the ideas of Menaldi, Robin, and Sun (1996) we transform the constrained into an unconstrained optimal stopping problem. The transformation replaces the original payoff by the value of a generalized barrier option. We suggest a new Monte Carlo method to numerically calculate the option value also for multidimensional Markov processes. Because of presence of stopping time constraints the classical Longstaff-Schwartz least-square Monte Carlo algorithm or its extension introduced in Egloff (2005) cannot be directly applied. We adapt the Longstaff-Schwartz algorithm to solve the stochastic Cauchy-Dirichlet problem related to the valuation problem of the barrier option along a set of simulated trajectories of the underlying Markov process.

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1 Introduction

We consider a new type of American options with constrained exercise rules. American options allow for early exercise before the contract expires. Well-known examples of constrained American options are Bermudan options, where exercise is restricted to a deterministic and predefined discrete set of dates. In contrast to Bermudan options, we introduce a stochastic constraint on the exercise rule. This constraint is expressed in terms of the states of a Markov process. As an example, we can link the exercise rule to a stochastic performance condition. Such performance-based constraints not only play an important role for structuring new investment products, but also for the design of executive stock option plans with exercise constraints based, e.g., on the out-performance of a reference index.

We show that the pricing problem of American options with stopping time constraints is the solution of an unconstrained optimal stopping problem. To do so, we adapt the technique of Menaldi, Robin, and Sun (1996). In particular, if the Markov process is a Feller process, the constrained optimal stopping problem can be transformed into an unconstrained problem by replacing the original payoff with the value process of a generalized barrier option. The valuation of this barrier option is related to a stochastic Cauchy-Dirichlet problem. Once we have transformed the constrained into an unconstrained pricing problem, we can adapt the classical Longstaff-Schwartz least-square Monte Carlo method (see Longstaff and Schwartz (2001), or its extensions Egloff (2005)) to determine the value of the generalized barrier option and from there the price of the constrained American option contract. The algorithm we suggest is a nested Monte Carlo simulation, which calculates the value of the barrier option along a set of simulated trajectories of the underlying Markov process.

We structure the remainder of the paper as follows. Section 2 introduces the notation and the model setup. In Section 3, we derive the arbitrage-free pricing formula for the American option pricing problem with stochastic constraints on the exercise rule. Section 4 illustrates the numerical implementation based on Monte Carlo approximation. Section 5 concludes.
2 Model Setup

Consider a financial market of \( d \) traded securities whose price process \( X_t \) is a Markov process with values in \( \mathbb{R}^d \). To represent the stopping time constraints, we assume that there exists an \( \mathcal{A} \)-valued stochastic process \( A_t \) where \( \mathcal{A} \) is a locally compact Hausdorff space with countable base (LCCB) such that the joint process

\[
Y_t^y = \left( s + t, X_{s+t}^{(s,x)}, A_{s+t}^{(s,a)} \right), \quad y = (s, x, a)
\]

is a time-homogeneous Feller process on the state space \( \mathbb{E} = \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{A} \).

According to the general theory of Markov processes, \( Y_t^y \) has a realization on the space \( \Omega = D([0, \infty), \mathbb{E}) \) of right continuous \( \mathbb{E} \)-valued functions \( \omega \) with left-hand limits. Let \( Y_t = \omega(t) \) denote the coordinate projection and introduce the \( \sigma \)-algebras \( \mathcal{F}^\omega = \sigma(Y_s | s \geq 0) \) and \( \mathcal{F}_t^\omega = \sigma(Y_s | 0 \leq s \leq t) \). Then, there exists a unique probability measure \( Q^y \) on \( (\Omega, \mathcal{F}^\omega) \) such that if \( \mathcal{F}_\tau^\omega \) is the \( Q^y \)-completion of \( \mathcal{F}_\tau^\omega \) and \( \mathcal{F}_t^\omega \) is the \( Q^y \) completion of \( \mathcal{F}_t^\omega \), then the process

\[
(\Omega, \mathcal{F}^\omega, \mathcal{F}_t^\omega = (\mathcal{F}^\omega_t)_{t \geq 0}, Q^y, Y_t)
\]

is a right continuous, quasi-left continuous strong Markov process with \( Q^y(Y_0 = y) = 1 \). The quasi-left continuity means that if a jump occurs at a stopping time, then this stopping time must be totally inaccessible.

**Definition 2.1.** Let \( \mathcal{A} \subset \mathbb{E} \) be a closed subset of states admissible for stopping. A *constrained American option* with underlying \( X_t \), constraint process \( A_t \), admissible stopping states \( \mathcal{A} \), and payoff function \( f : \mathbb{E} \to [0, \infty) \) is a contract, which can be exercised at time \( t \) to pay the cash flow \( f(Y_t) \) to the option holder, if and only if the condition \( Y_t \in \mathcal{A} \) is satisfied.

To exclude integrability issues, we assume from now on that payoff processes \( f(Y_t) \) are at least of class D, which means that the families of random variables \( \{ f(Y_\tau) | \tau \in T_{[0,T]} \} \) are uniformly integrable.\(^1\)
Definition 2.2. The set of $A$-admissible or just admissible $\mathbb{F}^y$-stopping times is defined by

$$T_{[t,T]}(A) = \{ \tau \in T_{[t,T]} \mid Y_\tau \in A \text{ a.s.} \},$$

where $T_{[t,T]}$ are the $\mathbb{F}^y$-stopping times $\tau$, satisfying $t \leq \tau \leq T$.

Let $\theta_t : \Omega \to \Omega$ be the canonical shift operator defined as $\theta_t(\omega) = \omega(t + \cdot)$. If $\tau \in T_{[t,T]}$ is a $\mathbb{F}^y$-stopping time and $D_A = \inf\{ t \geq 0 \mid Y_t \in A \}$ is the first entrance time of $A$, then because $A$ is closed, $X_{D_A} \in A$ such that

$$\tau + D_A \circ \theta_\tau \in T_{[t,\infty]}(A)$$

is $A$-admissible. Note however that $\tau + D_A \circ \theta_\tau \in T_{[t,T]}(A)$ is generally false, unless for instance $Y_T \in A$ a.s.

At maturity $T$, the option contract expires regardless of the stopping constraints. Put differently, the option holder can always exercise at maturity, however, receives a zero payoff, if the constraints are not met. For this reason, we extend the state space $E$ of $Y_t$ to $\hat{E} = E \cup \{ \Delta \}$ by adding an additional state $\Delta$ and replace $Y_t$ by

$$\hat{Y}_t = Y_t \quad \forall t < T, \quad \hat{Y}_T = \begin{cases} Y_T & \text{if } Y_T \in A, \\ \Delta & \text{else.} \end{cases}$$

We also add $\Delta$ to the set $A$ of admissible stopping states and extend any payoff function

$$f : \hat{E} \to [0, \infty)$$

by setting $f(\Delta) = 0$. By this construction, we can always assume that $Y_T \in A$, which we assume from now on without further mentioning.
3 Arbitrage Valuation

In complete markets, the existence of a unique equivalent martingale measure and arbitrage arguments suffice to derive the unique prices for European as well as American derivative contracts. Early contributions to arbitrage pricing of American options in complete markets in continuous time are Bensoussan (1984) and Karatzas (1988) (see also the survey by Myneni (1992)). There, the well-known result emerges that the American option equals the supremum over all implied European options with their exercise times fixed by the stopping times, i.e.,

\[ V_0 = \text{ess sup}_{\tau \in \mathcal{T}_{[0,T]}} E_Q \left[ e^{-r\tau} f(\tau, X_{\tau}) \right] , \tag{3.1} \]

where \( Q \) is the unique equivalent martingale measure. However, in an incomplete market, the no arbitrage principle alone does not induce a unique equivalent martingale measure \( Q \). Instead, each \( Q \) defines a viable price of the contingent claim. Therefore, different approaches have been pursued to define additional criteria, either economically or mathematically motivated, to select an appropriate martingale measure. Examples of such approaches are superhedging (see, e.g., El Karoui and Quenez (1991), Kramkov (1996)), the minimal measure (see, e.g., Föllmer and Sondermann (1986)), the optimal variance martingale measure (see, e.g., Delbaen and Schachermayer (1994), Schweizer (1992)), and approaches based on risk measures (see Föllmer and Leukert (1999), Cvitanić and Karatzas (1995)) and utility considerations (see, e.g., Davis and Zariphopoulou (1995) and Kallsen and Kuehn (2004) for American options).

For American option pricing in incomplete markets, Kallsen and Kuehn (2004) show that we face a severe conceptual problem. In particular, the price of an American option no longer coincides with the supremum over all implied European options, i.e., the representation (3.1) does not hold in general. However, by introducing the concept of neutral derivative pricing, they are able to identify a pricing measure where (3.1) holds again.

For our paper, we proceed as follows. First, we present a general result that allows us to transform the constrained optimal stopping time problem into an unconstrained optimal stopping time problem. This result is independent of market completeness. Since our focus is not on the choice of the equivalent
martingale measure in incomplete markets, we assume that we can single out an equivalent martingale measure such that representation (3.1) is valid (see Kallsen and Kuehn (2004) for details). Then, in Section 3.2 we consider a complete market setup. We derive the lower and upper hedging prices and prove that these hedging prices collide.

3.1 General Result

The value $V_0$ of the constrained optimal stopping problem

$$V_0 = \text{ess sup}_{\tau \in \tau_{0,T}(A)} E_Q \left[ e^{-r\tau} f(Y_\tau) \right]$$  \hspace{1cm} (3.2)

defines an viable price of the American option with stopping time constraints, which does not lead to an arbitrage opportunity. The argument follows similar lines as in the case of an ordinary American option without stopping time constraints.

We assume that we have singled out an appropriate martingale measure $Q$, which we keep fixed from now on and write $E = E_Q$ for the expectation under $Q$. Let

$$v(y) = \sup_{\tau \in \tau_{0,T}(A)} E^y \left[ e^{-r\tau} f(Y_\tau) \right]$$  \hspace{1cm} (3.3)

denote the value function of the constrained optimal stopping problem (3.2).

Remark 3.1. Recall our setting $f(\Delta) = 0$. Since $f$ is positive, the supremum in (3.3) remains unchanged, if we restrict ourselves to the set $\{Y_\tau \neq \Delta\}$. Therefore we have in fact

$$v(y) = \sup_{\tau \in \tau_{0,T}(A)} E^y \left[ e^{-r\tau} f(Y_\tau) 1_{\{Y_\tau \neq \Delta\}} \right].$$  \hspace{1cm} (3.4)

This remark will allow us to apply in the proof of Theorem 3.3 the strong Markov property.

To reduce (3.2) to an unconstrained problem, we introduce the function

$$g(y) = E^y \left[ e^{-rD_A} f(Y_{DA}) \right],$$  \hspace{1cm} (3.5)
where
\[ D_A = \inf \{ s \geq 0 \mid Y_s \in A \} \] (3.6)
is the first entrance time of \( A \). Note that \( D_A \leq T \) by the construction (2.5). In mathematical terms, \( g \) is the solution of a stochastic Cauchy-Dirichlet problem. In financial terms, \( g \) is the value of a barrier option, which pays the exercise value of the American option as soon as the stopping constraints are fulfilled. We will use this option when we construct a hedging strategy in the case of complete markets (see Section 3.2), but first we elaborate on the relation between \( f \) and \( g \).

**Lemma 3.2.** For all \( \tau \in T_{[0,T]}(A) \) it follows that
\[ g(Y_\tau) = f(Y_\tau). \] (3.7)

**Proof.** By the strong Markov property
\[
g(Y_\tau) = \mathbb{E}^{Y_\tau}[e^{-rD_A} f(Y_{D_A})]
= \mathbb{E}^y[e^{-rD_A \circ \theta_\tau} f(Y_{\tau + D_A \circ \theta_\tau}) \mid \mathcal{F}_\tau]. \] (3.8)

But for \( \tau \in T_{[0,T]}(A) \), we have \( D_A \circ \theta_\tau = 0 \), so that
\[
\mathbb{E}^y[e^{-rD_A \circ \theta_\tau} f(Y_{\tau + D_A \circ \theta_\tau}) \mid \mathcal{F}_\tau] = f(Y_\tau).
\]

We now consider \( g \) as a new payoff and introduce the value function
\[
\hat{v}(y) = \sup_{\tau \in T_{[0,T]}} \mathbb{E}^y[e^{-r\tau} g(Y_\tau)] \] (3.9)
of an unconstrained optimal stopping problem for payoff \( g \). We can now solve the valuation problem (3.2) of American options with state space stopping time constraints. The following result is due to Menaldi, Robin, and Sun (1996), which we reproduce for completeness.
Theorem 3.3. Let $Y_t$ be a Feller process. Assume that the payoff process $f(Y_t)$ is a non-negative right continuous left limit process of class $D$. Then, the solution of the optimal stopping problems (3.3) and (3.9) coincide.

Proof. Adding the additional state $\Delta$ to guarantee that $T \in T_{[0,T]}(A)$ does not change the value of the optimal stopping problem, cf. also Remark 3.1. We may apply the strong Markov property of $Y_t$ to show the equality $v(y) = \hat{v}(y)$. For any stopping time $\tau$, equation (3.8) implies that

$$
\mathbb{E}^y \left[ e^{-r\tau} g(Y_\tau) \right] = \mathbb{E}^y \left[ e^{-r\tau} \mathbb{E}^{Y_\tau} \left[ e^{-rD_A f(Y_{D_A})} \right] \right] = \mathbb{E}^y \left[ e^{-r\tau} \mathbb{E}^y \left[ e^{-rD_A \circ \theta_{\tau}} f(Y_{\tau + D_A \circ \theta_{\tau}}) \mid \mathcal{F}_\tau \right] \right] = \mathbb{E}^y \left[ e^{-r(\tau + D_A \circ \theta_{\tau})} f(Y_{\tau + D_A \circ \theta_{\tau}}) \right].
$$

(3.10)

From (3.10) we get

$$
\hat{v}(y) \geq \sup_{\tau \in T_{[0,T]}(A)} \mathbb{E}^y \left[ e^{-r\tau} g(Y_\tau) \right] = \sup_{\tau \in T_{[0,T]}(A)} \mathbb{E}^y \left[ e^{-r(\tau + D_A \circ \theta_{\tau})} f(Y_{\tau + D_A \circ \theta_{\tau}}) \right] = \sup_{\tau \in T_{[0,T]}(A)} \mathbb{E}^y \left[ e^{-r\tau} f(Y_\tau) \right] = v(y).
$$

(3.11)

because $Y_\tau \in A$ implies $D_A \circ \theta_{\tau} = 0$. For the reverse inequality, note that $\tau + D_A \circ \theta_{\tau}$ satisfies the constraints for an arbitrary stopping time $\tau$. Here it is used that $A$ is a closed set implying $Y_{D_A} \in A$.

Therefore, by equation (3.10),

$$
\hat{v}(y) = \sup_{\tau \in T_{[0,T]}} \mathbb{E}^y \left[ e^{-r(\tau + D_A \circ \theta_{\tau})} f(Y_{\tau + D_A \circ \theta_{\tau}}) \right] \leq \sup_{\tau \in T_{[0,T]}(A)} \mathbb{E}^y \left[ e^{-r\tau} f(Y_\tau) \right] = v(y).
$$

(3.12)

\qed

To give the intuition behind Theorem 3.3 we briefly review what we have done so far. First, we replaced the payoff $f$ of the constrained American option by a new payoff $g$, which is the value process of a barrier option. This barrier option is alive once the set of admissible stopping states is hit. Second,
we showed that by solving the unconstrained stopping time problem for the payoff \( g \), we can solve the original constrained American option pricing problem. Therefore, in financial terms, the constrained American option can be interpreted as an ordinary American option with a transformed payoff given by the value process of a barrier option.

Next we analyze the existence of an optimal constrained stopping time. The existence of an optimal stopping time for the unconstrained problem (3.9) requires some sort of regularity on the process \( g(Y_t) \). The continuity of \( g \) would certainly be sufficient and is also desirable from a numerical standpoint, because it improves its approximation capabilities. However, the Feller property of \( Y_t \) alone is not sufficient to guarantee that \( g \) is continuous. On the other hand the representation (3.5) suggests that \( g \) is the solution of a PIDE boundary value problem provided that its existence, uniqueness, and proper regularity can be established. Because such a program is difficult to realize, we settle for minimal regularity requirements.

**Proposition 3.4.** If \( Y_t \) is a Feller process and \( f : \mathbb{E} \rightarrow [0, \infty) \) a continuous payoff function satisfying

\[
E \left[ \sup_{0 \leq t \leq T} \left( e^{-rt} f(X_t) \right)^p \right] < \infty
\]  

(3.13)

for some \( p > 1 \), then \( g(Y_t) \) is a right continuous quasi-left continuous process of class D and

\[
\tau^*_0 + D_A \circ \theta^*_\tau = \inf \{ s \geq \tau^*_0 \mid Y_s \in A \}
\]  

(3.14)

is an optimal stopping time, where

\[
\tau^*_0 = \inf \{ s \geq 0 \mid g(Y_s) \geq \hat{v}(Y_s) \}
\]  

(3.15)

is the optimal stopping time of the unconstrained problem.

**Proof.** Let \( \tau_n \) be an increasing sequence of stopping times converging to \( \tau \). By localizing \( \tau_n \) and \( \tau \), we may assume that \( \tau \) is bounded. We need to prove that \( g(Y_{\tau_n}) \rightarrow g(Y_{\tau}) \) on \( \{ \tau < \infty \} \). But

\[
g(Y_{\tau_n}) = \mathbb{E}^y \left[ e^{-rD_A \circ \theta_{\tau_n}} f \left( Y_{\tau_n + D_A \circ \theta_{\tau_n}} \right) \mid \mathcal{F}_{\tau_n} \right]
\]  

(3.16)
and

$$g(Y_\tau) = \mathbb{E}^y \left[ e^{-rD_A \circ \theta_t} f \left( Y_\tau + D_A \circ \theta_t \right) \mid \mathcal{F}_\tau \right]. \quad (3.17)$$

Clearly $\tau_n + D_A \circ \theta_{\tau_n}$ is an increasing sequence of stopping times converging to $\tau + D_A \circ \theta_\tau$. Further, with $Y_t$ quasi-left continuous, using the continuity of $f$ we have that

$$e^{-rD_A \circ \theta_{\tau_n}} f \left( Y_{\tau_n + D_A \circ \theta_{\tau_n}} \right) \to e^{-rD_A \circ \theta_{\tau}} f \left( Y_{\tau + D_A \circ \theta_\tau} \right) \ a.s.,$$

hence it is sufficient to show that $\mathbb{E}^y \left[ e^{-rD_A \circ \theta_{\tau_n}} f \left( Y_{\tau_n + D_A \circ \theta_{\tau_n}} \right) \mid \mathcal{F}_{\tau_n} \right] \to g(Y_\tau)$. But the martingale convergence theorem and the boundedness of $\tau$ imply

$$\mathbb{E}^y \left[ e^{-rD_A \circ \theta_{\tau_n}} f \left( Y_{\tau_n + D_A \circ \theta_{\tau_n}} \right) \mid \mathcal{F}_{\tau_n} \right] \to \mathbb{E}^y \left[ e^{-rD_A \circ \theta_{\tau}} f \left( Y_{\tau + D_A \circ \theta_\tau} \right) \mid \bigvee_{n \in \mathbb{N}} \mathcal{F}_{\tau_n} \right].$$

Finally, Meyer’s theorem on the quasi-left continuity of the filtration $(\mathcal{F}_t)_{t \geq 0}$ implies that $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{\tau_n} = \mathcal{F}_{\tau}$, so that $g(Y_{\tau_n}) \to g(Y_\tau)$. Representation $(3.17)$ and Jensen’s inequality shows that under the assumption $(3.13)$ for any stopping time $\tau \leq T$

$$\mathbb{E} \left[ |g(Y_\tau)|^p \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( e^{-rt} f(X_t) \right)^p \right]. \quad (3.18)$$

Consequently, the family of random variables $\{g(Y_\tau) \mid \tau \in T_{[0,T]} \}$ is bounded in $L_p$, hence, uniformly integrable and therefore of class D.

The quasi-left continuity implies that $g(Y_t)$ is continuous in expectations in the sense $E[g(Y_{\tau_n})] \to E[g(Y_\tau)]$. It follows from Bismut and Skalli (1977) or El Karoui (1981, 2.18) that the Snell envelope $J_t$ is continuous in expectation as well and $\inf \{ t \mid J_t = g(Y_t) \}$ is an optimal stopping time.

Remark 3.5. Note that the existence of optimal stopping times can be guaranteed under even weaker conditions. If $g(Y_t)$ is upper semi-continuous in expectation, e.g., $E[g(Y_\tau)] \geq \limsup E[g(Y_{\tau_n})]$ for every sequence $\tau_n \to \tau$, and if $J_t = M_t - A_t$ is the Doob-Meyer decomposition, then $\inf \{ t \mid J_t = g(Y_t) \}$ is the smallest and $\inf \{ t \mid J_t \neq M_t \}$ the largest optimal stopping time, see El Karoui (1981).
Remark 3.6. It is worth mentioning that in El Karoui (1981, 2.21-2.50) the optimal stopping problem is addressed in the very general context of Meyer filtrations and stopping times that form a so called chronology. A chronology is family of stopping times with respect to some Meyer filtration, which is stable under the supremum and infimum. The properties of a chronology are used to aggregate the supermartingale system formed by the conditional optimal rewards to a proper process, see also Dellacherie and Lenglart (1981). However, our constrained stopping times (2.2) do not form a chronology. They are stable under the infimum, due to the right continuity of the path, but not necessarily stable under the supremum, if the process $Y_t$ exhibits jumps.

3.2 Complete Markets

Hedging American options with exercise constraints is more subtle than it is in the case of ordinary American options. In our setup, the general specification in Definition 2.1 of American options with exercise constraints may lead to an incomplete market. The stopping time constraints may give rise to additional risks that cannot be completely hedged. However, and this is the situation we will consider in this section, the completeness of an underlying market is retained, if the constraints just depend on the states of the instruments of the underlying market. Nevertheless, we face an additional problem caused by the constraints themselves. The replicating portfolio cannot be deduced directly from the Doob-Meyer decomposition of the Snell envelope. In fact, we need to find a slightly more general procedure.

We consider a complete market of $d$ risky assets $X_t = (X_{1,t}, \ldots, X_{d,t})$ with the dynamics given by a non-degenerate diffusion

$$dX_t = \text{diag}(X_t) (r 1 dt + \sigma_t dW_t),$$

(3.19)

where $W_t$ is a $d$-dimensional Brownian motion under the equivalent martingale measure $Q$. We assume the stopping time constraint $A_t$ to satisfy the SDE

$$dA_t = \mu_A(t, X_t) dt + \sigma_A(t, X_t) \cdot dW_t$$

(3.20)
for some functions $\mu_A(t, X_t) \in \mathbb{R}$ and $\sigma_A(t, X_t) \in \mathbb{R}^d$. Then, $Y^y_t$ is the same as in (2.1) with state space $E = \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$. Stopping time constraints that are expressed in terms of the SDE (3.20) retain the market completeness, as we will see below.

Let $\pi$ be a portfolio process, $C$ a consumption-investment process of finite variance with $C_0 = 0$ and $V^{v, \pi, C}$ be the corresponding wealth process with initial endowment $x$, i.e.,

$$e^{-rt}V^{v, \pi, C}_t = v + \int_0^t e^{-rs} dC_s + \int_0^t e^{-rs} \pi_s^\top \sigma dW_s.$$  \hfill (3.21)

Let

$$M^{\pi}_t = \int_0^t e^{-rs} \pi_s^\top \sigma dW_s$$ \hfill (3.22)

be the corresponding local martingale part. A portfolio process is said to be tame, if the associated discounted gains process is almost surely bounded from below. This implies that the local martingale (3.22) is in fact a supermartingale. We refer to Karatzas and Shreve (1998) and Oksendal (1998) for precise statements and additional details.

Consider a constrained American option according to Definition 2.1 with payoff $f : E \to \mathbb{R}$. The upper hedging price $h_{\text{up}}(f, A)$ is defined as

$$h_{\text{up}}(f, A) = \inf \{v \mid \exists \pi \text{ tame and } C \geq 0 \text{ s.t. } V^{v, \pi, C}_\tau \geq f(Y_\tau) \text{ } \forall \tau \in T[0, T] \}(A) \}. \hfill (3.23)$$

The upper hedging price is the smallest amount of capital required by the option seller to set up a risk free hedging strategy $(\pi, C)$ such that the wealth $V^{v, \pi, C}_\tau$ is sufficient to cover the liability $f(Y_\tau)$ for every admissible stopping time $\tau$. On the other hand, the lower hedging price $h_{\text{low}}$ is defined by

$$h_{\text{low}}(f, A) = \sup \{v \mid \exists \pi, \text{ } C \geq 0, \text{ } \text{ and } \tau_0 \in T[0, T](A), \text{ such that } M^{\pi}_{t \land \tau_0} \text{ is a supermartingale and } V^{-v, \pi, -C}_{\tau_0} + f(Y_{\tau_0}) \geq 0, \text{ } C_{\tau_0} = 0 \}. \hfill (3.24)$$
It is the largest initial debt that the option buyer is willing to accept to purchase the contract. If he then applies the borrowing strategy \((\pi, -C)\) to manage his debt, he does not need to invest additional cash up to the admissible time \(\tau_0\) and if he exercises the option at \(\tau_0\), the payoff is sufficient to close the borrowing strategy.

**Theorem 3.7.** Assume that the payoff process \(f(Y_t)\) is a non-negative right continuous left limit process of class \(D\). Then,

\[
\mathbb{E}[e^{-rT}f(Y_T)] \leq h_{up}(f, A) = v(Y_0) = \hat{v}(Y_0).
\]  

(3.25)

*Proof.* The equality \(v(Y_0) = \hat{v}(Y_0) \equiv v_0\) follows from Theorem 3.3. The inequality \(v(Y_0) \leq h_{up}(f, A)\) is a direct consequence of Doob’s optional sampling theorem applied to the supermartingale \(V_{t^{v, \pi, C}}\) in (3.24).

To prove \(v(Y_0) \geq h_{up}(f, A)\), we construct a portfolio consumption process \((\pi, C)\). Let denote \(J_t\) the Snell envelope of \(e^{-rt}g(Y_t)\), which by El Karoui, Lepeltier, and Millet (1992, Theorem 3.4) can be expressed as \(J_t = \hat{v}(Y_t)\). The Doob-Meyer decomposition \(J_t = v_0 + M_t - \Lambda_t\) together with the martingale representation theorem proves that

\[
J_t = v_0 + M_t - \Lambda_t = e^{-rt}V_{t^{v_0, \pi, C}}
\]  

(3.26)

is the wealth of a hedging portfolio \((\pi, C)\) with \(C_t = \exp(rt)\Lambda_t\). Note that \(M_t \geq -v_0\), hence \(\pi\) is tame. Because the Snell envelope is the least supermartingale dominating the discounted payoff \(e^{-rt}g(Y_t)\), we get for every stopping time \(\tau\)

\[
V_{\tau^{v_0, \pi, C}} \geq g(Y_\tau).
\]  

(3.27)

But Lemma 3.2 implies that, for all \(\tau \in \mathcal{T}_{[0, T]}(A)\),

\[
V_{\tau^{v_0, \pi, C}} \geq g(Y_\tau) = f(Y_\tau).
\]  

(3.28)

\(\square\)

The next result, Theorem 3.8, essentially relies on the existence of optimal stopping times.
Theorem 3.8. Assume that the payoff function $f$ is as in Proposition 3.4. Then,

$$h_{\text{low}}(f, A) = h_{\text{up}}(f, A) = v(Y_0) = \hat{v}(Y_0).$$  \hspace{1cm} (3.29)

Proof. From Theorem 3.7 we have $h_{\text{up}}(f, A) = v(Y_0) = \hat{v}(Y_0) \equiv v_0$. Because $g(Y_t)$ is right continuous quasi-left continuous process of class D the stopping time

$$\tau^*_0 = \inf\{s \geq 0 \mid g(Y_s) \geq \hat{v}(Y_s)\}$$  \hspace{1cm} (3.30)

is optimal for the unconstrained problem (3.9) and, therefore,

$$\hat{\tau}^*_0 = \tau^*_0 + D_A \circ \theta_{\tau^*_0}$$  \hspace{1cm} (3.31)

is optimal for (3.3). Let $J_t$ be the Snell envelope of $e^{-rt} g(Y_t)$ and $J_t = v_0 + M_t - \Lambda_t$ the Doob-Meyer decomposition. The optimality of $\tau^*_0$ implies that

$$J_{\tau^*_0} = e^{-r\tau^*_0} g(Y_{\tau^*_0}), \quad J_t \wedge \tau^*_0 \text{ is a martingale, } \Lambda_{\tau^*_0} = 0.$$  \hspace{1cm} (3.32)

The martingale representation theorem proves the existence of a portfolio strategy and a consumption process

$$e^{-rt} V_{t \wedge \tau^*_0}^{-v_0, -\pi^f, 0} = -v_0 - M_{t \wedge \tau^*_0}$$
$$= -v_0 - \int_0^{t \wedge \tau^*_0} e^{-rs} \pi_s^f \sigma dW_s$$  \hspace{1cm} (3.33)

up to time $\tau^*_0$. To proceed up to $\hat{\tau}^*_0$, we need to switch at time $\tau^*_0$ from the strategy $\pi^f_l$ to a strategy replicating the payoff $f(Y_{\hat{\tau}^*_0})$ at time $\hat{\tau}^*_0$. Introduce the martingale

$$V^f_t = \mathbb{E} \left[ e^{-r\tau^*_0} f(Y_{\tau^*_0}) \mid \mathcal{F}_t \right]$$
$$= \mathbb{E} \left[ e^{-r\tau^*_0} f(Y_{\tau^*_0}) \right] + \int_0^t e^{-rs} \pi_s^f \sigma dW_s.$$  \hspace{1cm} (3.34)
Because $V_{\tau_0}^f = e^{-r\tau_0} g(Y_{\tau_0}^*)$ by definition, we can switch at time $\tau_0^*$ to the portfolio strategy $\pi^f_t$, so that

$$\pi_t = \pi^f_t 1_{\{ t < \tau_0^* \}} + \pi^f_t 1_{\{ t \geq \tau_0^* \}}$$

(3.35)

results in a self-financing portfolio strategy and

$$e^{-rt} V_{t \wedge \tau_0^*}^ {- v_0, - \pi, - C} = -v_0 - \int_0^{t \wedge \tau_0^*} e^{-rs} \pi_s^T \sigma dW_s$$

(3.36)

with $C_t = 0$ on $\{ t \leq \tau_0^* \}$ defines a wealth process such that

$$V_{\tau_0}^ {- v_0, - \pi, - C} + f(Y_{\tau_0}^*) \geq 0.$$  

This shows $h_{\text{low}} (f, A) = v_0$.

4 Numerical Approximation

In this section, we present the numerical implementation of American options with exercise restrictions. To numerically solve the optimal stopping problem (3.3), we make use of the Longstaff-Schwartz algorithm Longstaff and Schwartz (2001). A recent textbook exposition on Monte Carlo methods for American option pricing can be found in Glasserman (2004). Further generalizations of the Longstaff-Schwartz algorithm are developed for instance in ?.

For the numerical calculations, we assume a simple performance measure that is based on the relative return of the underlying stock $S_t$ and a correlated benchmark $M_t$. We assume that the stock and benchmark follow a geometric Brownian motion

$$d \begin{pmatrix} S_t \\ M_t \end{pmatrix} = \text{diag} (S_t, M_t) (r dt + \sigma dW_t),$$

(4.1)

with diffusion matrix

$$\sigma = \begin{pmatrix} \sigma_S \rho & \sigma_S \sqrt{1 - \rho^2} \\ \sigma_M & 0 \end{pmatrix},$$

(4.2)
where \( W_t = (W_1^t, W_2^t) \) is a standard Brownian motion in \( \mathbb{R}^2 \) under the martingale measure \( Q \). We define the performance condition as

\[
A_t = \frac{S_t - S_0}{S_0} - \frac{M_t - M_0}{M_0} - \varepsilon(t), \quad \varepsilon(0) = -a,
\]  

(4.3)

with performance threshold \( \varepsilon(t) \) a piece-wise continuous deterministic function. Having defined the performance condition (4.3), \( A = \mathbb{R} \) and we can specify the set of admissible stopping states as

\[
A = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.
\]  

(4.4)

[ Figure 1 about here ]

In Figure 1, we illustrate the basic mechanism behind the above contract specification. Figure 1 plots a simulation of the stock price and the performance measure over a time horizon of 350 days. The admissible exercise region is the area between the two dashed vertical lines. The performance measure turns positive for the first time after day 199. After day 330, the performance condition is no longer fulfilled (see Panel (B)). Between these two dates, the option holder has the right to (American) exercise the option. At day 199, the option is still out-of-money \((K = 100, \text{dashed horizontal line in Panel (A)})\), so the option holder has no incentive to exercise the option. However, during the period in which the performance condition is fulfilled, the option becomes in-the-money and the option holder has to decide whether he wants to exercise the option or not. After day 330, the option, although in-the-money, can no longer be exercised, since the performance condition is no longer fulfilled.

Remark 4.1. A performance condition as in (4.3) would be suitable for a constrained American option in an executive stock option plan. The company can define \( \varepsilon(t) \) as a relative return target over a prespecified time horizon \( T \). If this performance target is already met before the expiration date \( T \), the manager has the right to exercise the option as long as the relative return stays above \( \varepsilon(t) \). More complicated performance measures that fit into our framework could be considered, e.g., performance measures of Asian type or lookback type.

We start our algorithm by first simulating \( n_0 \) trajectories for the discretized stock price and the benchmark dynamics, starting from \( t = 0 \) up to the expiration date \( T \). We assume equidistant time
steps of length $\Delta t$. Given these trajectories, we directly obtain the trajectories of the performance condition (4.3). Next, beginning at time $t = 0$, we calculate at each time slice $t = 0 < \ldots < t + \frac{j}{m} = k_j < \ldots < T, \quad j = 0, \ldots, m,$

de the values of

$$
g(Y_{k_j}) = \mathbb{E}^{Y_{k_j}} [e^{-r D_A f (Y_{D_A})}] = \mathbb{E}^y [e^{-r D_A \circ \theta_{k_j}} f \left(Y_{k_j} + D_A \circ \theta_{k_j}\right)], \quad (4.5)
$$

where $D_A \circ \theta_{k_j}$ is the first entrance time after time $k_j$ of the set $A$ determined by the performance condition (4.3).

To calculate the values $g(Y_{k_j})$ in (4.5), we simulate $n_j$ stock and benchmark trajectories up to time $D_A \circ \theta_{k_j}$ with their starting values at $k_j$ taken from the first simulation with sample size $n_0$. Continuing to work forward through the trajectories obtained by the first simulation, we get $n_0$-values of the function $g$ at each time slice $k_j, \quad j = 0, \ldots, m$. These values correspond to the value of forward start European options with a stochastic time-to-maturity (smaller or equal to $T$), where the forward start dates collide with the predetermined time slices.

[ Figure 2 about here ]

Finally, to obtain the price of the constrained American option, we have to identify the supremum of the payoff from the European options over all stopping times, i.e., over the forward start dates. To identify this optimal unconstrained stopping time, we can directly apply the Longstaff-Schwartz algorithm. The only difference to the traditional Longstaff-Schwartz algorithm is that we do not use the stock price $S_{k_j}$ as the underlying process, but the simulated option prices $g(Y_{k_j})$. As basis functions, we take a constant (with regression coefficient $\beta_0$), the first three powers of the stock price and the benchmark ($\beta_i, \quad i = 1, 2, 3$ and $\beta_j, \quad j = 4, 5, 6$, respectively), the cross product ($\beta_7$), the cross product of the squared stock price with the benchmark ($\beta_8$), and the values of $g(Y_t)$ itself ($\beta_9$).

We simulate the prices of constrained American call options with strike $K = 100$ and a yearly time horizon, i.e., $T = 1$. For the simulation, we discretize the underlying processes and assume equidistant time intervals of length $\Delta t = 1/350$. Furthermore, we slice the time horizon into fourteen time intervals.
by setting $m = 15$.\footnote{We simulate the stock and benchmark prices with a sample size $n_j = 5000$, for all $j = 0, \ldots, m$. For the stock and benchmark processes, we assume $S_0 = M_0 = 100$, $r = 0.05$, $\sigma_S = \sigma_M = 0.2$, and a correlation of $\rho = 0.6$.}

Panel (A) of Figure 2 plots the function $g(Y_{k_j})$ with $m = 15$ for a constrained American call option assuming $\varepsilon(t) = \varepsilon = -10$. The solid line in the middle represents the mean value, the two dashed lines are the 95% confidence intervals, and the outer solid lines display the maximum and minimum values of the performance process at each time slice. For the case $\varepsilon = -10$, the lower confidence bound and the minimum values are very close to zero. Therefore, they can hardly be seen from the figure (also compare with Figure 3).

With the choice $\varepsilon = -10$, the probability that the performance condition is not fulfilled is negligibly small. To doublecheck, Panel (B) plots the performance measure $P_t$ defined in (4.3). Again, the solid line in the middle represents the mean value, the two dashed lines are the 95% confidence intervals, and the outer solid lines display the maximum and minimum values of the performance process at each time slice. By inspection, the performance condition is always fulfilled in our simulation. Consequently, at each time slice, the options $g(Y_{k_j})$ are immediately exercised, since their time-to-expiration degenerates to zero, i.e., $D_A \circ \theta_{k_j} = k_j$. Hence, we are back in the classical unconstrained case. Since we assume no dividends, the above specification for the constrained American option should lead to a value equal to the Black-Scholes price of a European call option. Indeed, Panel (A) of Table 1 shows that the price we obtain from our simulation is equal to 10.5010. This price is reasonably close to the analytical value of 10.4506.

Figure 3 plots $g(Y_t)$ for a constrained American call option when, e.g., $\varepsilon = 0.20$. Compared to Figure 2, we observe two differences in the behavior of $g(Y_t)$. First, since in Figure 2 the performance condition is always fulfilled, the value of $g(Y_0)$ equals zero since the option is at-the-money at time $t = 0$. This is not the case in Figure 3. The $g(Y_0)$ takes a nonzero value. Second, in the benchmark case illustrated in Figure 2 we observe positive terminal values $g(Y_T)$. However, in Figure 3 the value of $g(Y_T)$ is always zero. Indeed, this holds true for all constrained American options with performance condition (4.3) and $\varepsilon(T) < 0$. 
In Table 1 Panel (A), we list the prices of constrained American call options with different target returns set equal to \( \varepsilon = 0, 0.05, 0.10, \) and 0.20, respectively, together with the values of the regression coefficients \( \beta_i, \ i = 0, \ldots, 9. \) For comparison, Panel (B) of Table 1 lists the prices of constrained American put options. Again, as the benchmark case, we calculate the constrained American put option with \( \varepsilon = -10. \) Using a trinomial tree with a depth of 10'000 time steps, we obtain a value of 6.0903 for the price of the American put, which is close to the value of 6.0937 that we obtain using our Longstaff-Schwartz algorithm.

We note that the sensitivities of the constrained American option are in line with intuition. For instance, if we increase the volatility of the stock price process, the value of the constrained American call option will increase (for the case \( \varepsilon = 0.05, \) an increase in volatility from \( \sigma_S = 0.20 \) to \( \sigma_S = 0.25 \) increases the option price from 6.9586 to 9.1533). Similarly, the price of the option will increase, if the correlation between stock and benchmark decreases (for the case \( \varepsilon = 0.05, \) a decrease of correlation from \( \rho = 0.6 \) to \( \rho = -0.3 \) increases the option price from 6.9586 to 8.5328).

5 Conclusion

We consider the pricing of constrained American options. For the constraints, we consider stochastic stopping time constraints expressed in terms of the states of a Markov process. We present the transformation of the original constrained problem into an unconstrained problem. Once we achieved this transformation, we can use an adapted Longstaff-Schwartz algorithm to price constrained American options. Such option contracts may be used to design executive stock option plans as an alternative to more traditional options. We present a simple numerical example with geometric Brownian motions. Note however that that our theoretical results extend to the class of so called Hunt processes, and therefore also hold for Lévy processes. A path for future research is the extension of our model to more general constraints, in particular with non-Markovian features.
Figure 1: Panel (A) displays the evolution of the stock price process and plots the level of the exercise price \( K \) (dashed line). Panel (B) shows the evolution of the corresponding performance measure. The area between the two dotted vertical lines represents the admissible exercise region (from day 199 until day 330). We assume \( \Delta t = 1/350, S_0 = M_0 = 100, K = 100, r = 0.05, \sigma_S = \sigma_M = 0.2, \rho = 0.6 \).
Figure 2: Constrained American call option. Panel (A) displays the function $g(Y_{k_j})$ for the $m = 15$ time slices when $\varepsilon(t) = \varepsilon = -10$. The resulting price for the option is $v(0, X_0) = 10.5010$, see Table 1. Panel (B) shows the evolution of the corresponding performance measure, where the solid line in the middle is the mean value, the two dashed lines are the 95% confidence intervals, and the outer solid lines display the maximum and minimum values at each time slice. For the simulation, we set $n_j = 5000$, $\forall j$, $\Delta t = 1/350$, $S_0 = M_0 = 100$, $K = 100$, $r = 0.05$, $\sigma_S = \sigma_M = 0.2$, $\rho = 0.6$. 
### Table 1: The table lists the prices of the constrained American call and put option and the different weights for the regression coefficients of the Longstaff-Schwartz least square algorithm.

For the simulation, we set $n_j = 5000$, $\forall j$, $\Delta t = 1/350$, $S_0 = M_0 = 100$, $K = 100$, $r = 0.05$, $\sigma_S = \sigma_M = 0.2$, $\rho = 0.6$. For all contracts, we use the same random seed.

#### Panel (A): Call Option

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>option price</th>
<th>$\beta$-weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$10</td>
<td>10.5010</td>
<td>$\beta_0$ -0.2823, $\beta_1$ -120.7148, $\beta_2$ 1.0747, $\beta_3$ -0.0004, $\beta_4$ -44.1416, $\beta_5$ -0.4796, $\beta_6$ 0.0015, $\beta_7$ 1.7856, $\beta_8$ -0.0084, $\beta_9$ -92.4878</td>
</tr>
<tr>
<td>0</td>
<td>10.0054</td>
<td>$\beta_0$ -0.3495, $\beta_1$ -97.1943, $\beta_2$ 0.8440, $\beta_3$ -0.0005, $\beta_4$ -13.6239, $\beta_5$ -0.4968, $\beta_6$ 0.0016, $\beta_7$ 1.2513, $\beta_8$ -0.0059, $\beta_9$ -62.2881</td>
</tr>
<tr>
<td>0.05</td>
<td>6.9586</td>
<td>$\beta_0$ -7.2678, $\beta_1$ -7.4404, $\beta_2$ 0.0132, $\beta_3$ 0.0002, $\beta_4$ 7.8986, $\beta_5$ -0.1311, $\beta_6$ 0.0004, $\beta_7$ 0.1078, $\beta_8$ -0.0005, $\beta_9$ -0.1887</td>
</tr>
<tr>
<td>0.10</td>
<td>5.7744</td>
<td>$\beta_0$ 0.1096, $\beta_1$ -16.7794, $\beta_2$ 0.1096, $\beta_3$ -0.0002, $\beta_4$ 3.9447, $\beta_5$ -0.0906, $\beta_6$ 0.0003, $\beta_7$ 0.1023, $\beta_8$ -0.0005, $\beta_9$ -0.7993</td>
</tr>
<tr>
<td>0.20</td>
<td>4.3092</td>
<td>$\beta_0$ 0.0666, $\beta_1$ -2.7038, $\beta_2$ 0.0666, $\beta_3$ -0.0000, $\beta_4$ 1.2215, $\beta_5$ -0.0336, $\beta_6$ 0.0001, $\beta_7$ 0.0417, $\beta_8$ -0.0002, $\beta_9$ 0.2341</td>
</tr>
</tbody>
</table>

#### Panel (B): Put Option

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>option price</th>
<th>$\beta$-weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$10</td>
<td>6.0937</td>
<td>$\beta_0$ 159.8780, $\beta_1$ 6.3049, $\beta_2$ -0.0614, $\beta_3$ 0.0003, $\beta_4$ -21.1553, $\beta_5$ 0.1607, $\beta_6$ -0.0006, $\beta_7$ 0.1007, $\beta_8$ -0.0004, $\beta_9$ 5.6923</td>
</tr>
<tr>
<td>0</td>
<td>5.9364</td>
<td>$\beta_0$ 0.1529, $\beta_1$ 2.4396, $\beta_2$ -0.0756, $\beta_3$ 0.0009, $\beta_4$ -4.8469, $\beta_5$ 0.0221, $\beta_6$ -0.0059, $\beta_7$ 0.3570, $\beta_8$ -0.0018, $\beta_9$ 12.8467</td>
</tr>
<tr>
<td>0.05</td>
<td>4.4201</td>
<td>$\beta_0$ 298.9418, $\beta_1$ -3.6499, $\beta_2$ 0.0094, $\beta_3$ -4.8469, $\beta_4$ 0.0221, $\beta_5$ 0.1078, $\beta_6$ -0.0059, $\beta_7$ 0.1078, $\beta_8$ -0.0028, $\beta_9$ 0.0929</td>
</tr>
<tr>
<td>0.10</td>
<td>3.3129</td>
<td>$\beta_0$ 247.3308, $\beta_1$ -4.5467, $\beta_2$ -0.0018, $\beta_3$ 0.0001, $\beta_4$ -1.5746, $\beta_5$ 0.0746, $\beta_6$ -0.0059, $\beta_7$ 0.1023, $\beta_8$ -0.0028, $\beta_9$ -0.0007</td>
</tr>
<tr>
<td>0.20</td>
<td>1.7572</td>
<td>$\beta_0$ 89.3294, $\beta_1$ -1.3844, $\beta_2$ -0.0001, $\beta_3$ 0.0000, $\beta_4$ -0.7109, $\beta_5$ 0.0197, $\beta_6$ -0.0059, $\beta_7$ 0.0417, $\beta_8$ -0.0002, $\beta_9$ -0.0007</td>
</tr>
</tbody>
</table>
Figure 3: Constrained American call option. Panel (A) displays the function $g(Y_{k,j})$ for the $m = 15$ time slices when $\varepsilon(t) = \varepsilon = 0.20$. The resulting price for the option is $v(Y_0) = 4.3092$, see Table II. Panel (B) shows the evolution of the corresponding performance measure, where solid line in the middle is the mean value, the two dashed lines are the 95% confidence intervals, and the outer solid lines display the maximum and minimum values at each time slice. For the simulation, we set $n_j = 5000$, $\forall j$, $\Delta t = 1/350$, $S_0 = M_0 = 100$, $K = 100$, $r = 0.05$, $\sigma_S = \sigma_M = 0.2$, $\rho = 0.6$. 
Notes

1Meyer coined the term ‘class D’ in honor of J.L. Doob. For this and other anecdotes, see the historic review by Jarrow and Protter (2004).

2We also tested for larger values of $m$, but did not find much of an improvement.
References


