Efficient Portfolios with Endogenous Liabilities

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Efficient Portfolios with Endogenous Liabilities

Abstract

We study the optimal policies and mean-variance frontiers (MVF) of a multiperiod mean-variance optimization of assets and liabilities (AL). Our model allows for a contemporaneous optimization of the balance-sheet as a whole. This makes the analysis more challenging than in a setting based on purely exogenous liabilities. We show that under general conditions on the joint AL dynamics the arising optimal policies and MVF can be decomposed in an orthogonal set of basis returns. Such a decomposition is derived using a geometric formalism based on exterior algebra which simplifies the computations when liabilities are endogenous. As a special case, the geometric representation in Leippold, Trojani and Vanini (2004) for the exogenous liabilities case follows directly. We apply such a decomposition to study the structure of optimal policies and MVF under endogenous liabilities and show how to obtain MVF representations that substantially improve analytical descriptions and numerical analysis. We finally illustrate the methodology by studying the impact of the rebalancing frequency on the MVF and by highlighting in a numerical example the main differences arising when liabilities are exogenous and when they are endogenous.

JEL Classification Codes: G12, G13, E43.

Key Words: Assets and Liabilities, Mean-Variance Frontiers, Markowitz Model, Endogenous Liabilities, Grassmann Algebra.
Using a geometric approach, we study the optimal policies and mean-variance frontiers (MVF) of a multiperiod mean-variance optimization of assets and liabilities (AL). Our model allows for a contemporaneous optimization of the balance-sheet as a whole. This makes the analysis more challenging than in a setting based on purely exogenous liabilities. Using geometric techniques, we solve the joint optimization of assets and endogenous liabilities in a multiperiod mean-variance model for a given set of short-selling constraints.

A large part of research on dynamic portfolio selection has been based on an expected utility formulation of the objective function in the relevant optimization problem. In this paper, we focus on a model based on the mean-variance paradigm. It is well known that mean-variance and expected utility criteria are different approaches that lead to solutions which coincide only under restrictive assumptions. However, for a financial institution it is often difficult or unfeasible to define an investment objective in terms of some expected utility criterion based on some specific concept of risk aversion. Instead, it is often much more intuitive and easy to work with concepts of risk and return defined in terms of the variance and the mean return of a portfolio strategy.

Our model is based on a separate specification of the assets and liabilities dynamics. The surplus results as the final aggregate difference between assets and liabilities. Such a modelling approach has several advantages. Most importantly, it can account for different distributional properties of each AL component separately. In practice, these differences do arise, since each component of the balance sheet can differ with respect to, e.g., duration, liquidity, and embedded optionalities. A direct specification of the surplus dynamics as the only relevant state variable would make it impossible to account for such differences.

The contemporaneous optimization of the balance-sheet as a whole makes the analysis of AL optimal portfolios and MVF more challenging than in a setting based on purely exogenous liabilities. Indeed, increasing the number of state variables and controls in the model may have the drawback of making the optimization problem more difficult to solve. By introducing an appropriate geometric formalism we show that this problem can be circumvented to a large extent.
Under general conditions on the joint AL dynamics, we show that the optimal policies and MVF can be decomposed in an orthogonal set of basis returns. Such a decomposition is derived using a geometric formalism based on exterior algebra which simplifies the computations when liabilities are endogenous.\footnote{For a treatment of exterior algebra we refer to, e.g., chapter 6 of Abraham, Marsden and Ratiu (1991), Darling (1999), and the original and recently translated work of Grassmann (Peano and Kannenberg (2000)).} The geometric representation in Leippold et al. (2004) for the exogenous liabilities case follows directly from these results. In the endogenous liabilities case, however, the arising orthogonal decompositions are based on more involved projection operators. We propose a geometric formalism based on exterior algebra to make such operators manageable both for analytical and numerical purposes.

The paper is structured as follows. The next section shortly reviews the literature on mean-variance portfolio selection and AL optimization. This relates more directly the paper’s contribution to the vast literature on portfolio optimization for AL. Section 2 introduces our model and formulates a general orthogonal decomposition theorem for the optimal final surplus in a multiperiod mean-variance optimization of AL. This decomposition holds both under exogenous and endogenous liabilities and is the main result of the paper. Section 3 proves the general theorem and elaborates on the geometric distinction between AL optimizations under exogenous or endogenous liabilities. Section 4 illustrates the methodology by studying the impact of the rebalancing frequency on the MVF and by highlighting in some more detailed numerical examples the main differences between exogenous and endogenous liabilities. Section 5 concludes.

1 Background

We formulate our portfolio selection problem in a mean-variance setting. This extends the basic intuition behind the static Markowitz (1952), (1959) model to a dynamic setting where portfolios of assets and liabilities are jointly optimized. Several authors have tried to generalize Markowitz’s seminal work to a multiperiod context using different modelling techniques.\footnote{See also Steinbach (2001) for a nice review of the literature on mean-variance models in financial portfolio analysis.} Early essays in this direction were for instance proposed in Smith (1967), Mossin (1968), Merton
(1969), Samuelson (1969), Chen, Jen and Zions (1971), Hakansson (1971), Merton (1971), and Grauer and Hakansson (1993). More recently Bajeux-Besnainou and Portait (1998) obtained an explicit dynamic asset allocation in a continuous-time mean-variance framework. That paper is based on the martingale techniques (see also Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987)) and is therefore only applicable to a complete markets setting. Since short-selling constraints typically render markets incomplete, the martingale approach is not directly applicable to solve portfolio optimization problems under short-selling types of constraints. Li, Zhou and Lim (2002) study the solutions of a continuous time mean-variance portfolio optimization problem under short-selling constraints on stocks. That approach adopts the embedding technique first proposed by Li and Ng (2000) in a discrete-time mean-variance model.\footnote{Such a technique has been extended to a unified framework in Zhou and Li (2000).} In particular, Li et al. (2002) show that short-selling constraints on stocks induce non-smooth solutions of the corresponding Hamilton-Jacobi-Bellman equations.

Models that include explicitly liabilities in a static Markowitz-type optimization were proposed already in the early nineties; see for instance Sharpe and Tint (1990), Elton and Gruber (1992), Leibowitz, Kogelman and Bader (1992). In a static context, such an inclusion of liabilities is mathematically quite a direct extension of the standard Markowitz model. In a multiperiod model, however, incorporating liabilities in a portfolio selection problem is more challenging, because the relevant Hamilton Jacobi Bellman equations depend on a multivariate state dynamics. Leippold et al. (2004) propose a geometric approach to multiperiod mean-variance AL optimization that largely simplifies the mathematical analysis and the economic interpretation of such model settings. Using a geometric approach to dynamic mean-variance portfolio optimization, closed form solutions in the iid setting can be obtained for AL portfolios when liabilities are modelled as a separate exogenous state variable. As a special case, the solution of the asset only case in Li and Ng (2000) can be written in a very simple and easily interpretable geometric form. In this paper we extend previous research on multiperiod mean-variance AL optimization by allowing for a simultaneous optimization of the balance-sheet as a whole. This renders the assets and the liabilities dynamics endogenous and makes the analysis more complex than under purely exogenous liabilities. Therefore, we propose a geometric formalism based on exterior algebra that makes the model with endogenous liabili-
ties manageable both for analytical and numerical purposes. As we show below, this approach can be applied also in the presence of short-selling constraints on the assets or the liabilities side.

An alternative route to AL optimization is based on approaches that put the management of assets and liabilities at the core of enterprise-wide risk management for financial institutions. This often requires models that can handle at the same time a large number and variety of constraints and rich state variable dynamics. Rosen and Stavros (2003) discuss qualitatively within such a general framework the role of asset liability management. Several important contributions to asset-liability optimization models for the banking and insurance industry are available today. In particular, multi-stage stochastic programming approaches to solve multiperiod investment problems have been proved very useful in the context of enterprise-wide AL management. These approaches build on the generation of scenario trees. The objective is to minimize the expected costs along the scenario paths. The cost functions are tailored to the circumstances and goals of the financial entity under consideration. Early contributions to such multiperiod models include Cohen and Thore (1970), Both (1972), Both and Dash (1979), Brodt (1979), Brodt (1984), Kallberg, White and Ziemba (1982), and Kusy and Ziemba (1986). More recent models, some of them implemented and tested in the industry, include Cariño, Kent, Myers, Stacy, Sylvanus, Turner, Watanabe and Ziemba (1994), Consigli and Dempster (1998), Mulvey (1999), Pflug (2000), Zaremba (2000), and Siegmann and Lucas (2001).  

2 Model and Main Results

Let the aggregate value of assets at time \( t \) be \( A(t) \) and the aggregate value of liabilities at time \( t \) be \( L(t) \). The firm’s aggregate surplus is defined as \( S(t) = A(t) - L(t) \) and the joint aggregate assets and liabilities state vector is \( s(t) = (A(t), L(t))^{\top} \). For simplicity of exposition, we consider two assets and two liabilities in the balance sheet. One of the assets and one of the liabilities define a reference benchmark for the assets and the liabilities side. The gross

\footnote{Zenios (1995) classifies asset-liability management models into static, stochastic one-period, and stochastic multiperiod models.}

\footnote{See also Ziemba and Mulvey (1998), Zenios and Ziemba (2003), and the overview in Kouwenberg and Zenios (2003) for a complete exposition of these approaches to multiperiod AL optimization.}
benchmark-return for assets and liabilities are $r_A(t)$ and $r_L(t)$. The joint gross benchmark return vector for assets and liabilities is $\mathbf{r}(t) = (r_A(t), r_L(t))^\top$. The joint excess returns vector for the second asset and the second liability relative to the benchmark is $\varphi(t) = (\varphi_A(t), \varphi_L(t))^\top$. Similarly, $\mathbf{R}(t) = \text{Diag}[\mathbf{r}(t)]$ and $\Phi(t) = \text{Diag}[\varphi(t)]$ are diagonal matrices of benchmark and excess AL returns. We write $\mathbf{d} = (1, -1)^\top$ for the “difference vector”, i.e., the vector which translates assets and liabilities into a surplus. All return variables in the model are assumed to be elements of the space $L_2$ of random variables with finite second moments. The scalar product in $L_2$ is denoted by $\langle \cdot, \cdot \rangle$ and we write $\mathbf{1}$ for the unit vector in $L_2$. Conditional expectations at time $t$ are denoted by $\mathbb{E}_t(\cdot)$ while unconditional expectations are denoted by $\mathbb{E}(\cdot)$.

Consider an investor at time $t = 0$ with initial assets $A(0)$ and initial liabilities $L(0)$, starting to invest dynamically over a time horizon of length $T$. Transactions can take place at the times $t = 0, \ldots, T-1$. For a given value of aggregate assets $A(t)$ at time $t$, we denote by $W_A(t) \geq 0$ the wealth amount invested in the excess return $\varphi_A(t)$. The amount $A(t) - W_A(t) \geq 0$ is invested at time $t$ in the asset benchmark return $r_A(t)$. Similarly, $W_L(t) \geq 0$ is the amount invested at time $t$ in the excess liability return $\varphi_L(t)$ while $L(t) - W_L(t) \geq 0$ is the amount invested in the liability benchmark return $r_L(t)$. Finally, $\mathbf{w}(t) = (W_A(t), W_L(t))^\top$ is the control vector of amounts invested in the assets and liabilities excess returns. Based on the above notations, the balance sheet at time $t$ takes the stylized form:

<table>
<thead>
<tr>
<th>Balance Sheet</th>
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</thead>
<tbody>
<tr>
<td><strong>Benchmark Asset:</strong></td>
</tr>
<tr>
<td><strong>Other Assets:</strong></td>
</tr>
<tr>
<td><strong>Benchmark Liability:</strong></td>
</tr>
<tr>
<td><strong>Other Liabilities:</strong></td>
</tr>
<tr>
<td><strong>Surplus:</strong></td>
</tr>
</tbody>
</table>

The joint dynamics for assets and liabilities are directly implied by the self financing condition on a candidate AL portfolio policy:

$$A(t + 1) = r_A(t)A(t) + \varphi_A(t)W_A(t), \quad L(t + 1) = r_L(t)L(t) + \varphi_L(t)W_L(t). \quad (2.1)$$
Therefore, the vector dynamics for the bivariate state process of assets and liabilities is

\[ s(t + 1) = R(t)s(t) + \Phi(t)w(t). \] \hspace{1cm} (2.2)

Remark that the dynamics (2.2) cannot be reduced to a single surplus dynamics unless \( r_A = r_L \).

We study a multiperiod version of a Markowitz portfolio selection problem for portfolios of AL. The investor maximizes the expected value of the final surplus \( S(T) = A(T) - L(T) \) subject to some given variance penalization:

\[ P(1) : \begin{cases} 
\max_w \left[ E(S(T)) - \theta \text{VAR}(S(T)) \right] \\
\text{s.t.} \; (2.2)
\end{cases}, \]

where \( \theta > 0 \). Problem (1) can be substantially simplified by considering the problem:

\[ P(2) : \begin{cases} 
\max_w E \left[ (\psi S(T) - \theta S(T)^2) \right] \\
\text{s.t.} \; (2.2)
\end{cases}, \]

where \( \psi > 0 \) is an auxiliary parameter. In particular, Leippold et al. (2004) prove that (i) any solution of \( P(1) \) is also a solution of \( P(2) \) and (ii) if \( w^* \) is a solution of \( P(2) \) for given \( (\psi^*, \theta) \), then it is also a solution for \( P(1) \), if the condition

\[ \psi^* = 1 + 2\theta E(S(T))|_{w^*} \] \hspace{1cm} (2.3)

is satisfied. Although \( P(1) \) and \( P(2) \) are equivalent problems under condition (2.3), problem \( P(2) \) is separable in the sense of dynamic programming. Therefore, problem \( P(2) \) can be solved by means of standard dynamic programming techniques while problem \( P(1) \) cannot.

We formulate the optimization problem under a set of short-selling constraints on assets and liabilities. More specifically, \( \mathcal{A}(\cdot) \) is the state dependent set of constraints defined by

\[ \mathcal{A}(s) = \{ w \mid 0_{2 \times 1} \leq w \leq s \}. \] \hspace{1cm} (2.4)
Further linear constraints, such as liquidity constraints or transaction costs, could be added to \( \mathcal{A}(\cdot) \)'s definition. Although such constraints are of high practical relevance, we do not introduce them for notational simplicity.

The relevant optimization problem is formulated as

\[
P(3) = \begin{cases} 
    \max_{w \in \mathcal{A}(s)} \mathbb{E} \left[ \gamma S(T) - \frac{1}{2} S(T)^2 \right] \\
    \text{s.t. } (2.2), \ w(t) \in \mathcal{A}(s(t)); \ t = 0, ..., T - 1 
\end{cases},
\]

where \( \gamma = \frac{\psi}{2\theta} \). To solve this problem, we follow a geometric approach where multiperiod optimal policies are decomposed in a linear combination of policies with orthogonal terminal date pay-offs in \( \mathcal{L}_2 \). This decomposition fully characterizes the arising solutions and permits an easier description of the implied MVF for AL portfolios. We state this orthogonal decomposition result in the next theorem. This is the main result of the paper.

**Theorem 1.** Given the mean-variance AL optimization problem \( P(3) \), the optimal final surplus \( S^*(T) \) can be decomposed into a sum of \( T - t + 1 \) pay-offs in \( \mathcal{L}_2 \):

\[
S^*(T) = r_s(T - t)^\top s(T - t) + \gamma \sum_{i=1}^{t} r_\gamma(T - i). \tag{2.5}
\]

Such a decomposition is orthogonal in \( \mathcal{L}_2 \), i.e., the set \( \{r_s(T - t)^\top s(T - t), r_\gamma(T - t), r_\gamma(T - t + 1), ..., r_\gamma(T - 1)\} \) is an orthogonal subset of \( \mathcal{L}_2 \).

A more explicit description of the single pay-offs in the decomposition given in Theorem 1 will be provided in the following sections using projection operators. To clarify the structure of the optimal surplus in Theorem 1, we first consider the single-period model under different assumptions on the existing binding constraints. With the insights gained from this preliminary analysis, we introduce a geometric symbology based on orthogonal projections and wedge products on Grassman algebras. This symbolism allows us to understand the structure of the AL optimal policies and MVF also in the more general multiperiod setting with endogenous liabilities.
3 Derivation of the Main Results

We start from a single-period setting and highlight the geometric structure of the prevailing optimal surplus. Under a binding constraint on assets or liabilities, we show that the solution of the model is a special case of the one in Leippold et al. (2004). Hence, for these cases a natural geometric decomposition of the surplus arises directly. The extension to the multiperiod setting follows in a second step.

3.1 Single-Period Optimization

Consider first the solution of the optimization problem $P(3)$ at time $T-1$, where we drop all time indexes $T-1$ for brevity. The value function $J(s)$ of this problem is defined by:

$$J(s) = \max_{w \in A(s)} \mathbb{E} \left[ \gamma S(T) - \frac{1}{2} S(T)^2 \right]$$

$$= \max_{w \in A(s)} \mathbb{E} \left[ \gamma d^T s(T) - \frac{1}{2} (d^T s(T))^2 \right]$$

$$= \max_{w \in A(s)} \mathbb{E} \left[ (\gamma d^T - s^T Rdd^T) \Phi w - \frac{1}{2} w^T \Phi dd^T \Phi w \right] + C(s), \quad (3.6)$$

using the state dynamics (2.2), where $C(s)$ is some constant that does not depend on $w$. From the first-order condition, the optimal policy is

$$w^* = \left[ \mathbb{E} \left( \Phi dd^T \Phi \right) \right]^{-1} \left[ \gamma \mathbb{E}(\Phi d) - \mathbb{E} \left( \Phi dd^T R \right) s + \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_3 - \alpha_4 \end{pmatrix} \right],$$

where $\alpha_i \leq 0$, $i = 1, \ldots, 4$, are the Kuhn-Tucker multipliers for the constrains in the set $A(s)$. More precisely, $\alpha_1$ and $\alpha_2$ are the multipliers for the short-selling assets constraints while $\alpha_3$ and $\alpha_4$ are the multipliers for the liabilities constraints. By the slackness conditions, the optimal policy is specified. If no constraint is binding, then the optimal policy is given by

$$w^*_0 = \left[ \mathbb{E} \left( \Phi dd^T \Phi \right) \right]^{-1} \left[ \gamma \mathbb{E}(\Phi d) - \mathbb{E} \left( \Phi dd^T R \right) s \right], \quad (3.7)$$
and the optimal surplus $S^*(T)$ is

$$S^*(T) = d^\top (Rs + \Phi w_0^*) = r_s^\top s + \gamma r_\gamma, \quad (3.8)$$

where

$$r_s^\top = d^\top \left( R - \Phi \left( E \left( \Phi dd^\top \Phi \right) \right)^{-1} E \left( \Phi dd^\top R \right) \right), \quad (3.9)$$

$$r_\gamma = d^\top \Phi \left[ E \left( \Phi dd^\top \Phi \right) \right]^{-1} E(\Phi d) \quad (3.10)$$

We see from (3.8) that the optimal surplus $S^*(T)$ has been decomposed in a linear combination of two “returns” in $r_s$ and an “excess returns” $r_\gamma$. The weights in the linear combinations are the single components of the state variable $s$ and the risk aversion related coefficient $\gamma$.

The goal in the sequel is to identify $r_s$ and $r_\gamma$ as two particular $L_2$ projections in some orthogonal subspaces of $L_2$. Such a type of orthogonal decomposition holds irrespectively of the fact that some constraints on assets or liabilities are binding. However, the situation where constraints are binding is more easy to analyze and can be handled using a standard projection formalism in $L_2$, as in Leippold et al. (2004). Therefore, in the next sections we discuss first this latter case. In a second step, we study the unconstrained case by introducing an exterior algebra formalism which simplifies the computations.

### 3.1.1 Binding Short-Selling Constraints on Liabilities

It is sufficient to study two cases. In the first case, the constraint $W_L \geq 0$ binds while the constraint $L - W_L \geq 0$ is slack. Then $W_L = 0$ follows. In the second case, the constraint $L - W_L \geq 0$ binds while the constraint $W_L \geq 0$ is slack, implying $W_L = L$. Hence, the optimal policies for the two cases are

$$w^* = (W_A^*, 0)^\top, \text{ or } w^* = (W_A^*, L)^\top. \quad (3.11)$$
As a consequence, we can interpret the solution for the first and the second case as the solution of a problem with exogenous liabilities. In the first case, the liabilities amount \( L \) is invested exclusively in the gross benchmark liability return \( r_L \). In the second case it is invested exclusively in the gross liability return \( r_L + \varphi_L \).

In particular, in both cases \( \mathbf{w}^* \) can be written as the solution to an exogenous benchmark liability optimization of the form studied in Leippold et al. (2004). Using the scalar product in \( \mathcal{L}_2 \) the optimal policy can be then written as:

\[
W_A^* = \arg\max_{W_A} \mathbb{E} \left[ \gamma S(T) - \frac{1}{2} S(T)^2 \right] \bigg|_{W_L=0 \text{ or } W_L=L}
= \begin{cases} 
\frac{1}{\langle \varphi_A, \varphi_A \rangle} \left( \gamma \langle \mathbb{1}, \varphi_A \rangle - \langle r_A, \varphi_A \rangle A + \langle r_L, \varphi_A \rangle L \right) & \text{if } W_L = 0 \\
\frac{1}{\langle \varphi_A, \varphi_A \rangle} \left( \gamma \langle \mathbb{1}, \varphi_A \rangle - \langle r_A, \varphi_A \rangle A + \langle r_L + \varphi_L, \varphi_A \rangle L \right) & \text{if } W_L = L
\end{cases}
\]

The optimal surplus \( S^*(T) \) becomes

\[
S^*(T) = r_s^\top \mathbf{s} + \gamma r_\gamma,
\]

where

\[
r_\gamma = \frac{\langle \mathbb{1}, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A
\]

and

\[
r_s^\top = \begin{cases} 
\left( r_A - \frac{\langle r_A, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A, \right. & \left. - \left( r_L - \frac{\langle r_L, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A \right) \right) & \text{if } W_L = 0 \\
\left( r_A - \frac{\langle r_A, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A, \right. & \left. - \left( r_L + \varphi_L - \frac{\langle r_L + \varphi_L, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A \right) \right) & \text{if } W_L = L
\end{cases}
\]

In particular, using an orthogonal projection notation, the above expressions can be written more compactly. Indeed, we have:

\[
r_\gamma = \frac{\langle \mathbb{1}, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A = \mathbb{P}_\varphi_A(\mathbb{1}),
\]

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where $\mathbb{P}_x(y)$ is the orthogonal projection of $y \in \mathcal{L}_2$ on the subspace spanned by $x \in \mathcal{L}_2$. Similarly, we can write $r_s$ as

$$r_s^\top = \begin{cases} \left( \mathbb{P}_{\varphi_A^\perp}(r_A), -\mathbb{P}_{\varphi_{A}^\perp}(r_L) \right) & \text{if } W_L = 0 \\ \left( \mathbb{P}_{\varphi_A^\perp}(r_A), -\mathbb{P}_{\varphi_{A}^\perp}(r_L + \varphi_L) \right) & \text{if } W_L = L \end{cases} \tag{3.16}$$

where $\mathbb{P}_{\varphi_A^\perp}(y)$ is the orthogonal projection of $y \in \mathcal{L}_2$ on the orthogonal subspace to the subspace spanned by $x \in \mathcal{L}_2$.

Since $r_\gamma$ and the components of $r_s$ are projections on orthogonal subspaces of $\mathcal{L}_2$, it follows that the surplus $S^*(T)$ has been decomposed as an orthogonal sum in $\mathcal{L}_2$. We summarize this result in the next corollary.

**Corollary 2.** *When the short-selling constraints on liabilities are binding, the optimal surplus $S^*(T)$ allows an orthogonal decomposition into two $\mathcal{L}_2$-orthogonal returns $r_\gamma$ and $r_s^\top s$. These returns are projections on the excess asset return and its orthogonal complement.*

The pay-off difference

$$r_s^\top s = A\mathbb{P}_{\varphi_A^\perp}(r_A) - L\mathbb{P}_{\varphi_{A}^\perp}(r_L) \tag{3.17}$$

is the single-period global minimum-second-moment (MSM) final surplus, which is obtained as the difference between a single-period assets-only and a single-period liabilities-only MSM pay-off (corresponding to $\gamma = 0$, $L = 0$, and $\gamma = 0$, $A = 0$, respectively). On the other hand, the pay-off $\mathbb{P}_{\varphi_{A}^\perp}(\mathbb{I})$ is the asset excess return which is closest to the fictive risk-free pay-off $\mathbb{I}$ in the $\mathcal{L}_2$-norm.

While the global MSM pay-off in equation (3.28) is the final surplus of a portfolio with a generally non-zero initial position in both assets and liabilities, the pay-off $\mathbb{P}_{\varphi_{A}^\perp}(\mathbb{I})$ is an asset excess return and can be generated by a zero initial cost portfolio investing only in the available asset returns. A similar type of decomposition and interpretation as the one obtained in Corollary 2 arises when the short-selling constraints on assets are binding. The next section gives the projection operators that make such a representation explicit in the case of binding constraints on assets.

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3.1.2 Binding Short-Selling Constraints on Assets

With constraints on assets binding, we again have to study two cases. In the first case, the constraint $W_A \geq 0$ is binding and $W_A = 0$. In the second case, the constraint $A - W_A \geq 0$ is binding and $W_A = A$. Hence, the optimal policies for the two cases satisfy

$$w^* = (0, W_L^*)^\top, \text{ or } w^* = (A, W_L^*)^\top.$$  

We can thus interpret the solution for the first and the second case as the solution of an endogenous liabilities problem with exogenous assets. In the first case, the assets amount $A$ is invested exclusively in the gross benchmark asset return $r_A$. In the second case it is invested exclusively in the gross asset return $r_A + \varphi_A$.

Such an optimization problem can be handled analogously to the one in the previous section. In particular, using a notation based on projections the optimal surplus $S^*(T)$ becomes

$$S^*(T) = r_s^\top s + \gamma r_\gamma,$$  

where

$$r_\gamma = P_{\varphi_L}(1),$$  

and

$$r_s^\top = \begin{cases} \left( P_{\varphi_L}(r_A), -P_{\varphi_L}(r_L) \right) & \text{if } W_A = 0 \\ \left( P_{\varphi_L}(r_A + \varphi_A), -P_{\varphi_L}(r_L) \right) & \text{if } W_A = A \end{cases}.$$  

Since $r_\gamma$ and the components of $r_s$ are projections on orthogonal subspaces of $L_2$ the surplus $S^*(T)$ is again decomposed as an orthogonal sum in $L_2$. By contrast with the previous section, projections are now on subspaces related to the excess liability return rather than the excess asset return.

**Corollary 3.** When the short-selling constraints on assets are binding, the optimal surplus $S^*(T)$ allows an orthogonal decomposition into two $L_2$-orthogonal returns $r_\gamma$ and $r_s$. These returns are projections on the excess liability return and its orthogonal complement.
The pay-off difference

\[ \mathbf{r}_s^T \mathbf{s} = A\mathbb{P}_{\varphi_A^+} (r_A) - L\mathbb{P}_{\varphi_L^+} (r_L) \]

is again a single-period global MSM final surplus, which is obtained as the difference between a single-period assets-only and a single-period liabilities-only MSM pay-off. The pay-off \( \mathbb{P}_{\varphi_L^+} (1) \) is now the liabilities excess return which is closest to the fictive risk-free pay-off \( \mathbb{1} \) in the \( L_2 \)-norm.

### 3.1.3 No Binding Constraints

We now come back to the situation where no constraint is binding. The returns \( r_s \) and \( r_\gamma \) defined in (3.9) and (3.10) for the single-period problem can be written explicitly as:

\[
\begin{align*}
\mathbf{r}_s &= \left( r_A - \frac{(r_A, \varphi_A) - (r_A, \varphi_L)}{(\varphi_A, \varphi_A)} \varphi_A - \frac{(r_A, \varphi_A) - (r_A, \varphi_L)}{(\varphi_A, \varphi_A)} \varphi_L \right), \\
\mathbf{r}_\gamma &= \frac{(\varphi_L, 1)(\varphi_A, \varphi_L) - (\varphi_A, \varphi_L) (\varphi_L, \varphi_A)}{(\varphi_A, \varphi_L)^2 - (\varphi_A, \varphi_A)(\varphi_L, \varphi_L)} \varphi_L.
\end{align*}
\]

(3.22)

(3.23)

Apparently, (3.22) and (3.23) are more complex expressions than those obtained for the case where some constraint is binding. Moreover, in a true multiperiod setting or when several assets and liabilities are present such types of expressions will become even more complicated and difficult to interpret and to handle.

In the case of orthogonal assets and liabilities excess returns, i.e., \( \langle \varphi_A, \varphi_L \rangle = 0 \), equations (3.22) and (3.23) imply that \( r_s \) and \( r_\gamma \) are again projections on a space of excess returns and its orthogonal complement. In the non-orthogonal case, the complexity of the arising expressions makes the structure of the optimal surplus less transparent. One could then apply first a change of basis and then express the optimal surplus with respect to an orthogonal basis of excess assets and liabilities returns. In fact, this approach if very useful for computational purposes in the multiperiod case. However, in order to get more insight into the financial structure of the optimal surplus a formalism that allows to work directly with the initial basis
of assets and liabilities excess returns is necessary. We therefore propose a formalism based on exterior algebra which renders the relevant expressions manageable also in a multiperiod context and in the presence of several assets and liabilities.

Let \( \Lambda^n(L_2) \) be the \( n \)-fold antisymmetric tensor product space of \( L_2 \) and set \( \Lambda^0(L_2) = \mathbb{R} \). Taking the infinite direct sum over \( n \) defines the infinite dimensional Grassmann algebra\(^6\). For details on the general construction, see Berezin (1966). In the sequel it will be sufficient to work with finite dimensional subsets of the Grassmann algebra, and specifically with the 2-fold antisymmetric tensor product space \( \Lambda^2(L_2) \) of \( L_2 \). However, the methodology is in principle applicable also to an infinite dimensional setting, i.e., a setting with an infinite number of assets or liabilities.

We denote the wedge product by ‘\( \wedge \)’ and the inner product on \( \Lambda^2(L_2) \) by \( \langle \cdot, \cdot \rangle_2 \). This product is defined by

\[
\langle w \wedge x, y \wedge z \rangle_2 = \left| \begin{array}{cc} \langle w, y \rangle & \langle w, z \rangle \\ \langle x, y \rangle & \langle x, z \rangle \end{array} \right| ; \quad w, x, y, z \in L_2 ,
\]

where \( | \cdot | \) is the determinant of a 2×2 matrix. In particular, the scalar product in \( L_2 \) is an inner product of elements in \( \Lambda^1(L_2) = L_2 \). This feature allows us to interpret the AL solutions when some constraint is binding as a geometric special case of the solutions when no constraint is binding.

To write compactly the returns (3.22) and (3.23) we finally define the following operator, given the two one-forms \( \varphi_A, \varphi_L \):

\[
[x, y] = \frac{\langle x \wedge y, \varphi_A \wedge \varphi_L \rangle_2}{\langle \varphi_A \wedge \varphi_L, \varphi_A \wedge \varphi_L \rangle_2} ; \quad x, y \in L_2 . \tag{3.24}
\]

Using this notation, the returns (3.22) and (3.23) are given by:

\[
\mathbf{r}_s = \left( \begin{array}{c} r_A - ([r_A, \varphi_L] \varphi_A - [r_A, \varphi_A] \varphi_L) \\
-(r_L - ([r_L, \varphi_L] \varphi_A - [r_L, \varphi_A] \varphi_L)) \end{array} \right) , \quad \mathbf{r}_\gamma = [\mathbb{I}, \varphi_L] \varphi_A - [\mathbb{I}, \varphi_A] \varphi_L . \tag{3.25}
\]

\(^6\)If the appropriate topology is added.
We next show that also in the case of no binding constraints the components of \( r_s \) and the pay-off \( r_\gamma \) are orthogonal elements of \( L_2 \). Similarly to the previous sections we introduce some appropriate projection operators that highlight this point. To this end, we define for any \( x = (x_1, x_2)^\top \in L_2^2 \) the operator \( P_x \) by:

\[
P_x(y) = \begin{pmatrix} y_1 & y_2 \\ x_1 & y \\ x_2 & y_1 \\ x & 1 \end{pmatrix}.
\]

(3.26)

In particular, with this last definition we obtain for any \( y \in L_2^2 \):

\[
P_{[\varphi]}(P_{[\varphi]}(y)) = [(y, \varphi_L)\varphi_A - (y, \varphi_A)\varphi_L, \varphi_A] - [(y, \varphi_L)\varphi_A - (y, \varphi_A)\varphi_L, \varphi_A] \varphi_L.
\]

Using the bilinearity of the operator \([x, y]\) and since \([\varphi_A, \varphi_A] = [\varphi_L, \varphi_L] = 0, [\varphi_A, \varphi_L] = -[\varphi_L, \varphi_A] = 1\), we get \( P_{[\varphi]}(P_{[\varphi]}(y)) = P_{[\varphi]}(y) \). Similar calculations show that \( P_{[\varphi]}(\cdot) \) is a self-adjoint operator in \( L_2 \). Therefore, \( P_{[\varphi]}(\cdot) \) is an orthogonal projection on a finite dimensional subspace \( M \) of \( L_2 \). In our case, \( M \) is the linear space generated by \( \varphi_A, \varphi_L \).

Based on this last result the orthogonality of the returns in (3.25) becomes apparent. In fact, from (3.25) we have:

\[
\begin{pmatrix} r_s \\ r_\gamma \end{pmatrix} = \begin{pmatrix} P_{[\varphi]}^\perp(r_A) \\ -P_{[\varphi]}^\perp(r_L) \end{pmatrix}, \quad r_\gamma = P_{[\varphi]}(1).
\]

(3.27)

We summarize the relevant findings in the next corollary.

**Corollary 4.** When no short-selling constraint is binding the optimal surplus \( S^*(T) \) allows an orthogonal decomposition into two \( L_2 \)-orthogonal returns \( r_\gamma \) and \( r_s^\top s \). These returns are projections on the space generated by the excess assets and liabilities returns and its orthogonal complement.

The pay-off difference

\[
r_s^\top s = A P_{[\varphi]}^\perp(r_A) - L P_{[\varphi]}^\perp(r_L)
\]

(3.28)
is also in this general setting a single-period global MSM final surplus, which is obtained as the
difference between a single-period assets-only and a single-period liabilities-only MSM pay-off.
Similarly, the pay-off $P_{[\varphi]}(I)$ is now the asset and liability excess return which is closest to the
fictive risk-free pay-off $I$ in the $L_2$-norm. The exact contribution of the excess returns $\varphi_A, \varphi_L$
and the benchmark returns $r_A, r_L$, to $r_s$ and $r_\gamma$ is explicitly given by the coefficients $[r_A, \varphi_A],$
$[r_L, \varphi_A], [r_A, \varphi_L], [r_L, \varphi_L], [I, \varphi_A]$ and $[I, \varphi_L]$.

The orthogonal decomposition result implied by (3.25) is very useful to compute recursively
the value function of the AL optimization problem. Indeed, we have:

$$J(s) = Eg S^*(T) - \frac{1}{2} S^*(T)^2$$
$$= E \left[ \gamma \left( r_s^T s + \gamma r_\gamma \right) - \frac{1}{2} \left( r_s^T s + \gamma r_\gamma \right)^2 \right]$$
$$= \gamma^2 E \left[ r_\gamma - \frac{1}{2} r_\gamma^2 \right] + \gamma E \left[ (1 - r_\gamma) r_s^T s \right] - \frac{1}{2} s^T E \left( r_s r_s^T \right) s .$$

Using the further properties

$$\langle r_s, r_\gamma \rangle = 0 \quad , \quad \langle r_\gamma, r_\gamma \rangle = \langle r_\gamma, I \rangle , \tag{3.29}$$

the value function simplifies to:

$$J(s) = \frac{\gamma^2}{2} E \left[ r_\gamma \right] + \gamma E \left[ r_s^T \right] s - \frac{1}{2} s^T E \left[ r_s r_s^T \right] s . \tag{3.30}$$

The same type of decomposition of the value function holds when some constraint is binding
at time $T - 1$. In this case, the relevant returns $r_s$ and $r_\gamma$ are those defined in (3.13), (3.14)
and (3.19), (3.20).

### 3.2 Two-period Optimization

To clarify the differences between single-period and multiperiod optimizations we consider
next the two-period problem. Using the above geometric formalism the structure behind the
solution for the more general case will then be evident. For brevity we drop in the sequel all
time indices $T - 2$. Moreover, since the case with binding constraints can be again treated as a problem with exogenous assets or liabilities we focus in the sequel on the situation where no short-selling constraint is binding. From (3.30), the value function at time $T - 2$ is

$$J(s) = \max_{w \in A(s)} \mathbb{E}[J(s(T - 1))]$$

$$= \max_{w \in A(s)} \mathbb{E} \left[ \gamma r_s(T - 1)^\top s(T - 1) - \frac{1}{2} s(T - 1)^\top (r_s(T - 1)r_s(T - 1)^\top)s(T - 1) + \frac{\gamma^2}{2} r_s \right]$$

$$= \max_{w \in A(s)} \mathbb{E} \left[ \left( \gamma d^\top - s^\top \tilde{R}d^\top \right) \tilde{\Phi}w - \frac{1}{2} w^\top \tilde{\Phi}dd^\top \tilde{\Phi}w \right] + C(s) ,$$

using the state dynamics (2.2). The term $C(s)$ in the value function does not depend on $w$ while $\tilde{R}, \tilde{\Phi}$ are defined by:

$$\tilde{R} = RRR_s(T - 1) \ , \ \tilde{\Phi} = \Phi R_s(T - 1) \ ,$$

with:

$$R_s(T - 1) = \text{Diag} \left( \begin{array}{c} P_{[\varphi(t - 1)]^\perp}(r_A(T - 1)) \\ P_{[\varphi(t - 1)]^\perp}(r_L(T - 1)) \end{array} \right) . \quad (3.31)$$

Therefore, the functional form of the value function at time $T - 2$ is the same as the one at time $T - 1$ and the optimal policy at time $T - 2$ is given by

$$w^* = \left[ \mathbb{E} \left( \tilde{\Phi}dd^\top \tilde{\Phi} \right) \right]^{-1} \left[ \gamma \mathbb{E}(\tilde{\Phi}d) - \mathbb{E}(\tilde{\Phi}dd^\top \tilde{R})s \right] . \quad (3.32)$$

Then,

$$S^*(T) = r_s(T - 1)^\top s(T - 1) + \gamma r_s(T - 1)$$

$$= d^\top R_s(T - 1)(Rs + \Phi w^*) + \gamma r_s(T - 1)$$

$$= r_s(T - 2)^\top s + \gamma(r_s(T - 2) + r_s(T - 1)) \ , \quad (3.33)$$
where

\[
\mathbf{r}_s^\top(T - 2) = d^\top \left( \tilde{\mathbf{R}} - d^\top \Phi \left[ \mathbb{E} \left( \tilde{\Phi} d d^\top \tilde{\Phi} \right) \right]^{-1} \mathbb{E} \left( \tilde{\Phi} d d^\top \tilde{R} \right) \right) ,
\]

(3.34)

\[
\mathbf{r}_\gamma(T - 2) = d^\top \tilde{\Phi} \left[ \mathbb{E} \left( \tilde{\Phi} d d^\top \tilde{\Phi} \right) \right]^{-1} \mathbb{E} (\tilde{\Phi} d) .
\]

(3.35)

In particular, defining \( \tilde{\varphi} = (\tilde{\varphi}_A, \tilde{\varphi}_L)^\top \) and \( \tilde{\mathbf{r}} = (\tilde{r}_A, \tilde{r}_L)^\top \) for the vectors of the diagonal elements of \( \tilde{\Phi} \), it follows:

\[
\mathbf{r}_s^\top(T - 2) = \left( \tilde{r}_A - [\tilde{r}_A, \tilde{\varphi}_L] \tilde{\varphi}_A + [\tilde{r}_A, \tilde{\varphi}_A] \tilde{\varphi}_L, -(\tilde{r}_L - [\tilde{r}_L, \tilde{\varphi}_L] \tilde{\varphi}_A + [\tilde{r}_L, \tilde{\varphi}_A] \tilde{\varphi}_L) \right) \\
= \left( \mathbb{P}_{[\tilde{\beta}]^\top}(\tilde{r}_A), -\mathbb{P}_{[\tilde{\beta}]^\top}(\tilde{r}_L) \right) ,
\]

(3.36)

\[
\mathbf{r}_\gamma(T - 2) = \left[ \mathbb{I}, \tilde{\varphi}_L \right] \tilde{\varphi}_A - \left[ \mathbb{I}, \tilde{\varphi}_A \right] \tilde{\varphi}_L = \mathbb{P}_{[\tilde{\beta}]}(\mathbb{I}) .
\]

(3.37)

Hence, \( \mathbf{r}_s(T - 2) \) and \( \mathbf{r}_\gamma(T - 2) \) are again projections on orthogonal subspaces of \( \mathcal{L}_2 \). Moreover, since by construction \( \varphi_A(T - 1) \) and \( \varphi_L(T - 1) \) are orthogonal to \( \tilde{\varphi}_A, \tilde{\varphi}_L \) and \( \tilde{r}_A, \tilde{r}_L \), the single-period excess return \( \mathbf{r}_\gamma(T - 1) \) is also orthogonal to \( \mathbf{r}_s(T - 2) \) and \( \mathbf{r}_\gamma(T - 2) \). As a consequence, the two-period optimal surplus \( S^*(T) \) is written in (3.33) as a linear combination of the orthogonal pay-offs \( \mathbf{r}_s(T - 2)^\top \mathbf{s} \), \( \mathbf{r}_\gamma(T - 2) \) and \( \mathbf{r}_\gamma(T - 1) \).

**Corollary 5.** The two-period optimal surplus \( S^*(T) \) allows an orthogonal decomposition into three \( \mathcal{L}_2 \)-orthogonal pay-offs \( \mathbf{r}_\gamma(T - 1) \), \( \mathbf{r}_\gamma(T - 2) \) and \( \mathbf{r}_s(T - 2)^\top \mathbf{s} \).

The same general geometric structure as for the single-period case arises. Indeed, single projections are replaced by compositions of projections on suitable subspaces of random variables measurable with respect to information at time \( T - 1 \) and \( T - 2 \). Precisely, \( \mathbf{r}_s(T - 2)^\top \mathbf{s} \) is the two-period global MSM return for AL portfolios, obtained as the difference between a two-period assets-only and a two-period liabilities-only MSM pay-off. Further, \( \mathbf{r}_\gamma(T - 2) \) is the projection of the final risk free pay-off \( \mathbb{I} \) on the space of two-period assets and liabilities excess returns orthogonal to the final period excess returns \( \varphi_A(T - 1) \) and \( \varphi_L(T - 1) \). This space is generated by the two-period assets and liabilities excess return \( \tilde{\varphi}_A \) and \( \tilde{\varphi}_L \). \( \varphi_A(T - 2) \) and \( \varphi_L(T - 2) \) are excess returns of a zero initial cost investment from \( T - 2 \) to \( T - 1 \), while the components of \( \mathbf{r}_s(T - 1) \) are assets and liabilities returns of a single-period MSM AL portfolio.
Therefore, \( r_\gamma(T - 2) \) can be interpreted as the two-period assets and liabilities excess return of a particular zero initial cost strategy, which is exposed to the second asset and the second liability risk only in the transaction period from \( T - 2 \) to \( T - 1 \). Roughly speaking, this portfolio produces a local exposure to the second asset and the second liability returns relatively to their benchmarks. Therefore, we can interpret \( r_\gamma(T - 2) \) as a local, two-period, excess assets and liabilities return. Summarizing, the two-period final surplus has been decomposed as the orthogonal sum of a two-period MSM surplus and two local assets and liabilities excess returns. The same type of decomposition arises in the general multiperiod setting.

### 3.3 Multiperiod Optimization

From the above orthogonal decomposition for time \( T - 2 \) we proceed recursively to derive the corresponding orthogonal decomposition for time \( T - t \). In particular, the value function at time \( T - t \) is of the same functional form as in the two-period model, with return matrices \( \tilde{R}(T - t) \) and \( \tilde{\Phi}(T - t) \) given by:

\[
\tilde{R}(T - t) = R(T - t)R_s(T - t + 1) \quad , \quad \tilde{\Phi}(T - t) = \Phi(T - t)R_s(T - t + 1) \quad ,
\]

where

\[
R_s(T - t + 1) = \text{Diag} \left[ \begin{pmatrix} \mathbb{P}[\Phi(T - t + 1)] (\tilde{r}_A(T - t + 1)) \\ \mathbb{P}[\Phi(T - t + 1)] (\tilde{r}_L(T - t + 1)) \end{pmatrix} \right] . \tag{3.38}
\]

The resulting orthogonal final surplus components are implied by the pay-offs

\[
r_s(T - t)^T = \begin{pmatrix} \mathbb{P}[\Phi(T - t)] (\tilde{r}_A(T - t)) , -\mathbb{P}[\Phi(T - t)] (\tilde{r}_L(T - t)) \end{pmatrix} \tag{3.39}
\]

and

\[
r_\gamma(T - t + i) = \mathbb{P}[\Phi(T - t + i)] (1) \quad , \quad i = 0, ..., T - t - 1 \quad . \tag{3.40}
\]
Iterating the same arguments as in the two-period case, it follows at once that all surplus components are orthogonal elements in $L_2$. Since the optimal surplus $S^*(T)$ is obtained recursively as

$$S^*(T) = s(T - t)^\top r_s(T - t) + \gamma \sum_{i=1}^{t} r_\gamma(T - i),$$

the orthogonal decomposition in Theorem 1 follows for arbitrary $T - t$.

For any $t = 1, \ldots, T$ the return $r^\top (T - t)s(T - t)$ is the $(T - t)$—period MSM return. Similarly, the return $r_\gamma(T - t)$ can be interpreted as a local $(T - t)$—period AL excess return (see again the discussion for the two-period setting in the previous section). Summarizing, the final surplus $S^*(T)$ is thus decomposed in an orthogonal sum of a $(T - t)$—period MSM AL surplus and a set of $T - t$ local AL excess returns.

In particular, for states where short-selling constraints are binding, the relevant projections map returns on some lower dimensional spaces than in the unconstrained case. If for example the constraint $W_L = 0$ binds at time $T - t$ the relevant return components degenerate to

$$r_s^\top (T - t) = (P_{\tilde{\varphi}_1(T-t)^\perp}(\bar{r}_A(T - t)), -P_{\tilde{\varphi}_1(T-t)^\perp}(\bar{r}_L(T - t))) ,$$

$$r_\gamma(T - t) = P_{\tilde{\varphi}_1(T-t)^\perp}(1) ,$$

where $\tilde{\varphi}_1$ is the first component of $\tilde{\varphi}$. Finally, when short-selling constraints are simultaneously binding on assets and liabilities no optimization has to be performed and the resulting orthogonal projections collapse to the identity operator. This concludes the proof of Theorem 1.

4 Application: Exogenous vs. Endogenous Liabilities

The assumption of exogenous liabilities is realistic for some types of applications, as for instance those related to the AL optimization of pension funds. However, such an assumption is less motivated for financial institutions like banks. Indeed, such institutions typically have to
optimize not only their assets mix, but also their liabilities mix. The orthogonal decomposition in Theorem 1 can be used to analyze several issues related to AL portfolio optimization, as for instance the determination of the optimal funding ratio or the analysis of the impact of the rebalancing frequency. This section discusses some of these topics with a particular emphasis on the distinction between AL optimizations under exogenous or endogenous liabilities. For simplicity of exposition we abstract from the existence of binding short selling constraints.

4.1 Geometric structure

One issue is the relation between MVF’s under exogenous and endogenous liabilities. Based on the orthogonal decomposition in Theorem 1 we can focus without loss of generality on the single-period case. Any MVF can be represented in the \((\text{VAR}(S), \text{E}(S))\)-space by a curvature parameter \(a\), a vertical shift parameter \(b\) and a horizontal shift parameter \(c\). The description of MVF frontiers for the exogenous and endogenous liabilities case is provided by the next corollary.

**Corollary 6.** The single-period MVF is a quadratic form

\[
\text{VAR}(S^*(T)) = a\text{E}(S^*(T))^2 + 2b\text{E}(S^*(T)) + c,
\]

where under endogenous liabilities the coefficients \(\{a, b, c\}\) are

\[
a = \frac{1}{\langle P_{[\phi]}(1), 1 \rangle} - 1,
\]

\[
b = -\frac{\langle P_{[\phi]}^+(r)^{\top}s, 1 \rangle}{\langle P_{[\phi]}(1), 1 \rangle},
\]

\[
c = \langle P_{[\phi]}^+(r)^{\top}s, P_{[\phi]}^+(r)^{\top}s \rangle + \frac{\langle P_{[\phi]}^+(r)^{\top}s, 1 \rangle^2}{\langle P_{[\phi]}(1), 1 \rangle}.
\]

\(^7\)In particular, in the multiperiod setting the moments of the sum \(\sum_{i=1}^{t} r_i(T - i)\), instead of those of \(r_i(T - 1)\) (see the corollary below), will affect the quadratic form describing the MVF’s. Similarly, the relevant MSM returns \(r_i^s\) and their moments will also be different.
Under exogenous liabilities the coefficients \{\tilde{a}, \tilde{b}, \tilde{c}\} of the quadratic form are

\[
\tilde{a} = \frac{1}{\langle \varphi_A(1), 1 \rangle} - 1, \\
\tilde{b} = \frac{\langle \varphi_A^\perp(\mathbf{r})^\top \mathbf{s}, 1 \rangle}{\langle \varphi_A(1), 1 \rangle}, \\
\tilde{c} = \langle \varphi_A^\perp(\mathbf{r})^\top \mathbf{s}, \varphi_A^\perp(\mathbf{r})^\top \mathbf{s} \rangle + \frac{\langle \varphi_A^\perp(\mathbf{r})^\top \mathbf{s}, 1 \rangle^2}{\langle \varphi_A(1), 1 \rangle}.
\]

Proof. The first and second moment of the optimal surplus are given by

\[
\mathbb{E}[S^*(T)] = \mathbb{E}[\mathbf{r}_s^\top \mathbf{s}] + \gamma \mathbb{E}[r_\gamma], \\
\mathbb{E}[(S^*(T))^2] = \mathbb{E}\left[(\mathbf{r}_s^\top \mathbf{s} + \gamma r_\gamma)^2\right].
\]

Solving for \(\gamma\) and plugging in the arising expressions into the definition of the variance, we obtain the coefficients \{a, b, c\} using (3.29). The coefficients \{\tilde{a}, \tilde{b}, \tilde{c}\} are obtained by exploiting the projection properties \(\langle \varphi_A(1), \varphi_A(1) \rangle = \langle \varphi_A^\perp(1), \varphi_A^\perp(1) \rangle = 0\).

The corollary implies that in an AL optimization under exogenous liabilities the implied MVF is affected by liabilities in only two ways. First, through a vertical shift caused by the parameter \(\tilde{c}\) and, second, by a sidewise shift caused by the parameter \(\tilde{b}\). Therefore, the introduction of liabilities induces a pure translation of the MVF in the mean-variance space, caused by a modified global MSM surplus, relatively to a pure assets optimization. Under endogenous liabilities the curvature parameter \(a\) depends on the structure of the liabilities returns. Hence, in this case the AL optimization also produces a change in the curvature of the relevant MVF. As expected, the curvature under endogenous liabilities is lower. This is proved in the next proposition.

**Proposition 7.** The curvature of the MVF under endogenous liabilities is always less or equal to the curvature of the MVF under exogenous liabilities. This leads to a better risk-return tradeoff in the \((\text{VAR}(S), \mathbb{E}(S))\)-space.
Proof. We have to show that
\[ \frac{1}{\langle P_{\varphi A}(I), I \rangle} \geq \frac{1}{\langle P_{\varphi L}(I), I \rangle}, \]
which, after some algebraic manipulation, is equivalent to
\[ \frac{\langle \varphi A, \varphi A \rangle}{(I, \varphi A)^2} \geq \frac{\langle \varphi A, \varphi A \rangle \langle \varphi L, \varphi L \rangle - \langle \varphi A, \varphi L \rangle^2}{(I, \varphi L)^2 (I, \varphi A)^2} - 2(I, \varphi A)(I, \varphi L) + (I, \varphi A)^2 (I, \varphi L)^2. \] (4.43)

But, this last inequality is equivalent to
\[ (\langle \varphi A, \varphi L \rangle (I, \varphi A) - (I, \varphi L) \langle \varphi A, \varphi A \rangle)^2 \geq 0. \] (4.44)

This completes the proof.

Interestingly, the change in MVF because of endogenous liabilities is not arbitrary. In fact, the MVF under exogenous and endogenous liabilities have to share one tangential point.

**Proposition 8.** The MVF under endogenous and exogenous liability share a tangential point in \((\mathbb{E}(S), \text{VAR}(S))\)-space.

Proof. The claim is proved, if the discriminant of the parabola vanishes, i.e.,
\[ (b - \bar{b})^2 = (a - \bar{a})(c - \bar{c}). \] (4.45)

This is shown after some lengthy but straightforward calculations.\(^8\)

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\(^8\)These calculations can be obtained from the authors.
4.2 Numerical Example

We analyze in a numerical example the differences in the MVF’s under exogenous and endogenous liabilities. We assume $A(0) = 1.05$ and $L(0) = 1$. The vector of the mean returns and the vector of volatilities are set equal to

$$\begin{align*}
(\mu(r_A), \mu(\varphi_A), \mu(r_L), \mu(\varphi_L)) &= (1.04, 0.20, 1.06, 0.12), \\
(\sigma(r_A), \sigma(\varphi_A), \sigma(r_L), \sigma(\varphi_L)) &= (0.085, 0.272, 0.083, 0.195).
\end{align*}$$

For the correlation matrix we assume:

<table>
<thead>
<tr>
<th></th>
<th>$r_A$</th>
<th>$\varphi_A$</th>
<th>$r_L$</th>
<th>$\varphi_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_A$</td>
<td>1</td>
<td>0.178</td>
<td>0.445</td>
<td>-0.010</td>
</tr>
<tr>
<td>$\varphi_A$</td>
<td>1</td>
<td>0.446</td>
<td>-0.157</td>
<td></td>
</tr>
<tr>
<td>$r_L$</td>
<td></td>
<td>1</td>
<td>0.270</td>
<td></td>
</tr>
<tr>
<td>$\varphi_L$</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

The upper panel of Figure 1 plots single-period MVFs for a one year investment horizon. As expected, the MVF under endogenous liabilities (the straight curve) is an envelope of the MVF under exogenous liabilities (the dotted curve). Under the given model parameters the tangential point of the two curves lies on the inefficient part of the endogenous liabilities MVF. This needs not be the case for other model inputs.

The lower panel of Figure 1 plots multiperiod MVFs. To this end, we assume iid returns. In such a setting, the multiperiod surplus components in Theorem 1 can be expressed in closed-form. We consider the same investment horizon of one year. Dotted curves represent the MVFs under exogenous liabilities when (from right to left) the rebalancing frequency is 1, 2, 6, 12 (monthly), and 360 (daily). Straight curves represent MVFs under endogenous liabilities. In particular, when the rebalancing frequency increases the MVFs are shifted to the right. This rightward shift results from intertemporal diversification opportunities. The benefit from intertemporal diversification is most pronounced at low frequencies: Switching from a static...
Figure 1: MFVs for exogenous and endogenous liabilities. The upper panel plots the single-period MVF for the exogenous (dotted) and exogenous (straight) liability case. The lower panel plots multiperiod MVFs. The dotted lines represents the one-year MFVs for exogenous liabilities when (from right to left) the rebalancing frequency is 1, 2, 6, 12 (monthly), and 360 (daily). The straight lines represent the corresponding MVFs for endogenous liabilities.

to a semi-annual optimization shifts substantially the frontiers. However, a switch from a monthly to a daily rebalancing only produces slight MVF’s changes. Finally, Figure 1 shows that for any given surplus variance the benefits from intertemporal diversification are more pronounced under endogenous liabilities.
5 Conclusions

We propose a geometric approach for the joint multiperiod mean-variance optimization of assets and liabilities (AL). With the geometric formalism of exterior algebra we show that the arising optimal surpluses and minimum variance frontiers (MVF) can be decomposed in an orthogonal set of basis returns. This decomposition holds both under binding and not binding short selling constraints on AL. We obtain the orthogonal basis returns by applying projection operators which can be represented by means of inner products on Grassmann algebras. If short-selling constraints are binding, the optimization problem reduces to one under some exogenous assets or liabilities. In this case, the effect of a binding constraint on the optimal surplus is reflected by a simpler structure of the projection operators arising in the orthogonal surplus decomposition. Finally, we illustrate the usefulness of the orthogonal decomposition by studying the structure of MVF under endogenous liabilities and by quantifying in a numerical example the impact of the rebalancing frequency on the prevailing dynamic risk and return trade-off.
References


