Equilibrium Impact of Value-At-Risk

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Equilibrium Impact of Value-At-Risk *

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Abstract

We study in a general perspective the partial equilibrium incentives and the general equilibrium asset pricing implications of Value-at-Risk (VaR) regulation in continuous time economies with intermediate consumption, stochastic opportunity set, and heterogeneous attitudes to risk. Our findings show that the partial equilibrium incentives of VaR regulation can lead banks to increase their risk exposure precisely in “high volatility” states, because of an anticipatory effect of VaR constraints on the optimal hedging demand. In general equilibrium, VaR regulation affects equity volatility and equity expected returns ambiguously, depending on the given model dynamics and heterogeneity structure. On the other hand, VaR constraints tend to produce unambiguously lower interest rates and higher equity Sharpe ratios.

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Regulation aims at maintaining and improving the safety of the financial industry by defining minimal capital standards. In the 1996 Amendment of the Basle 1988 Accord, the Bank for International Settlements (BIS) extended the regulatory framework for credit risk to market risk. The Amendment recognizes the complexity of correctly assessing capital reserves and allows qualified banks to use their internal models for regulatory reporting. In 1996 Value-at-Risk (VaR) based risk management had already emerged as common market practice. Therefore, the BIS chose VaR as the regulatory reporting tool and is about to extend this practice to credit risk.

This paper studies the partial equilibrium incentives and the general equilibrium asset pricing implications of VaR regulation for economies with stochastic opportunity set dynamics and heterogeneous attitudes to risk. Incorporating these features is crucial for a full understanding of the regulatory incentives and the policy implications of VaR regulation. Indeed, strengthening one of these two assumptions to either a constant opportunity set or a representative agent economy drastically simplifies, and in some cases trivializes, the emerging regulatory effects. Our analysis reveals some unexpected results and findings, which certainly neither the financial industry nor the regulators expected. In a naive way, one would expect to have a clear-cut statement about the outcome of applying regulatory standards to the financial sector. From our analysis, however, we conclude that such a statement strongly depends on market factors, regulatory control variables, the market participants, and finally on the chosen model.

Our main results are as follows. First, the partial equilibrium incentives of VaR regulation are ambiguous. Indeed, they can be quite perverse, depending on the structure of the given opportunity set dynamics and the resulting intertemporal hedging behavior. We characterize and provide explicit examples where such intertemporal hedging behavior leads banks to increase their risk exposure precisely in high-volatility states. However, these distorted incentives are not due to the lack of coherency of VaR. They also arise under Expected Shortfall constraints. Second, the endogenous general equilibrium impact of VaR regulation on equity volatility and expected returns is ambiguous, as it may bind investors with heterogeneous risk aversions in a whole variety of possible ways. Finally, VaR regulation produces unambiguous effects on interest rates and equity Sharpe ratios. Interest rates decrease, while Sharpe ratios increase. In this respect, the general equilibrium impact of VaR regulation rationalizes some well-known puzzles in the financial literature (see e.g. Cochrane (1997)). Against the backdrop of these results, the regulator’s aim to improve the safety of the financial industry, regardless of the state-dependent structure of the economy, seems highly questionable.

Surprisingly, academic literature providing a thorough analysis of both the partial equilibrium incentives and general equilibrium impacts of VaR regulation in a dynamic setting is almost non-existent. The first issue has been studied in Basak and Shapiro (2001), Cuoco, He, and Issaenko (2001), and - more recently - by Cuoco and Liu (2002). All these papers are based
on a constant opportunity set assumption. To our knowledge, only two contributions have addressed the second issue explicitly: Basak and Shapiro (2001) and Danielsson, Shin, and Zigrand (2001). As mentioned, the first one is based on a constant opportunity set assumption. The second one assumes a very simplified and particular dynamic setting consisting of a sequence of myopic (single period) general equilibrium economies. Therefore, neither paper studies the general implications of dynamic VaR regulation in the joint presence of stochastic opportunity set dynamics and heterogeneities of risk attitudes.

Basak and Shapiro (2001) show analytically within a model with a static VaR constraint over terminal wealth that investors are induced to take on a larger risk exposure than in the unconstrained setting, thereby deepening and prolonging market downturns. They ascribe this unappealing effect to the non-coherency of VaR. In their model, when VaR is replaced by a coherent risk measure like Expected Shortfall, such unattractive effects disappear. Cuoco, He, and Issaenko (2001) attribute the findings of Basak and Shapiro (2001) to the static definition of the VaR constraint. In a partial equilibrium, they derive numerical solutions showing that, when VaR limits are dynamically updated, no unappealing incentives arise. Cuoco and Liu (2002) incorporate both the optimal trading and the optimal VaR reporting behavior of the financial institution and find that VaR regulation can be very effective in curbing the risk of trading portfolios and in inducing truthful revelation of this risk. Finally, Danielsson, Shin, and Zigrand (2001) analyze by simulation the effects of VaR regulation, finding that it affects prices, liquidity and volatility adversely, when compared with a benchmark unregulated economy. However, the very simple myopic structure of their economy eliminates by construction the effects of some key variables like intermediate consumption, the intertemporal hedging demand, and all investment horizon effects.

A thorough analysis of VaR-based regulation is a challenging problem, since not only the bank’s behavior but also the impacts on the financial market have to be derived within a “realistic” model. We are convinced that such an analysis rests upon three minimal requirements at least. First, the model should be based on a stochastic opportunity set allowing for stochastic volatility and several forms of realistic intertemporal relations between Sharpe ratios and volatilities. Indeed, assuming only normally distributed asset returns is insufficient for analyzing the role of regulation when market conditions worsen. It is in these situations that regulation has to prove its effectiveness. These situations are typically characterized by high and fast-changing volatilities and expected equity returns. Second, banks should solve an intertemporal optimization problem (thereby explicitly taking investment horizon effects into consideration) under a properly defined VaR constraint. By “properly defined” we mean that the VaR constraint should mimic the regulator’s specification and, specifically, should be dynamically updated. Thirdly, individual risk management behavior must be embedded within a general equilibrium analysis with heterogeneous banks, where the relevant VaR constraints
are endogenously determined in equilibrium. This allows to study the aggregate effect of VaR
regulation. The restriction to a myopic economy misses a broad set of relevant equilibrium
feedback effects.

We contribute to the existing literature by encompassing the above requirements in a
continuous-time model where VaR-regulated investors derive utility both from intermediate
consumption and terminal wealth. Interest rates, drift and volatility of asset price processes
depend on a stochastic state variable. Closed-form solutions for optimal policies are obviously
not available in such a general setting. We overcome this problem by making use of asymptotic
approximation procedures relying on perturbation theory to analyze the arising equilibria
(see also Kogan and Uppal (2001), as well as Trojani and Vanini (2001)). Using asymptotic
analysis in the study of VaR regulation is a necessary compromise in order to study the arising
equilibrium effects analytically, both under general conditions on the economy structure and
in the presence of endogenous VaR constraints. By applying approximate solutions to a broad
class of realistic models, we find that the general economic implications of VaR regulation
are very complex and cannot be characterized irrespective of the given economic setting. As
a consequence, the intuition implied by exact solutions of particular models seems to be too
limited for the purpose of understanding the general economic consequences of VaR regulation.

The remainder of the paper is structured as follows. The next section introduces the
basic model set-up. Section 2 elaborates on the bank’s optimization problem and presents
its solution in the presence of VaR constraints. Section 3 investigates general equilibria with
VaR-constrained investors and derives the implications for interest rates, the dynamics of the
price of equity and the implied portfolio policies. Section 4 concludes.

1 The Model

The financial market consists of a risky asset with price $P_t$ and an instantaneous risk-free
money-market account with value $B_t$ at time $t$. The dynamics of $B_t$ and $P_t$ are

$$
\begin{align*}
\text{d}B_t &= r(X_t)B_t \text{d}t, \quad B_0 = 1, \\
\text{d}P_t &= \alpha(X_t)P_t \text{d}t + \sigma(X_t)P_t \text{d}Z_t, \quad P_0 = p.
\end{align*}
$$

(1)

with $r(X_t)$ the risk-free rate process, and

$$
\begin{align*}
\text{d}X_t &= \mu_X(X_t) \text{d}t + \sigma_X(X_t) \text{d}Z_t^X, \quad X_0 = x.
\end{align*}
$$

(3)
Uncertainty is modelled by a two dimensional Brownian motion \((Z_t, Z_t^X)\) having correlation
\[E[dZ_t dZ_t^X] = \rho dt.\]
In the sequel, we split the drift of the asset price process into the short rate component \(r(X_t)\) and a risk premium component \(\lambda(X_t)\), i.e. \(\alpha(X_t) = r(X_t) + \lambda(X_t)\).

We consider a bank selecting a portfolio fraction \(^3 w_t\) of current wealth \(W_t\) invested in the risky asset, and a fraction \(1 - w_t\) invested in the riskless asset. Thus the bank’s wealth dynamics are
\[
\frac{dW_t}{W_t} = (w_t \lambda(X_t) + r(X_t))dt + w_t \sigma(X_t)dZ_t. \tag{4}
\]

The regulator is supposed to define the bank’s constraint on market risk by means of a VaR risk measure (for an overview on VaR see Jorion (1997) and Duffie and Pan (1997)).

Definition 1. The time-\(t\) regulatory VaR of a portfolio \(w_t\) for a given \(\mathbb{P}\)-probability level \(\nu \in (0, 1)\) and for a fixed time-horizon \(\tau > 0\) is defined by
\[
\text{VaR}_{t}^{\nu,w} = \inf\{L \geq 0|\mathbb{P}(W_t - W_{t+\tau} \geq L|\mathcal{F}_t) < \nu\}, \tag{5}
\]
where \(W_{t+\tau}\) is the portfolio value at time \(t + \tau\) of a fixed-weight strategy with initial weight \(w_t\) at time \(t\).

Definition 1 is consistent with the VaR concept adopted by regulators to measure market risk. For reporting purposes the time-horizon \(\tau\) is typically 1 day or 10 days. In our model, the bank’s VaR is bounded at time \(t\) by an exogenous limit \(\overline{\text{VaR}}_t\) for the given time-horizon \(\tau\). In the sequel we work with a VaR limit \(\overline{\text{VaR}}_t\) proportional to current wealth, i.e.,
\[
\overline{\text{VaR}}_t = \beta W_t, \beta \in [0, 1]. \tag{6}
\]

The wealth dynamics (4) depend on the stochastic opportunity set \(X_t\). Thus, we cannot expect to obtain closed-form solutions for the bank’s intertemporal decision problem in the presence of VaR constraints. To retain analytical tractability, we therefore carefully approximate the VaR constraint implied by (5), (6). In particular, we apply the Itô Taylor formula to define the first-order approximation,
\[
\log W_{t+\tau} \approx \log W^{(1)}_{t+\tau} = \log W_t + \left(r(X_t) + w_t \lambda(X_t) - \frac{1}{2} w_t^2 \sigma(X_t)^2\right) \tau. \tag{7}
\]
The accuracy of the approximation (7) is quantified by the next proposition.

Proposition 1. The approximation error of the first-order approximation \(W^{(1)}_{t+\tau}\) for the value \(W_{t+\tau}\) of a fixed-weight portfolio with initial weight \(w_t\) is bounded by
\[
\mathbb{P}\left|\log W^{(1)}_{t+\tau} - \log W_{t+\tau}\right| \geq M |\mathcal{F}_t| \leq \frac{1}{M} \mathbb{E}[\|R\|], \tag{8}
\]
where
\[
\mathbb{E}[|R|] = \left| \int_r^{r+\tau} \mathbb{E}\left[ Lr(X_u) + w_t L \lambda(X_u) + \frac{1}{2} w_t^2 L \sigma(X_u)^2 | \mathcal{F}_t \right] \ duds \right|
\]
and \( L = \mu X \frac{\partial}{\partial X} + \frac{1}{2} \sigma^2 X \frac{\partial^2}{\partial X^2} \) is the infinitesimal generator of \( X \).

Proposition 1 can be interpreted as follows: \( \mathbb{P}(\cdot | \mathcal{F}_t) \) is the conditional probability that the logarithmic difference between the approximated wealth and the true wealth exceeds \( M \) at time \( t + \tau \). If we choose, say, \( M = 0.10 \), Proposition 1 offers a bound on the probability that \( \log W_{t+\tau}^{(1)}(X_t) \) and \( \log W_{t+\tau} \) differ by more than 10%. The accuracy of the VaR approximation (8) is illustrated in Table 1, where for illustration purposes we assumed a mean-reverting geometric Brownian motion for the volatility process. In this case, \( \mathbb{E}[|R|] \) is obtained in closed form. The results are presented for two different conditioning values of the state variable, \( X_t = 1 \) and \( X_t = 3 \). For lower values of \( X_t \) the approximation is even more accurate. According to Table 1 the approximation bounds are generally very tight. For instance, the bound on the error probability for \( M = 1\% \), \( X_t = 1 \) and a horizon \( \tau = 10 \) is always below 0.01%. The bounds for \( X_t = 3 \) are always below 1% and in many cases below 0.05%. Hence, the quality of the above approximation results suggests that the VaR approximation in (7) can be reasonably used to investigate the theoretical properties of the constrained portfolio selection problem. Moreover, common market practice usually confines itself to VaR figures reported based on a conditional normal distribution. This further motivates our approach.

The advantage of using (7) as an approximate VaR constraint is that it implies some direct portfolio bounds on the optimal policy of the VaR-constrained bank. These are given in the next proposition.

Proposition 2. To first order, the constraint \( \text{VaR}_t^{\text{w}, \text{v}} \leq \text{VaR}_t \) is equivalent to the following upper and lower bounds on the fraction \( w_t \) of wealth invested in the risky asset,

\[
w_t^- (X_t) \leq w_t \leq w_t^+ (X_t),
\]

where
\[
w_t^\pm (X_t) = \frac{\lambda(X_t)}{\sigma(X_t)^2} \pm \frac{\nu}{\sigma(X_t) \sqrt{\tau}} 
\]

\[
\pm \sqrt{(\lambda(X_t) \tau + \sigma(X_t) \sqrt{\tau} \nu)^2 + 2 \sigma(X_t)^2 \tau (r(X_t) \tau - \log(1 - \beta))} \frac{2}{\sigma(X_t)^2 \tau} ,
\]

with \( \nu = N^{-1}(\nu) \) the \( \nu \)-quantile of the standard Normal distribution.

The functional form (6) for the VaR limit implies a bound on the optimal portfolio fraction that is wealth-independent\(^4\). By inspection of equation (10) we have \( w_t^+ (X_t) \geq 0, \) and
$w_t(X_t) \leq 0$ for any $X$. This holds for all functional forms $\lambda(X_t)$ and $\sigma(X_t)$. Finally, observe that the portfolio bounds (9) are functions of interest rates $r(X_t)$ and equity expected returns and volatilities $\alpha(X_t)$ and $\sigma(X_t)$. Therefore, in a general equilibrium analysis of VaR regulation the relevant portfolio constraints will have to be determined in equilibrium, together with all the other equilibrium quantities.

2 Partial Equilibrium

2.1 Optimal Policies

To define the bank’s partial equilibrium optimization problem under VaR constraints we start from the following standard assumption on preferences.

**Assumption 1.** The bank derives utility from final wealth $W_T$ according to a CRRA-utility function

$$u(W) = \frac{W^{\gamma} - 1}{\gamma}, \gamma < 1.$$ 

According to Assumption 1, the bank derives utility from terminal wealth $W_T$ only. In Section 2.4 the model will be extended to allow for the presence of intermediate consumption streams. Here we are interested in the partial equilibrium impact of VaR constraints in the presence of a stochastic opportunity set. To this end, we impose the following parametric assumptions.

**Assumption 2.** $X_t$ follows a mean reverting process given by

$$dX_t = (\theta - \kappa X_t)dt + \sigma_X X_t^m dZ_t^X,$$

where $\kappa, \theta, \sigma_X, m$ are positive constants. In the sequel we assume that the model parameters are chosen in order to ensure that $(X_t)$ is a strictly positive process. The risk premium, the risky asset volatility and the interest rate processes are of the form

$$\lambda(X_t) = \lambda X_t^{n_1}, \sigma(X_t) = \sigma X_t^{n_2}, r(X_t) = r,$$

where $\lambda, n_1, n_2, r$ are non-negative constants.

Notice that the parameters $n_1$ and $n_2$ will have to be selected with some care to avoid trivial settings where VaR constraints are either vacuous or binding a-priori. Further, Assumption 2 comprises model settings where volatility is stochastic or where variance-in-mean effects are
present, for instance when \( n_1 \neq 0, n_2 \neq 0 \) and \( m \neq 0 \). On the other hand, when \( n_1 = 0 \) we can investigate the impact of VaR constraints under a pure stochastic volatility setting. In the case \( n_1 = 2n_2 \), the unconstrained Merton policy of a log-utility investor is a constant

\[
\lambda(X_t)/\sigma(X_t)^2 = \lambda/\sigma^2 .
\]

Given Assumption 2 the functional form of the portfolio bounds \( w_b^\pm(X_t) \) can be studied more explicitly. Using the definition of the VaR constraint in equation (10), it follows from the Implicit Function Theorem and Assumption 2 that

\[
\frac{\partial w_b^+}{\partial X} = \frac{w_b^+}{X} \cdot \frac{(n_1 - n_2)\lambda X^{n_1}\tau - n_2 D(X)}{D(X)} ,
\]

where

\[
D(X) = \pm \sqrt{(\lambda X^{n_1}\tau + \sigma X^{n_2}\sqrt{\tau}v)^2 + 2\sigma^2 X^{2n_2}\tau (r\tau - \log(1 - \beta))} .
\]

Since, as remarked above, \( w^+ > 0 \) and \( w^- < 0 \), the sign of (11) is determined by the sign of

\[
(n_1 - n_2)\lambda X^{n_1}\tau - n_2 D(X) .
\]

This allows us to discuss, for an explicit model setting, the impact of a change in volatility on the tightness of the VaR constraints. In particular, we observe that

\[
\frac{\partial w_b^\pm}{\partial X} < 0 \text{, if } n_1 \leq n_2 .
\]

Therefore, we see that a higher volatility state \( X \) implies lower absolute values of \( w_b^\pm \) and thus a tighter VaR constraint for both long and short portfolio exposures. For more general choices of \( n_1 \) and \( n_2 \) no such absolute statement about the sign of \( \frac{\partial w_b^\pm}{\partial X} \) can be made for the whole support of \( X \). However, the derivative in (13) is generally negative for reasonable values of \( X \). For example, this is the case when \( n_1 = 2, n_2 = 1 \). In Section 2.5, this particular choice is discussed in more detail.

When restricting the parameter \( m \) to \( m = \frac{1}{2} \) or \( m = 1 \) we obtain a model class that encompasses the most popular stochastic volatility models, like the lognormal, the square-Gaussian, or the square-root model of Heston (1993). At the same time this model class allows for closed forms of the moments and cross-moments of \( X_t \). Such closed form expressions are necessary to obtain analytical asymptotic characterizations of the relevant optimal policies and equilibrium quantities in the presence of VaR constraints. For the bank’s optimization problem we base the VaR constraint on the approximate VaR constraint implied by Proposition
Denoting by $\text{BR}(W, X)$ the budget constraints (3), (4) and by $C(X)$ the approximate VaR constraint (10), the optimization problem becomes

$$(P1): J(W, X, t) = \max_{w \in \text{BR}(W, X)} \mathbb{E}[u(W) | F_t],$$

where $\mathcal{R}(W, X) = \{ w \in \text{BR}(W, X) \} \cap C(X)$. Intuitively, the solution of problem (P1) has to provide optimal investment strategies characterized by a region where the VaR constraint binds and a region where it does not bind. This is the content of the next proposition.

**Proposition 3.** Consider the control problem (P1) under the Assumptions 1 and 2. In a region where the $J$-function is increasing and jointly strictly concave in $W$ and $X$, the policy solving (P1) is

$$w^*(X_t, t) = \begin{cases} w^+_b(X_t), & \text{if } w_f(X_t) \geq w^+_b(X_t), \\ w^-_b(X_t), & \text{if } w_f(X_t) \leq w^-_b(X_t), \\ w_f(X_t), & \text{else}, \end{cases}$$

where

$$w_f(X_t, t) = -X_t^{n_1-2n_2} \frac{\lambda J_W}{\sigma^2 W J_W} - \rho X_t^{m-n_2} \frac{\sigma X J_{WX}}{\sigma W J_W},$$

and $w^+_b(X_t)$ is given in (10).

Apparently, inserting the optimal policies back into the Hamilton-Jacobi-Bellman (HJB) equation for problem (P1) leads to a non-explicitly solvable PDE. We therefore apply an asymptotic analysis that computes the value function and the implied optimal policies as a power series in $\gamma$.

### 2.2 Perturbation with respect to Risk Aversion

From the homogeneity properties of problem (P1), the value function $J$ has to be of the form

$$J(W, X, t) = e^{\gamma g(X, t)} W^\gamma - 1 \frac{1}{\gamma},$$

for some unknown function $g(X, t)$. Expanding $g(X, t)$ in the parameter $\gamma$, i.e.

$$g(X, t) = g_0(X, t) + \gamma g_1(X, t) + \frac{1}{2} \gamma^2 g_2(X, t) + O(\gamma^3),$$

the optimal policy up to first order in $\gamma$ is easily obtained from (14), (15), and (16) as

$$w^{(1)}_f(X_t, t) = (1 + \gamma)X_t^{n_1-2n_2} \frac{\lambda}{\sigma^2} + \gamma X_t^{m-n_2} \frac{\rho \sigma X}{\sigma} \frac{\partial g_0(X, t)}{\partial X}.$$
Therefore, if \( g_0(X, t) \) can be determined, the first-order policy of an investor with risk aversion \( 1 - \gamma \) is also determined. Equation (17) gives the approximate optimal policy as a sum of two approximate sub-policies. A myopic portfolio

\[
(1 + \gamma)X_t^{n_1-2n_2} \frac{\lambda}{\sigma^2},
\]

and an intertemporal hedging position

\[
\gamma X_t^{n_2-n_2} \frac{\rho \sigma_X}{\sigma} \frac{\partial g_0(X, t)}{\partial X}.
\]

In general, it is not possible to determine the optimal intertemporal hedging policy, given a stochastic opportunity set dynamics. Therefore, we apply perturbation theory to analyze the asymptotic impact of VaR constraints for a broad range of model dynamics in partial and general equilibrium. In the perturbation approach, the problem of characterizing the first-order intertemporal hedging demand boils down to calculating the function \( g_0 \). By construction, \( g_0 \) is fully determined as soon as we can compute the value function of a log-utility investor.

**Proposition 4.** Consider the optimization problem \((P1)\). The function \( g_0 \) in (17) reads

\[
g_0(X, t) = g_0^f(X, t) - \frac{1}{2} \int_t^T \mathbb{E} \left[ \mathbb{I}_{\{\phi(X_s) < 0\}} \phi(X_s)^2 \mid \mathcal{F}_t \right] ds,
\]

where

\[
\phi(X_t) = \frac{v}{\sqrt{\tau}} + \frac{\sqrt{(\lambda X_t^{n_1} \tau + \sigma X_t^{n_2} \sqrt{\tau} v)^2 + 2\sigma^2 X_t^{2n_2 \tau} (r \tau - \log(1 - \beta))}}{\sigma X_t^{n_2 \tau}}.
\]

and \( g_0^f \) is the “\( g_0 \)-function” prevailing in the absence of VaR constraints.

The quality of the first-order approximation in Proposition 4 can be improved by considering higher-order terms in the perturbation series. We show in the next section how such higher-order approximations can be obtained and how convergence to the underlying solution can be achieved under appropriate assumptions.

### 2.3 Convergence and Accuracy of the Perturbation Approach

We proceed in three steps. First, we show how higher-order corrections for the optimal policies of problem \((P1)\) can be obtained. Second, we demonstrate that for some given reasonable a-priori bounds on the terms in the perturbation series the whole series converges. Third, we
compute different finite-order approximations and illustrate by means of a numerical example
the speed of convergence of the perturbation series.

2.3.1 Convergence

Higher-order policy approximations are obtained from (14), (15), and (16). The nth-order approximation of $w_f$ in Proposition 3 is given as

$$w_f^{(n)}(X, t) = X^{n_1-2n_2} \frac{\lambda}{\sigma^2} \frac{1 - \gamma^{n_1+1}}{1 - \gamma} + X^{m-n_2} \rho \sigma X \frac{X^{m-n_2}}{\sigma} \sum_{j=0}^{n-1} \frac{\gamma^{j+1}}{j!} \frac{\partial g_j(X, t)}{X} \frac{1 - \gamma^{n-j}}{1 - \gamma},$$

(21)

where $g_j$ is the jth order function in the expansion of $g(X, t)$ with respect to $\gamma$. Corollary 1 below gives a natural a-priori bound on the unknown functions $g_j(X, t)$, $j \in \mathbb{N}_0$, which ensures convergence of the perturbation theory.

**Corollary 1.** Consider the control problem (P1) for $\gamma > -1$. The optimal nth-order approximation $w_f^{(n)}$ converges to $w_f$ if $|\partial g_j(X, t)/\partial X| \leq K X^{2j(n_1-n_2)}$, $K > 0$, $j \in \mathbb{N}_0$. The limit policy is bounded by

$$|w_f(X, t)| = \lim_{n \to \infty} |w_f^{(n)}(X, t)| \leq X^{n_1-2n_2} \frac{\lambda}{\sigma^2} \frac{1}{|1 - \gamma|} + X^{m-n_2} \frac{\rho \sigma X e^{\gamma X^{2(n_1-n_2)}} K}{\sigma |1 - \gamma|}.$$

(22)

In the sequel we focus on model settings where convergence of the series of policy approximations $w_f^n$ can be granted. The constraint $\gamma > -1$ is not restrictive for our purposes, because we can reasonably assume banks in our model to be less risk averse than individual investors.

2.3.2 Accuracy

To compute the higher-order functions $g_j$, we define $\log W_T = \log W_t + H_{t,T}$ for the wealth dynamics, where

$$H_{t,T} := \int_t^T \left( r + \lambda X_s^\gamma w_s - \frac{1}{2} w_s^2 \sigma^2 X_s^{2n_2} \right) ds + \int_t^T w_s \sigma X_s^{n_2} dZ_s.$$

(23)

Then, we expand $g(X, t)$ as a power series in $\gamma$ and insert it in the equality

$$\mathbb{E} \left[ \frac{W_T^\gamma - 1}{\gamma} \mid \mathcal{F}_t \right] = \frac{e^{\gamma g(X)} W_t^\gamma - 1}{\gamma}.$$

(24)

Finally, we expand the LHS and the RHS of (24) as a power series in $\gamma$. By matching the resulting coefficient for each power of $\gamma$ in the resulting power series, we obtain the following
characterizations for some of the higher-order functions $g_j(X, t)$.

**Corollary 2.** Consider problem (P1). The functions $g_0$, $g_1$, and $g_2$ are given by:

$$
\begin{align*}
    g_0(X, t) &= \mathbb{E}[H_{t,T} | \mathcal{F}_t], \\
    g_1(X, t) &= \frac{1}{2} \text{Var}[H_{t,T} | \mathcal{F}_t], \\
    g_2(X, t) &= \frac{1}{6} \left( \mathbb{E}[H_{t,T}^3 | \mathcal{F}_t] + 2\mathbb{E}[H_{t,T} | \mathcal{F}_t]^3 - 3\mathbb{E}[H_{t,T} | \mathcal{F}_t] \mathbb{E}[H_{t,T}^2 | \mathcal{F}_t] \right).
\end{align*}
$$

Corollary 2 shows the direct link between the order of the $g_j$-functions in the perturbation series and the higher moments of the return $H_{t,T}$ on optimally invested wealth. We observe that whenever the $n$th order of $g$ enters the approximation for the optimal portfolio policy, the approximation is taking into account the $(n+1)$th moment of $H_{t,T}$. Notice that the entity $H_{t,T}$ in (23) is itself a function of $\gamma$ through the optimal policy $w_t$. Computation of $g_n$ in Corollary 2 therefore needs a recursive procedure that expands also the optimal policy $w_t$ in (23) as a function of $\gamma$. To compute $g_1$ only the optimal policy of a log utility investor is needed. By contrast, higher-order functions $g_j$, $j > 2$, require a more involved recursive procedure.

### 2.3.3 Numerical Example

To illustrate the accuracy of the perturbative solutions we compare the finite-order policy approximations implied by Corollary 2 for $j = 0, 1$ with the corresponding exact portfolio policy. We do this for an unconstrained portfolio optimization and a model setting corresponding to $n_1 = 2$, $n_2 = 1$, $r = 0$ in Assumption 2. Exact solutions are obtained using the Monte Carlo simulation method recently proposed by Cvitanic, Goukasian, and Zapatero (2003) and Detemple, Garcia, and Rindisbacher (2003). Both approaches are restricted to a complete market model setting. We therefore adapt our model accordingly and work with the following parametric specifications:

$$
\begin{align*}
    \frac{dP_t}{P_t} &= 0.05X_t^2 dt + 0.25X_t dZ_t, \\
    dX_t &= (0.8 - 0.8X_t) dt + 0.2X_t dZ_t.
\end{align*}
$$

Figure 1 illustrates our findings. Point $A$ in Figure 1 gives the log-investor’s optimal portfolio policy. The bold straight line in the figure is the first-order approximation for the portfolio policy. The second-order approximation is represented by the bold dashed line. Finally, the circled line is the exact portfolio policy estimated by Monte Carlo simulation. As is apparent from Figure 1 the second-order approximation already produces very accurate results. The first-order approximation does not fit the convexity of the portfolio profile as a function of $\gamma$. However, it produces the correct sign about the direction of the effects obtained for risk aversion parameters that differ from the log utility case.
2.4 Incorporating Intermediate Consumption

This section extends the previous analysis to incorporate intermediate consumption in the bank’s optimal wealth dynamics. We interpret intermediate consumption as intermediate cash-outflows caused by the bank’s expenditure policy. This puts us back to the known portfolio problem where utility is derived from both terminal wealth and intermediate consumption. In this way we can analyze the impact of VaR constraints on the optimal expenditure policy of a bank. Moreover, introducing consumption is needed to endogenize the asset price dynamics in the presence of VaR-constrained investors (see Section 3). Assumption 1 is modified as follows.

Assumption 3. Banks derive utility from both an intermediate consumption process \((C_t)_{0 \leq t \leq T}\) and from terminal wealth \(W_T\). They maximize the expected value of the random variable

\[
V(W, C, t) = \int_t^T e^{-\delta s} \frac{C_s^2 - 1}{\gamma} ds + e^{-\delta T} \frac{W_T^2 - 1}{\gamma}, \quad 0 \leq \delta < 1, \tag{25}
\]

where \(\delta \geq 0\) is the subjective discount rate.

In the presence of intermediate consumption, wealth dynamics are

\[
dW_t = ((r - c_t) + w_t \lambda(X_t))W_t dt + w_t \sigma(X_t)W_t dZ_t, \tag{26}
\]

where \(c_t = C_t/W_t\) is the consumption rate. It follows from equation (26) that future current wealth in the presence of a consumption policy is given by

\[
\log W_T = \log W_t + H_{t,T}, \tag{27}
\]

with

\[
H_{t,T} := \int_t^T \left( w_s \lambda(X_s) + r - c_s - \frac{1}{2} w_s^2 \sigma(X_s)^2 \right) ds + \int_t^T w_s \sigma(X_s) dZ_s. \tag{28}
\]

Again, we define the relevant VaR limit as a fraction of current wealth as in (6). Remind that in (6) we assumed a constant portfolio \(w_t\) over the relevant time horizon \([t, t + \tau]\). In a similar vein, we lock-in the consumption rate \(c_t\) at the rate prevailing at time \(t\). Given Assumption 2 and Proposition 2, the Itô-Taylor approximated VaR constraint in the presence of intermediate consumption is given by

\[
Q(w, c) = r - c + \lambda X^{\alpha_1} w - \frac{1}{2} w^2 \sigma^2 X^{2\alpha_2} - \frac{1}{\tau} \log(1 - \beta) + \frac{1}{\sqrt{\tau}} w \sigma X^{\alpha_2} \leq 0. \tag{29}
\]

For \((c, w)\), we thus obtain a constrained set \(C(X)\) given by

\[
(c, w) \in C(X) = \{Q(w, c) \leq 0\}.
\]
Since $Q(w, c)$ is a quadratic function of $w$ and a linear function of $c$, the set $\{(c, w) : Q(w, c) = 0\}$ describes a parabola in $(w, c)$-space. The portfolio fractions $w_b^\pm$ are obtained as the solutions of the equation $Q(c, w) = 0$, given any feasible consumption rate $c > 0$.

Under Assumption 3, the relevant optimization of a VaR-constrained investor is now of the form

$$\text{(P2)}: \quad J(W, X, t) = \max_{(c, w) \in \mathcal{R}(W, X)} \mathbb{E}[V(W, C, t) | \mathcal{F}_t],$$

where $C(X)$ incorporates the presence of intermediate consumption in the VaR constraint.

Asymptotic solutions for this optimization problem are given next.

**Proposition 5.** Given Assumption 2, 3, and the optimization problem $(\text{P2})$, the first-order approximations of the optimal policies are given by

$$w^{(1)}(X, t) = (1 + \gamma)X^{n_1-2n_2} + \frac{\gamma X^{n_2} - \rho \sigma X}{\sigma} \frac{\partial g_0(X, t)}{\partial X},$$

$$c^{(1)}(X, t) = 1 - \frac{\gamma (g_0(X, t) + \log A_t)}{A_t},$$

where $A_t = e^{-\delta(T-t)} + \frac{1 - e^{-\delta(T-t)}}{\delta}$ and

$$g_0(X, t) = \frac{1}{A_t} e^{-\delta(T-t)} \mathbb{E}[H_{t,T} | \mathcal{F}_t] + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} (\mathbb{E}[H_{t,s} | \mathcal{F}_t] - \log A_s) ds.$$

The function $\mathbb{E}[H_{t,T} | \mathcal{F}_t]$ is defined as

$$\mathbb{E}[H_{t,T} | \mathcal{F}_t] = \mathbb{E}[H_{t,T}^f | \mathcal{F}_t] - \frac{1}{2} \int_t^T \mathbb{E}[(1_{\phi(X_s, s) < 0}) \phi(X_s, s)^2 | \mathcal{F}_t] ds,$$

$$\mathbb{E}[H_{t,T}^f | \mathcal{F}_t] = r(T-t) - \int_t^T A_s^{-1} ds + \frac{\lambda^2}{2\sigma^2} \int_t^T \mathbb{E}[X_s^{2(n_1-n_2)} | \mathcal{F}_s] ds,$$

with

$$\phi(X, t) = \frac{v}{\sqrt{\tau}} + \sqrt{(\lambda X^{n_1} + \sigma X^{n_2} \sqrt{\tau} v)^2 + 2\sigma^2 X^{2n_2} \tau ((r - A_t^{-1}) \tau - \log(1 - \beta))}.\quad (31)$$

Again, the function $g_0$ is decomposed in a term that corresponds to the solution of a problem without constraints (the term including $\mathbb{E}[H_{t,T}^f | \mathcal{F}_t]$) and a term reflecting the marginal impact of the VaR constraint (the one including $\mathbb{E}[H_{t,T} | \mathcal{F}_t] - \mathbb{E}[H_{t,T}^f | \mathcal{F}_t]$). The latter term is associated with the utility loss that is caused by the presence of regulatory VaR constraints and reflects the bank’s anticipations on the fact that VaR constraints might be binding in the future. This reduces the expected utility compared to an unconstrained optimal policy. It thereby
affects the current optimal consumption and investment decision. In particular, the existence of VaR constraints produces a non-trivial anticipative effect on the bank’s decision and alters the intertemporal hedging demand. This dynamic optimal behavior fundamentally differs from the one implied by a myopic portfolio optimization as e.g. in Basak and Shapiro (2001), Cuoco, He, and Issaenko (2001) and Danielsson, Shin, and Zigrand (2001). Finally, notice that the presence of intermediate consumption implies optimal VaR constraints that are state and time dependent, as is made explicit in the above proposition by the function $\phi(X,t)$ in (31).

2.5 Partial Equilibrium: Do VaR Constraints Distort Incentives?

This section studies the partial equilibrium effects of VaR regulation for the model class defined in Assumption 2. From Proposition 5, Corollary 3 below is easily obtained. It characterizes the impact of VaR constraints on the optimal policies of problem (P2), relative to the solutions for the unconstrained portfolio problem.

**Corollary 3.** Let Assumption 2 and 3 be satisfied and consider the partial equilibrium problem (P2) with its first-order optimal policy approximations. Before hitting the VaR constraints:

i) a bank with $\gamma > 0$ ($\gamma < 0$) increases (decreases) optimal consumption $c_t$.

ii) the bank’s optimal portfolio is unaffected if $n_2 = n_1$.

iii) the bank’s optimal portfolio is unaffected if $\gamma = 0$.

iv) given $n_1 \neq n_2$, the bank increases or decreases its exposure to the risky asset according to:

<table>
<thead>
<tr>
<th>$\gamma \rho$</th>
<th>$\frac{\partial g_0}{\partial X}$</th>
<th>$\frac{\partial g_0^f}{\partial X}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&gt; 0$</td>
<td>increase</td>
<td>decrease</td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>decrease</td>
<td>increase</td>
</tr>
</tbody>
</table>

Moreover, if there exists $X^* > 0$ such that $P(\sup_{t \leq s \leq T} X_s < X^*) = 1$ and

$$v \sigma X^{n_2} \sqrt{\tau} + \lambda X^{n_1} \tau + D \left( \sigma^2 X^{2n_2} \tau - 1 \right) \leq 0,$$

(32)

for all $X < X^*$ then:

$$\frac{\partial g_0}{\partial X} > \frac{\partial g_0^f}{\partial X} \iff n_1 < n_2,$$

$$\frac{\partial g_0}{\partial X} < \frac{\partial g_0^f}{\partial X} \iff n_1 > n_2.$$
Corollary 3 shows that the direction of an anticipative impact of VaR constraints on optimal portfolios depends on several quantities:

1) Pure preference parameters, via the risk aversion related term $\gamma$,

2) Market dynamics, via the correlation parameter $\rho$,

3) A joint interplay of preferences, market dynamics and regulatory parameters, via the sign of the derivative \( \frac{\partial g_0}{\partial X} - \frac{\partial g_0^f}{\partial X} \).

The driving force for 1)-3) is the impact of an anticipated binding VaR constraint in the future on today’s hedging policy of a risk averse investor. Indeed, for $\gamma = 0$, we see that no anticipatory effect due to the VaR constraint is present (see iii) in Corollary 3). However, as soon as $\gamma \neq 0$ intertemporal hedging arises optimally when $n_1 \neq n_2$ and the existence of VaR constraints does affect today’s optimal portfolios even before the constraint is hit (see iv) in Corollary 3). The case $n_1 = n_2$ corresponds to a situation where Sharpe ratios are state independent. In this case, no intertemporal hedging motives emerge even for $\gamma \neq 0$. Consequently, VaR constraints do not have any anticipatory impact on investors’ optimal risk exposures. The geometric Brownian motion model considered in Cuoco, He, and Issaenko (2001) and Basak and Shapiro (2001) is clearly a special case of this setting. This emphasizes the restrictiveness of the geometric Brownian motion assumption when studying the incentive compatibility of VaR regulation. Similarly, the myopic assumption in Danielsson, Shin, and Zigrand (2001) fully neglects the anticipative impact of VaR constraints.

In the absence of VaR constraints, the direction of the hedging demand depends on the impact of stochastic volatility on an investors’ opportunity set and on the sign of the hedging demand of an unconstrained investor, i.e. on the sign of $\frac{\partial g_0^f}{\partial X}$. More particularly, as soon as the equity Sharpe ratio

$$\frac{\lambda(X_t)}{\sigma(X_t)},$$

is an increasing function of $X$, an increase in volatility yields a better opportunity set in the future, thereby implying $\frac{\partial g_0^f}{\partial X} > 0$ in Proposition 4. By Assumption 2 this happens as soon as $n_1 > n_2$. In the case $n_1 < n_2$, one typically gets $\frac{\partial g_0^f}{\partial X} < 0$ because higher levels of volatility in the future imply a worsening of tomorrow’s stochastic opportunity set. Thus, depending on the relative magnitude of $n_1$ and $n_2$, high volatility is either a “good” or a “bad” state.

Introducing VaR constraints, their first-order effect is characterized by the difference

$$\frac{\partial g_0}{\partial X} - \frac{\partial g_0^f}{\partial X},$$
which is directly related to the marginal impact of VaR constraints on today’s hedging demand\textsuperscript{12}. The sign of this difference determines the direction of the VaR-induced excess hedging demand, given a parameter choice $\gamma, \rho$. Generally, we can expect that for models where volatility is a “good” (“bad”) state one will have

$$\frac{\partial g_0}{\partial X} - \frac{\partial g_f}{\partial X} > 0 \quad , \quad (< 0) ,$$

because VaR constraints reduce the marginal utility deriving from an optimal constrained hedging policy when compared with the unconstrained case. Under condition (32) the relation in (35) is equivalent to $n_1 < n_2$ ($n_1 > n_2$). From a practical viewpoint this holds for many explicit model settings when using realistic parameter choices. In fact, for all calibrations later in the paper - where we consider some more explicit model settings - we have found $X^*$ to imply volatility levels higher than 100%.

As Corollary 3 reveals, the anticipatory effect of VaR constraints can lead to either an increase or a decrease in the bank’s exposure in the risky asset, depending on the structure of the underlying opportunity set dynamics. To give an example, if correlation is negative and $\gamma > 0$, VaR regulation has an adverse (positive) effect on the riskiness of the banks portfolio decision for models where

$$\frac{\partial g_0}{\partial X} - \frac{\partial g_f}{\partial X} > 0 \quad , \quad (< 0).$$

In this setting, only a positive correlation causes VaR constraints to decrease exposures in the risky asset before the VaR constraint is hit. However, empirical evidence suggests that stock price movements are negatively correlated. Such evidence was already reported in Black (1976), Christie (1982), and Schwert (1989), and is commonly referred to as “leverage effect”. More recent empirical investigations in Anderson, Benzoni, and Lund (2002) have found a correlation\textsuperscript{13} of $\rho = -0.4$ for S&P 500 daily returns during the period from 1/3/1980 to 12/31/1996. Finally, we remark that while $\gamma$ and the difference (34) determine jointly the sign of the anticipatory impact of VaR regulation, the difference $m - n_2$ determines the absolute amount of the hedging demand since $\partial g_f(X,t)/\partial X$ is scaled by $X^{m-n_2}$ in (21).

The results of Corollary 3 are further illustrated within explicit model settings. We first consider the impact of VaR regulation in a model specified by

$$\lambda(X_t) = \lambda X_t^2, \quad \sigma(X_t) = \sigma X_t, \quad \sigma_X(X_t) = \sigma_X X_t.$$  \hspace{1cm} (36)

Thus, with the above specification volatility is a “good” state. Figure 2 plots the optimal portfolio strategies of a VaR-constrained and a VaR unconstrained investor for $\gamma = 0.5$ as functions of volatility $\sigma(X_t)$.\textsuperscript{14} Panels (A), (C), (E) assume a positive, panels (B), (D), (F) a negative correlation between $X_t$ and risky asset returns. For a one-year investment horizon
we observe in panels (A), (B) that the difference of the two portfolio strategies is small. However, in Panels (C) and (E) where \( T = 5 \) and \( T = 10 \) we see that the risk exposure of a VaR-constrained bank has been already substantially reduced before the VaR constraint is hit. Therefore, the anticipation of the fact that the VaR constraint might be hit in the future leads to a reduction of the bank's exposure in the risky asset. As expected, when correlation is negative (see e.g. panel (D)), a reversed anticipative effect is found. The bank increases the optimal risk exposure before the VaR constraint is actually hit. When volatility increases further, the constraint will eventually be hit and the bank will finally end up with a lower risk exposure compared to an unconstrained bank.

A conditionally normal dynamic VaR setting allows us to transform a VaR limit into an equivalent Expected Shortfall limit (see e.g. Cuoco, He, and Issaenko (2001) and Acerbi and Tasche (2001)). Therefore, imposing an Expected Shortfall limit is, in our setting, equivalent to imposing a tighter confidence bound on the VaR limit. The impact of tightening the confidence bound is illustrated in Figure 3 for model (36). Again, we distinguish between positive and negative correlation ((Panel (A), (C), (E) and (B), (D), (F), respectively). We adopt the confidence levels \( \nu = 0.01 \) (Panel (A) and (B)) and \( \nu = 0.05 \) (Panel (C) and (D)). Similarly to Figure 2 we observe for both confidence levels an increase in the bank's exposure when correlation is negative. In Panel (E) and (F) the differences between the constrained portfolio policies under the two confidence levels is plotted. In particular, in Panel (F) a tighter confidence level induces the bank to optimally increase its risk exposure for low and moderate volatilities. Only for volatilities larger than about 50% a tighter confidence level leads to a decrease in the bank's risk exposure.

A model setting where volatility is a bad state is obtained by specifying

\[
\lambda(X_t) = \lambda, \quad \sigma(X_t) = \sigma X_t, \quad \sigma_X(X_t) = \sigma X X_t.
\]

In this model excess expected returns are constant, implying Sharpe ratios that are inversely related to volatility. The implied optimal policies obtained for a VaR-constrained and an unconstrained investor with \( \gamma = 0.5 \) are plotted in panels (A) and (B) of Figure 4. A striking difference with the results from model (36) is that now the VaR constraint tends to become binding at low rather than at high volatility states \( \sigma(X_t) \). This feature derives directly from the dynamic structure of model (37), causing Sharpe ratios to be high precisely when volatility is low. This is in sharp contrast to the dynamic specification (36) where exactly the opposite happens. From a regulatory perspective, an important partial equilibrium implication of this model setting is that VaR constraints are going to limit the bank’s risk exposure in low volatility states, while in high volatility states they will affect optimal exposures only via the anticipatory effect on the intertemporal hedging demand. Moreover, since in model (37) volatility is a bad state, intertemporal hedging goes in the opposite direction to what happened in model (36).
This implies for the relevant support of $X_t$ in our model calibrations:

$$\frac{\partial g_0}{\partial X} - \frac{\partial g_0^f}{\partial X} < 0.$$  

In Corollary 3, the anticipative effect of VaR constraints then goes exactly in the opposite direction of what has been observed for model (36). Depending on whether the product $\rho \gamma$ is positive (in Panel (A)) or negative (in Panel(B)), the constrained bank either decreases or increases its risk exposure relative to the unconstrained bank. The latter case implies that in the presence of VaR regulation a large risk exposure is optimally selected precisely in the risky states of the world, i.e. when volatility is already at a high level. Clearly, this is a paradoxical partial equilibrium implication of VaR regulation in the presence of a stochastic opportunity set.

3 General Equilibrium

In the previous sections we studied a partial equilibrium setting. Intuitively, the aggregate effect of VaR regulation on individual portfolio demands eventually influences the equilibrium dynamics of asset prices and interest rates. Hence, it also has a feedback impact on the definition of the relevant VaR-constrained portfolios themselves. As a consequence, a complete analysis of VaR regulation requires the study of general equilibrium implications. This task is addressed in this section. The previous partial equilibrium results are the necessary starting point for this analysis.

Using perturbation theory, the computation of first-order general equilibria in a heterogeneous economy amounts to determining the optimal policies of a log-investor in a homogeneous economy. This gives us a way to provide analytical characterizations of the general equilibrium impact of VaR regulation and to study its dependence as a function of exogenous parameters such as regulatory variables by means of comparative statics. Moreover, the perturbative solutions will offer economic insights and interpretation for the emerging equilibrium quantities, which cannot be obtained by means of a pure numerical approach.

3.1 The Exchange Economy

We consider an exchange economy with a financial market consisting of two financial instruments, an instantaneous risk-free asset and a stock. The risk-free asset is available in zero net supply and the interest rate $r_t$ is determined in equilibrium together with the asset price
dynamics. The risky asset is a contingent claim on aggregate endowment, $e_t$, with dynamics

$$\frac{de_t}{e_t} = \mu_e(X_t)dt + \sigma_e(X_t)dZ^e_t, \quad (38)$$

where $Z^e_t$ is a standard Brownian motion. The state variable $X_t$ follows an Itô diffusion

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dZ^X_t. \quad (39)$$

The instantaneous correlation between the endowment process and the state variable process is given by $E(dZ^e_t dZ^X_t) = \rho_{eX}$. The economy is populated by two banks, bank I and II, equipped with heterogeneous risk aversion indices $\gamma_I$ and $\gamma_{II}$, respectively. Both banks solve the optimization problems $(P2)^I$ and $(P2)^{II}$ according to equation (30). We denote by $\omega_I^t$ and $\omega_{II}^t$ the cross-sectional wealth of bank I and II, respectively, i.e.

$$\omega^t_i = \frac{W^i_t}{W^I_t + W^{II}_t}, \quad i = I, II,$$

where $W^i_t$ is the current wealth of bank $i$. The general equilibrium in this economy can be described by means of the extended state variables process $\chi_t = (X_t, \omega^t_I)^\top$. Hence, the stock price $P_t$ equals aggregate wealth, implying a gross return dynamics given by

$$\frac{dP_t + e_t dt}{P_t} = \alpha(\chi, t)dt + \sigma_P(\chi, t)dz^X_t + \sigma_{P2}(\chi, t)dZ^e_t$$

$$= \alpha(\chi, t)dt + \sigma(\chi, t)^\top dZ_t,$$

where $Z_t = (Z^e_t, Z^X_t)^\top$. The drift vector $\alpha(\chi, t)$ and the volatility vector $\sigma(\chi, t)$ have to be determined in equilibrium.

**Definition 2.** We call a process $(\alpha(\chi, t), \sigma(\chi, t), r(\chi, t), w^I_t, w^{II}_t, c^I_t, c^{II}_t)$ an asymptotic pure exchange equilibrium if

a) individual portfolio rules $w^I_t, w^{II}_t$ are optimal to first-order, i.e., they are of the form given in Proposition 5.

b) the financial market clears,

$$w^I_t \omega^t_I + w^{II}_t \omega_{II}^t = 1 + O(\gamma^2). \quad (40)$$

b) the market for consumption goods clears,

$$c^I_t W^I_t + c^{II}_t W^{II}_t = e_t + O(\gamma^2). \quad (41)$$
This equilibrium definition will be used in the sequel to study the first-order general equilibrium effects of VaR regulation. We start by deriving, in Section 3.2, a set of general first-order asymptotics for the individual consumption/investment policies and the relevant equilibrium quantities in the above exchange economy. Based on these results we analyze, in Section 3.3, the first-order general equilibrium impact of VaR constraints for some explicit model settings.

### 3.2 General Equilibrium Asymptotics

The general structure of the portfolio and consumption policies (as functions of some unknown function $\partial g_0/\partial X$) is the same as the one obtained for the partial equilibrium solutions in Proposition 5. However, the function $g_0$ in Proposition 5 is determined endogenously by the general equilibrium as an equilibrium function $g_{0e}$, say, which depends on the extended state vector $\chi$ and the preference parameters $\gamma_I$ and $\gamma_H$ as well. This fact will motivate further on an expansion of $g_{0e}$ with respect to the risk aversion parameters $\gamma_I$, $\gamma_H$:

$$g_{0e}(\chi, \gamma_I, \gamma_H, t) = g_{00}(\chi, t) + \left( \begin{array}{c} \gamma_I \\ \gamma_H \end{array} \right) \frac{\partial g_{0e}(\chi, t)}{\partial X} + O(\gamma_I^2, \gamma_H^2).$$

In this case, the function $g_{00}$ is independent of $\gamma_I$ and $\gamma_H$ and is uniquely determined by the value function of a log utility investor in a representative agent economy. For brevity, we write in the sequel $g_{0e}(\chi, \gamma_I, \gamma_H, t) = g_{0e}(\chi, t)$, and $O(\gamma^2)$ for $O(\gamma_I^2, \gamma_H^2)$. Moreover, we denote by $\ast = f, c$ the optimal policies of an unconstrained and a constrained investor, respectively.

The first-order general equilibrium portfolio policies for bank $j$, $j = \{I, H\}$, are then of the form

$$\begin{align*}
w^{j\ast}(\chi, t) &= (1 + \gamma_j) \frac{\lambda^{\ast}(\chi, t)}{\|\sigma^{\ast}(\chi, t)\|^2} + \gamma_j \eta_t^\ast + O(\gamma^2), \\
c^{j\ast}(\chi, t) &= \frac{1}{A_t} - \gamma_j \xi_t^\ast + O(\gamma^2),
\end{align*}$$

where

$$\begin{align*}
\xi_t^\ast &= \xi^{\ast}(\chi, t) = \frac{g_{0e}^\ast(\chi, t)}{A_t} + \log A_t, \\
\eta_t^\ast &= \eta^{\ast}(\chi, t) = \frac{1}{\|\sigma^{\ast}(\chi, t)\|^2} \left( \begin{array}{c} \sigma_{P_X}^{\ast}(\chi, t) \\ \sigma_{P_\omega}^{\ast}(\chi, t) \end{array} \right) \frac{\partial g_{0e}^\ast(\chi, t)}{\partial X},
\end{align*}$$

and where $\sigma_{P_X}^{\ast}(\chi, t)$ and $\sigma_{P_\omega}^{\ast}(\chi, t)$ are the covariances between stock price changes and the state variables $X$ and $\omega^I$, respectively.

Before giving the first-order properties of the relevant unconstrained and constrained econo-
mies, we specify more explicitly the zero-order equilibrium function \( g_{00} \) in dependence of the exogenous dynamics for the endowment process \( e_t \). Based on this result, asymptotic solutions for the general equilibria of some explicit model setting can be then computed.

**Lemma 1.** With an endowment and state variable process (38) and (39) it follows

\[
g_{00}(X, t) = \frac{\int g_{00}(X, t) = g_{00}(X, t)}{e^{-\delta(T-t)} A_t} (\log A_t + \mathbb{E}[H^e_{t,T} | \mathcal{F}_t]) + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} \mathbb{E}[H^e_{t,s} | \mathcal{F}_t] ds, \quad (46)
\]

where

\[
\mathbb{E}[H^e_{t,T} | \mathcal{F}_t] = \int_t^T \mathbb{E}\left[\mu_e(X, s) - \frac{1}{2} \sigma_e(X, s)^2 | \mathcal{F}_t\right] ds .
\]

Remark that the general equilibrium function \( g_{00} \) in an homogenous log-utility economy is independent of the presence of VaR constraints, because the market clearing condition in such an economy restricts the equilibrium portfolio fractions to one. Moreover, \( g_{00} \) depends on the state \( X_t \) but not on cross-sectional wealth \( \omega_t \), which is a constant in representative agents economies. Furthermore, by Lemma 1, \( \xi_t := \xi^*_t \) is not affected by VaR regulation to first-order and is a function of \( X \) and \( t \), but not of \( \omega_t \). On the other hand, \( \eta^*_t \) depends on cross-sectional wealth via \( \sigma^* \) and \( \sigma^*_{P,X} \). However, since \( g_{00}(X, t) \) is independent of \( \omega \), equations (44) and (45) yield

\[
\xi_t = \frac{g_{00}(X_t, t) + \log A_t}{A_t}, \quad (47)
\]

\[
\eta^*_t = \frac{\sigma^*_{P,X}(X_t, t) \partial g_{00}(X, t)}{\|\sigma^*(\chi, t)\|^2} \partial X, \quad (48)
\]

without affecting to first order the implied equilibrium optimal policies. These are now given by (42) and (43), with the simplified expressions for \( \xi^*_t \) and \( \eta^*_t \) given in (47) and (48), respectively.

To express the equilibrium asymptotics relevant for our analysis we introduce a weighted aggregate risk aversion coefficient \( \Delta \) defined by:

\[
\Delta = \gamma_I \omega^I_t + \gamma_H (1 - \omega^I_t).
\]

The next proposition summarizes the properties of the first-order general equilibrium when for both banks the VaR constraint does not bind. Since, as mentioned, the anticipative impact of VaR constraints is of an higher order in general equilibrium, this is equivalent to characterizing the general equilibrium of an economy where no VaR constraint is present.

**Proposition 6.** With an endowment/state variable process (38), (39) and in the absence of
VaR-constraints it follows:

i) the equilibrium interest rate is given by

\[ r^f(\chi, t) = \alpha^f(\chi, t) - \|\sigma^f(\chi, t)\|^2 \left(1 - (1 + \eta^f_t)\Delta\right) + O(\gamma^2). \]

ii) the cross-sectional wealth dynamics of investor I is given by

\[ d\omega^I_t = (\gamma_I - \gamma_{II}) \omega^I_t \left(1 - \rho^I_t\right) \left(\xi_I dt + \left(1 + \rho^I_t\right) \sigma_e(X_t) d\xi^e_t\right) + O(\gamma^2). \]

iii) drift and volatility of the cumulative return process in (40) are given by

\[
\begin{align*}
\alpha^f(\chi, t) &= \delta + \mu_e(X_t) + (\partial_t A_t - 1)\xi_t \Delta + A_t \Delta \left(\frac{\partial \xi_t}{\partial t} + \sigma_e \frac{\partial \xi_t}{\partial X_t}\right) + O(\gamma^2), \\
\sigma^f(\chi, t) &= \left(\begin{array}{c}
\sigma^{f_{11}}(\chi, t) \\
\sigma^{f_{22}}(\chi, t)
\end{array}\right) = \left(\begin{array}{c}
A_t \Delta \frac{\partial \xi_t}{\partial X_t} \sigma_e(X_t) \\
\sigma_e(X_t)
\end{array}\right) + O(\gamma^2).
\end{align*}
\]

From Proposition 6 we see that for any given functional form of \( \xi, \) i.e. any functional form for the dynamics of \( e_t \) and \( X_t, \) the first-order asymptotics for the risky asset and cross-sectional wealth dynamics are given in closed form. Finally, note that since \( d\omega^I_t \) is of order \( O(\gamma) \) in an unconstrained economy, the drift and the variance of the equilibrium price process in Proposition 6 do not depend on \( \omega^I_t. \) On the other hand, the equilibrium interest rate \( r^f \) depends on \( \omega^I_t \) via the weighted risk aversion coefficient \( \Delta. \)

We now study general equilibrium first-order asymptotics in a VaR regulated economy for those states where the second bank does hit the VaR constraint while the first one does not. We assume without loss of generality that \( \gamma_I < \gamma_{II}. \) To obtain tractable expressions for the relevant quantities in the constrained economy we perform a further expansion in the parameter \( \tau, \) which is typically a short time interval (e.g. a day).

**Proposition 7.** Given an endowment and state variable process (38), (39), and an economy where bank II is VaR-constrained and bank I not, it follows in states where the VaR constraint is binding:

i) the equilibrium interest rate is given by

\[ r^e(\chi, t) = \alpha^e(\chi, t) - \|\sigma^e(\chi, t)\|^2 + \|\sigma^e(\chi, t)\| \frac{v(1 - \omega^I_t)}{\sqrt{\omega^I_t B}} + h(\gamma_I; \tau) + O(\gamma^2, \tau^2), \]

where \( B = \omega^I_t v^2 + 2 \log(1 - \beta)(\omega^I_t - 2) \) and the function \( h(\gamma_I; \tau) \) is given in the Appendix.
ii) the cross-sectional wealth dynamics of bank $I$ is given by
\[
d\omega_I^e = \left( \mu_0^{\omega e}(\chi, t) + \mu_0^{\omega c}(\chi, t) + \mu_\tau^{\omega c}(\chi, t) \right) dt + \sigma_0^{\omega c}(\chi, t) dZ_t^e + \left( \sigma_\tau^{\omega e}(\chi, t) + \sigma_\tau^{\omega c}(\chi, t) \right) ^\top dZ_t + O(\gamma^2, \tau^2),
\]
where \((\mu^{\omega e}(\chi, t), \sigma^{\omega e}(\chi, t))\) are functions of the state variables and the regulatory parameters$^{16}$.

iii) drift and volatility of the cumulative price process in (40) are given by
\[
\alpha^c(\chi, t) = \alpha^f(\chi, t) + (\gamma_I - \gamma_H) A_t \xi_t \left( \mu_0^{\omega c}(\chi, t) + \left( \rho_e X \sigma_X(X_t) \frac{\partial \log \xi_t}{\partial X} + \sigma_e(X_t) \right) \sigma_0^{\omega c}(\chi, t) \right)
+ O(\gamma^2, \tau^2),
\]
\[
\sigma^c(\chi, t) = \sigma^f(\chi, t) + (\gamma_I - \gamma_H) \left( \frac{0}{\xi_t \sigma_0^{\omega c}(\chi, t)} \right) + O(\gamma^2, \tau^2),
\]
where \(\alpha^f(\chi, t)\) and \(\sigma^f(\chi, t)\) are given in Proposition 6. The functions \(\mu_0^{\omega c}(\chi, t)\) and \(\sigma_0^{\omega c}(\chi, t)\) are given in (A.59) and (A.60).

The results i)-iii) in Proposition 7 allow a direct comparison of the general structure of the equilibrium impact of VaR regulation with the structure of the unconstrained economy in Proposition 6. A more concrete quantification of this impact can be given for model settings with explicit state dynamics, a task that is accomplished in Section 3.3. Before doing so, however, some general remarks on the above results are necessary.

First, the size of the difference \(\alpha^f - \alpha^c\) and \(\sigma^f - \sigma^c\) in the risky asset expected return and volatility depends on quantities such as
1. the heterogeneity in risk aversions, through \(\gamma_I - \gamma_H\),
2. the investment horizon, via \(A_t\),
3. cross-sectional wealth, via \(\mu_0^{\omega c}\) and \(\sigma_0^{\omega c}\),
4. the structure of the exogenous state dynamics, i.e. the exogenous opportunity set, via \(\rho_e X, \sigma_X, \sigma_e X\) and \(\xi_t\).

In particular, we see that a large heterogeneity in risk aversion amplifies the VaR impact on the drift and volatility of equity returns. Second, in the constrained economy cross-sectional wealth matters to first-order in determining equity dynamics. This is a direct consequence of the market clearing condition in the regulated economy, which also causes a non-linear dependence of equilibrium interest rates \(r^e\) on cross-sectional wealth. This nonlinearity is
absent in the unconstrained economy (cf. i) in Propositions 6 and 7, respectively). Finally, Proposition 7 implies that working with a homogenous general equilibrium economy implies strong restrictions on the way VaR regulation affects the equilibrium variables. In fact, in such economies VaR regulation impacts only interest rates, but not the equilibrium dynamics of equity. This property is summarized by the next immediate corollary.

**Corollary 4.** Given an endowment and state variable process given in (38) and (39) with bank II VaR-constrained and bank I unregulated, if \( \gamma_I = \gamma_{II} \) it follows:

i) drift and volatility of the cumulative price process in (40) are given by

\[
\alpha^c(\chi, t) = \alpha^f(\chi, t) = \delta + \mu_e(X_t) + O(\gamma^2, \tau^2),
\]

\[
\sigma^c(\chi, t) = \sigma^f(\chi, t) = \sigma_e(X_t) + O(\gamma^2, \tau^2).
\]

ii) The equilibrium interest rate is given by

\[
r^c(\chi, t) = r^f(\chi, t) + \|\sigma^f(\chi, t)\| \frac{\gamma(1 - \omega^f)}{\sqrt{\omega^f B}} + h(\gamma_I; \tau) + O(\gamma^2, \tau^2),
\]

\[
r^f(\chi, t) = \alpha^f(\chi, t) - \|\sigma^f(\chi, t)\|^2.
\]

### 3.3 General Equilibrium: Do VaR Constraints Distort the Economy?

This section studies the general equilibrium implications of VaR regulation for some more explicit model settings. In particular, we consider a class \( \mathcal{M} \) of model specifications given by

\[
\mathcal{M} : \begin{cases}
M_0 : & \mu_e(X_t) = \mu_e, \quad \sigma_e(X_t) = \sigma_e X_t, \quad \gamma_I = \gamma_{II} = 0; \\
M_1 : & \mu_e(X_t) = \mu_e, \quad \sigma_e(X_t) = \sigma_e X_t; \\
M_2 : & \mu_e(X_t) = \mu_e X_t, \quad \sigma_e(X_t) = \sigma_e X_t.
\end{cases}
\]

We assume that state variable \( X \) follows a mean-reverting geometric Brownian motion\(^\text{17}\), i.e. \( m = 1 \) in Assumption 2. Model \( M_0 \) is a representative agent economy with a pure stochastic volatility model for the endowment process \( (e_t) \). The representative agent is equipped with a log-utility function. Model \( M_1 \) extends model \( M_0 \) to account for heterogeneities in risk aversion in the presence of a pure stochastic volatility model for \( (e_t) \). As highlighted by Corollary 4, in the setting of model \( M_0 \) no first-order impact of VaR regulation on equity dynamics is present. By contrast, in model \( M_1 \) heterogeneity in risk aversion also affects the drift and the volatility of equity returns. Finally, model \( M_2 \) studies the impact of VaR regulation in the presence of a mean reversion in both expected endowment returns and endowment volatility. Figures 5-8 summarize graphically our findings for some calibrations of models in the class \( \mathcal{M} \).
3.3.1 Model $M_0$

Figure 5 presents results for model $M_0$ when the correlation $\rho_{eX}$ between endowment and volatility states is positive. Since in this economy drift and volatility of equity returns are unaffected, we focus on the resulting optimal portfolios, interest rates and cross sectional wealth dynamics. In the top left panel portfolio fractions are equals to 1 (because of the market clearing condition) for all states of $X_t$ left of point A, i.e. for “low” or “moderate” endowment volatility states (the straight line to the left of point A). For a “high” endowment volatility, bank II hits the VaR constraint (the dotted line) and is forced to reduce its risk exposure below one (the dashed line). At the same time, to achieve market clearing bank I has to increase its exposure above one (the straight line right of point A). This is achieved by lowering the interest rate $r$. In the top right panels, equilibrium interest rates (the straight line) on the left of point A are on the dotted curve. This is the equilibrium interest rate level for the unconstrained economy. The equilibrium interest rate level in the constrained economy is on the straight line right of point A. For such states of $X_t$, the interest rate level is below the unconstrained economy. The impact of the regulation on the cross sectional wealth dynamics is illustrated in the lower panels, where cross-sectional wealth drift and volatility are plotted as a function of $X$. On the left of point A both parameters are zero, since in the unregulated homogenous economy cross sectional wealth is constant. As soon as bank II hits the constraint, however, both drift an volatility of the cross sectional wealth dynamics increase monotonically with the underlying endowment volatility state. Taking into account Corollary 4, this affects in equilibrium the conditional variance in the interest rates dynamics. We therefore conclude that within the setting of a representative agent economy of the form given in model $M_0$ a VaR regulation can have a destabilizing equilibrium effect, though more on money markets than on equity markets.

3.3.2 Model $M_1$

Figure 6 illustrates the results for model $M_1$ when heterogeneity in risk aversion is introduced within the basic setting of model $M_0$. Bank II is assumed to be less risk averse than bank I ($\gamma_I = -0.1$, $\gamma_{II} = 0.4$, respectively). The top left panel illustrates the optimal policies of bank II (dashed line) and bank I (straight line), respectively, as functions of $X$. To the left of points A and B, both banks are unconstrained. Bank II invests in a leveraged risky portfolio financed in equilibrium by bank I, which is primarily investing in a long position in the riskless asset. As volatility increases to states $X_t$ corresponding to points on the right of point A, bank II hits the regulatory VaR constraint (the dotted line through point A) and is forced to reduce its risk exposure. As a consequence, Bank I reduces its equilibrium investment in the riskless asset and increases its risk exposure to equity. Notice that the VaR
constraint through point A is determined by the equilibrium interest rate, drift and volatility of the equity dynamics prevailing in an unconstrained economy. As soon as bank II hits point A, these equilibrium quantities change to their corresponding values in the constrained economy. Therefore, the relevant VaR constraint shifts immediately to the new equilibrium level (the dotted line below point A). This feature highlights the endogenous nature of VaR constraints in general equilibrium. The increased demand for bonds of Bank II leads to a decrease in equilibrium interest rates for volatility states on the right of point A. For these particular states, equilibrium interest rates in the constrained economy (the dashed line) are below those prevailing in an unconstrained economy (the dotted line). However, a decrease in interest rates is not a sufficient incentive for bank I to clear the equity excess supply of bank II. Therefore, equilibrium equity expected returns and volatility also change and jump both upwards over the given support of $X_t$ in the two bottom panels of the figure. For states where bank II is constrained they are both higher than in the unconstrained economy. Moreover, since interest rates, drift and volatility have to change in such a way that bank I increases its demand for stocks (point B in Figure 6), the overall effect is an increase in the implied equity Sharpe ratio. This is confirmed by Figure 7 which plots the implied Sharpe ratio as a function of the state $X_t$. At point A the Sharpe ratio jumps to a higher level and obtains a steeper slope as a function of $X_t$ (the straight line beyond point A) than the Sharpe ratio function in the unconstrained economy (the dotted line). Thus, the presence of heterogeneously VaR-regulated financial institutions produces relatively high Sharpe ratios, high volatilities and low interest rates, a stylized fact which several standard financial models are not able to explain.

3.3.3 Model $M_2$

In an economy where banks are subject to heterogenous VaR constraints, the above stylized facts obtained within model $M_1$ are by no means the only possible equilibrium implication. To emphasize this point, we present in Figure 8 the results obtained for the dynamics of model $M_2$, when the correlation parameter $\rho_{eX}$ is negative. In this case, the more risk averse bank II is less exposed in equity (the dashed bold line in the top left panel) than bank I (the straight bold line in the top left panel), because of the intertemporal hedging motive. Therefore, regulation constrains the less risk averse but also less exposed bank II. When bank II hits the VaR constraint at point A, it has to reduce its risk exposure by reallocating wealth to the risky asset, thereby inducing a lower interest rate (top right panel). In equilibrium, this excess supply for equity has to be cleared now by the more risk averse Bank I through a higher demand for equity starting from point B. Therefore, the implied Sharpe ratios prevailing in the constrained economy will be higher, just as in the previous model $M_1$. However, when considering the bottom panels of Figure 6, we observe that for states where Bank II is constrained, both the equity expected return and the volatility are lower than in the unconstrained economy. Hence,
by contrast with the above results within model $M_1$, we observe that in this last model setting an heterogenous VaR regulation lowers the aggregate volatility risk of the equity market, thereby achieving one of the tasks that VaR regulation is intended to set forth.

3.3.4 What do we learn?

Our first-order equilibrium analysis and the explicit model settings discussed in the last subsections have shown that the impact of VaR regulation strongly depends on the degree of heterogeneity in risk aversions. As soon as such heterogeneities are present, the impacts of VaR regulation on equilibrium variables are very complex and hardly predictable. They depend on several factors which we summarize along three basic dimensions:

1. Regulatory control variables: $\beta, \nu, \tau$.
2. Market factors: $X$’s dynamics, $\rho_{X_0}$.
3. Individual investor’s parameters: $\gamma_1, \gamma_H, T, \omega_t$.

As expected, moving from a partial to a general equilibrium analysis makes the analysis of VaR regulation considerably more complex. This is because in a general equilibrium setting risk regulation affects the bank’s optimization problem also indirectly, via its endogenous impact on the economy and its indirect effect on the relevant individual VaR constraints. Since VaR constraints are defined as functions of interest rates as well as drift and volatilities of the stock price processes, they are determined endogenously. Therefore, in equilibrium VaR regulation causes highly non-linear effects and jumps in the parameters of the asset price dynamics. There are some important conclusions to be drawn from the above analysis:

- The impact of VaR regulation depends strongly on the heterogeneity structure in risk attitudes.
- For unregulated institutions, an investment in the risky asset becomes more attractive, as VaR regulation tends to cause an increase in the equity Sharpe ratios.
- An increase in the Sharpe ratio can imply either an increase or a decrease in equity expected returns and volatilities. Interest rates decrease.
- When the regulated bank becomes constrained, equilibrium interest rates, asset price drift and volatility can jump.

Since the exact effects and directions of a VaR regulation cannot be established unambiguously for a sufficiently general class of realistic models, a natural question arises. Is there a regulatory
standard yielding a clear statement about its economic impacts? As current regulation fails to provide an answer to this question, it appears that there is an obvious need to justify the considerable costs caused to the financial industry by the implementation of regulatory standards.

4 Conclusion

We have analyzed the partial equilibrium incentives and the general equilibrium implications of VaR-based regulation within a continuous-time economy with stochastic opportunity set, intermediate consumption and heterogeneities in risk aversions. In partial equilibrium we have shown that VaR-induced incentives can enhance the bank’s risk exposure. Such a distortion of incentives is, however, totally unrelated to the lack of coherence of VaR. It depends, instead, on the optimal intertemporal hedging behavior arising out of the given opportunity set dynamics and degrees of risk aversion. Our general equilibrium analysis has revealed that the effectiveness and possible distortion of VaR-based regulation depend on several quantities, including i) regulatory control variables, ii) pure market factors, and iii) individual investors' parameters. When these issues are considered, it becomes apparent that the optimal design of regulatory standards for market risk remains an unsolved problem and an urgent issue for future research.
Table 1: *First-order Approximation*. The table displays the upper bound for the probabilities given in Proposition 1. Reported numbers are percentage numbers. We consider the following model specification: $dX = (\theta - \kappa X_t)dt + \sigma_X X_t dZ^X_t$, $r(X_t) = r$, $\lambda(X_t) = \lambda X^2$, and $\sigma(X_t) = \sigma X_t$. To calculate the values for different $M$, $X$, $w$ and time-horizons, we assumed $\theta = 0.2$, $\kappa = 0.2$, $\sigma_X = 0.15$, $\lambda = 0.05$, and $\sigma = 0.2$. The unconditional mean is set to $\theta/\kappa = 1$.

<table>
<thead>
<tr>
<th></th>
<th>1 day</th>
<th>10 days</th>
<th>1 day</th>
<th>10 days</th>
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<tbody>
<tr>
<td>$X = 1$</td>
<td></td>
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<tr>
<td>$M = 1%$</td>
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<tr>
<td>$w$</td>
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<td>0.000</td>
<td>0.001</td>
<td>0.002</td>
</tr>
<tr>
<td>$w$</td>
<td>0.5</td>
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<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>$w$</td>
<td>0.8</td>
<td>0.000</td>
<td>0.009</td>
<td>0.009</td>
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<tr>
<td>$M = 5%$</td>
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</tr>
<tr>
<td>$w$</td>
<td>0.2</td>
<td>0.000</td>
<td>0.003</td>
<td>0.000</td>
</tr>
<tr>
<td>$w$</td>
<td>0.5</td>
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<td>0.001</td>
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<tr>
<td>$w$</td>
<td>0.8</td>
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<td>$M = 10%$</td>
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<td>$w$</td>
<td>0.8</td>
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</table>
Figure 1: Comparison of the perturbative solution and the exact one computed by Monte Carlo simulation. The model parameters are: $\alpha(X_t) = 0.05X_t^2, \sigma(X_t) = 0.25X_t, \theta = \kappa = 0.8, \sigma_X = 0.2, T = 1$. For the Monte Carlo solution, we partitioned the investor’s time-horizon into 100 time steps. In each time step, we draw 100,000 random numbers using antithetic variates. The log investor’s policy is given in the middle of the $x$-axis for the value $\gamma = 0$. 
Figure 2: Portfolio policy under VaR regulation and influence of the investment horizon. Graphs on the left assume a positive correlation parameter $\rho = 0.4$. Graphs on the right assume $\rho = -0.4$. Graphs (A) and (B) assume an investment horizon of $T = 1$ year. Graphs (C) and (D) assume $T = 5$ while graphs (E) and (F) assume $T = 10$. The solid bold line represents the portfolio policy of the VaR-constrained bank. At the circled point, the bank runs into the VaR constraint represented by the dotted line. The dashed line represents the optimal policy in the absence of constraints. The parameters were chosen as follows: $\gamma = 0.5$, $\nu = 0.01$, $\beta = 0.05$, $\tau = 10/250$, $r = 0.05$, $\lambda = 0.03$, $\sigma = 0.32$, $\kappa_X = \theta_X = 0.2$, $\sigma_X = 0.6$. The calculations were performed using standard Monte Carlo methods.
Figure 3: Portfolio policy under VaR regulation and the influence of the confidence level. Graphs on the left side assume $\rho = 0.4$. Graphs on the right side assume $\rho = -0.4$. Graphs (A) and (B) assume a confidence bound of $\nu = 0.01$. Graphs (C) and (D) assume $\nu = 0.05$. Graphs (E) and (D) show the differences in the optimal policies when the confidence level is tightened, i.e. we plot the differences (C)-(A) and (D)-(B). We assumed the same parameters as in Figure 2. The time horizon is $T = 5$ years.
Figure 4: Portfolio policies in a pure volatility model. We plot the portfolio policies when $\lambda(X_t) = \lambda$ and $\sigma(X_t) = \sigma X_t$. The dashed line represents the optimal policy of an unconstrained investor. The bold straight line is the portfolio policy of the constrained investor. Panels (A) and (B) assumes a positive a negative correlation parameter $\rho = 0.4$ and $\rho = -0.4$, respectively. Further parameters are calibrated as $\gamma = 0.5$, $\nu = 0.01$, $\beta = 0.05$, $\tau = 10/250$, $r = 0.05$, $\lambda = 0.03$, $\theta = \kappa = 0.7$, $\sigma_X = 0.8$, $\sigma = 0.35$, $T = 5$. 
Figure 5: Impact of VaR regulation in the log-economy $M_0$. We assumed the following parameter values: $\sigma_c = 0.2, \mu_c = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.05, T = 5, \tau = 1/250, \beta = 2.5\%, v = 5\%, \rho_{cX} = 0.4, \gamma_I = \gamma_H = 0$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, the drift and the volatility of the cross-sectional wealth dynamics dotted lines on the right of point $A$ represent quantities in the unconstrained economy.
Figure 6: Impact of VaR regulation in model $M_1$ with a positive correlation $\rho_{eX}$. We assumed $de_t/e_t = \mu_e dt + \sigma_e X_t dZ_t^e$ and the following parameter values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.1, T = 2, \tau = 1/250, \beta = 5\%, v = 1\%, \rho_{eX} = 0.4, \gamma_I = -0.1, \gamma_{II} = 0.4$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, the drift and the volatility of the asset price process, dotted lines on the right of point $A$ represent quantities in the unconstrained economy.
Figure 7: Impact of VaR regulation on Sharpe ratios for model $M_1$ with a positive correlation $\rho_{\phi X}$. We assumed the same parameter values as for Figure 6. The bold line is the Sharpe ratio resulting from the constrained economy. The dotted line on the right of point $A$ represents the Sharpe ratio in the unconstrained economy.
Figure 8: Impact of VaR regulation for model $M_2$ with a negative correlation $\rho_{eX}$. We assumed $d\xi / e_t = \mu_e X_t dt + \sigma_e X_t dZ_t$ and the following parameter values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.1, T = 2, \tau = 1/250, \beta = 2.5\%, v = 1\%, \rho_{eX} = -0.4, \gamma_I = -0.1, \gamma_{II} = 0.4$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, the drift and the volatility of the asset price process, the dotted lines on the right of point $A$ represent the corresponding quantities in the unconstrained economy.
Appendix

Proof of Proposition 1

With portfolio weights fixed at time \( t = 0 \), the wealth dynamics under Assumption 2 reads

\[
\log W_t = \log W_0 + \int_0^t (r(X_s) + w_0 \lambda(X_s) - \frac{1}{2} w_0^2 \sigma^2(X_s)) ds + w_0 \int_0^t \sigma(X_s) dZ_s.
\]

Performing an Itô-Taylor expansion in \( X \), we obtain

\[
\log W_t = \log W_0 + R \text{ with remainder}
\]

where

\[
R = w_0 \sigma(X_0) \int_0^t \int_0^s dZ_u ds + \int_0^t \int_0^s \left( \mathcal{L}_0 h_1(X_u) du + \mathcal{L}_1 h_1(X_u) dZ_u^X \right) ds
\]

with

\[
\mathcal{L}_0 = \mu_X(X) \frac{\partial}{\partial X} + \frac{1}{2} \sigma_X^2(X) \frac{\partial^2}{\partial X^2} , \quad \mathcal{L}_1 = \sigma_X(X) \frac{\partial}{\partial X} .
\]

Using Markov’s inequality we get the bound

\[
P \left( \left| \log W_t^{(1)} - \log W_t \right| \geq M \right) \leq \frac{1}{M} \mathbb{E} \left[ \left| \log W_t^{(1)} - \log W_t \right| \right] = \frac{1}{M} \mathbb{E} \left[ ||R|| \right] .
\]

Since the first moment of a multiple Itô integral vanishes, if it has at least one integration with respect to a component of the Wiener process, \( \mathbb{E} \left[ ||R|| \right] \) follows as claimed.

Proof of Proposition 2

From Proposition 1,

\[
P \left( W_t^{(1)} - W_t \leq -L \mid \mathcal{F}_t \right)
\]

\[
= P \left( Z_{t+\tau} - Z_t \leq \log (1 - \beta) - \frac{\log (1 - \beta) - (r(X_t) + w_t \lambda(X_t) + \frac{1}{2} w_t^2 \sigma(X_t)^2) \tau}{w_t \sigma(X_t)} \frac{\tau}{\mathcal{F}_t} \right)
\]

\[
= \mathcal{N} \left( \log (1 - \beta) - \frac{(r(X_t) + w_t \lambda(X_t) + \frac{1}{2} w_t^2 \sigma(X_t)^2) \tau}{w_t \sigma(X_t) \sqrt{\tau}} \right),
\]

38
where $\beta = L/W_t$ and $N(\cdot)$ is the cumulative normal distribution function. The approximated VaR at the confidence level $\nu$ then reads

$$\text{VaR}_t^{\nu,w} = W_t \left(1 - e^{\left(r(X_t) + w_1\lambda(X_t) - \frac{1}{2} w^2_1\sigma(X_t)^2\right)\tau + \nu w_2\sigma(X_t)\sqrt{\tau}}\right), \quad (A.50)$$

which is equivalent to

$$\log(1 - \beta) - \left(r(X_t) + w_1\lambda(X_t) - \frac{1}{2} w^2_1\sigma(X_t)^2\right)\tau - \nu w_2\sigma(X_t)\sqrt{\tau} \leq 0.$$ 

This inequality is equivalent to the upper and lower bound $w^\pm_b$ stated in the proposition. □

**Proof of Proposition 4**

We calculate the value function for a log-investor as

$$J_{\text{log}}(W, X, t) = \mathbb{E}[\log W_T | \mathcal{F}_t], \quad (A.51)$$

$$= \log W_t + \int_t^T \left(r + \mathbb{E}[w_{\text{log}} \lambda(X_s) | \mathcal{F}_t] - \frac{1}{2} \mathbb{E}[w_{\text{log}}^2 \sigma^2(X_s) | \mathcal{F}_t]\right) ds,$$

with

$$w_{\text{log}} = \begin{cases} w^f_{\text{log}} & \text{if } w^f_b < w^f_{\text{log}} < w^f_b, \\ w^b_b & \text{if } w^f_{\text{log}} \geq w^b_b, \\ w^b_{\text{log}} & \text{if } w^f_{\text{log}} \leq w^b_b, \end{cases}$$

where $w^f_{\text{log}} = X^{\nu_1 - 2\nu_2} \frac{\lambda}{\sigma^2}$ is the log-investor’s portfolio policy in the unconstrained case. Since $X > 0$ the only constraint binding for the log investor given Assumption 2 is $w^f_b$. Since

$$J_{\text{log}}(W, X, t) = \lim_{\gamma \to 0} \frac{\gamma \left(g_0(X,t) + \gamma g_1(X,t)\right) W_t^\gamma - 1}{\gamma} = \log W_t + g_0(X, t), \quad (A.52)$$
formulas (A.51) and (A.52) for the unconstrained function \( g_0(X, t) \) follow directly. Accounting for the presence of constraints, we have

\[
g_0(X, t) = r(T - t) + \lambda \int_t^T \mathbb{E} \left[ w_{\log}^f(X_s) + I_{|w_{\log}^f| > |w^+_b|} \tilde{w}^+_b(X_s) X_s^{n_1} | \mathcal{F}_t \right] ds
\]

\[
- \frac{1}{2} \sigma^2 \int_t^T \mathbb{E} \left[ \left( w_{\log}^f(X_s) + I_{|w_{\log}^f| > |w^+_b|} \tilde{w}^+_b(X_s) \right)^2 X_s^{2n_2} | \mathcal{F}_t \right] ds
\]

\[
g_0(X, t) = r(T - t) + \lambda \int_t^T \mathbb{E} \left[ w_{\log}^f(X_s) + I_{\tilde{w}^+_b < 0} \tilde{w}^+_b(X_s) X_s^{n_1} | \mathcal{F}_t \right] ds
\]

\[
- \frac{1}{2} \sigma^2 \int_t^T \mathbb{E} \left[ \left( w_{\log}^f(X_s) + I_{\tilde{w}^+_b < 0} \tilde{w}^+_b(X_s) \right)^2 X_s^{2n_2} | \mathcal{F}_t \right] ds
\]

\[
g_0(X, t) - \frac{1}{2} \sigma^2 \int_t^T \mathbb{E} \left[ I_{\tilde{w}^+_b < 0} \left( \tilde{w}^+_b(X_s) X_s^{n_2} \right)^2 | \mathcal{F}_t \right] ds, \quad (A.53)
\]

where \( \tilde{w}^+_b = w^+_b - w_{\log}^f \). Since \( X_s > 0 \) for all \( s \in [t, T] \), we can define \( \phi(X) := \tilde{w}^+_b \sigma X_s^{n_2} \) and make the corresponding substitutions in (A.53). This concludes the proof.

\[\square\]

**Proof of Corollary 1**

Plugging the bound \( K X^{2(n_1-n_2)} \) into equation (21), we can use the Euler gamma function \( \Gamma(a) \) and the incomplete Gamma function, \( \Gamma(b, a) \) to obtain

\[
|w^{(n)}_f(X, t)| \leq \left| X^{n_1-n_2} \frac{\lambda}{\sigma^2} \frac{1 - \gamma^{n_1+1}}{1 - \gamma} \right| + \left| \frac{\rho \sigma \gamma}{\sigma} K X^{m-n_2} \frac{X^{2(n_1-n_2)} \gamma \Gamma(n, X_s^{2(n_1-n_2)} \gamma) - \gamma^n e X^{2(n_1-n_2)} \Gamma(n, X_s^{2(n_1-n_2)})}{(1 - \gamma) \Gamma(n)} \right|.
\]

Since \( \lim_{b \to \infty} \Gamma(b, a)/\Gamma(b) = 1 \) for finite \( a \), the claim follows.
Proof of Proposition 5

The HJB equation for the control problem (P2) is

\[
0 = \max_{w,c} \left\{ J_t + \mu_X(X) J_X + \frac{1}{2} \sigma_X(X)^2 J_{XX} + W (r - c + w \lambda(X)) J_W \\
+ w W \sigma(X) \sigma_X(X) J_{WX} + \frac{1}{2} w^2 \sigma^2(X)^2 J_{WW} \\
- \phi \left( \log(1 - \beta) - \left( r - c + w \lambda(X) - \frac{1}{2} w^2 \sigma(X)^2 \right) \tau - w \sigma(X) \sqrt{\tau} \right) \right\}
= \max_{w,c} \left\{ G J - \phi Q(w, c) \right\}.
\]

By the homogeneity properties of the utility function, the solution is of the form

\[
J(W, X, t) = \frac{A_t}{\gamma} \left( \frac{e^{\gamma g(X, t) W \gamma}}{W} - 1 \right). \tag{A.54}
\]

The first-order conditions and slackness imply unconstrained optimal policies today given by

\[
w^f = -X_t^{n_1 - 2n_2} \frac{\lambda J_W}{\sigma^2 W J_{WW}} - \rho X_t^{m - n_2} \frac{\sigma_X J_{WX}}{\sigma W J_{WW}} \\
= X_t^{n_1 - 2n_2} \frac{\lambda}{\sigma^2 (1 - \gamma)} + \frac{\gamma}{1 - \gamma} X_t^{m - n_2} \frac{\rho \sigma_X \partial g(X, t)}{\sigma} \frac{\partial X}{\partial X},
\]

\[
c^f = (J_W W)^{\frac{1}{1 - \gamma}} = \left( A_t e^{\gamma g(X, t)} \right)^{\frac{1}{1 - \gamma}}.
\]

We expand \( g(X, t) = g_0(X, t) + \gamma g_1(X, t) + O(\gamma^2) \). From (A.54) the log-investor’s value function reads

\[
J(W, X, t) = A_t \left( \log(W_t) + g_0(X, t) \right).
\]

Expanding the optimal portfolio weight and optimal consumption rate up to first order we obtain \( w_f^{(1)}(X, t) \) and \( c_f^{(1)}(X, t) \) as claimed. To obtain the function \( g_0(X, t) \), recall the wealth dynamics in the presence of intermediate consumption given in equation (27). A second representation for the value function \( J \) is then

\[
J(W, X, t) = e^{-\delta(T-t)} \left( \log W_t + E \left[ H_{t,T} | \mathcal{F}_t \right] \right) \\
+ \int_t^T e^{-\delta(s-t)} \left( \log W_t + E \left[ H_{t,s} | \mathcal{F}_t \right] + E \left[ \log c_s | \mathcal{F}_t \right] \right) ds.
\]
Equating the last two expressions for the value function it thus follows

\[
A_t = e^{-\delta(T-t)} + \frac{1 - e^{-\delta(T-t)}}{\delta},
\]

\[
g_0(X, t) = \frac{1}{A_t} e^{-\delta(T-t)} \mathbb{E}[H_t, T | \mathcal{F}_t] + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} (\mathbb{E}[H_{t,s} | \mathcal{F}_t] + \mathbb{E}[\log c_s | \mathcal{F}_t]) \, ds.
\]

We still need to determine the function \(H_{t,T}\). Its general form is given in equation (28). Similarly as in Proposition 4, the structure of \(w^+_b(X, t)\) in the presence of intermediate consumption leads to the decomposition of \(H_{t,T}\) in the theorem. \(\square\)

**Proof of Corollary 3**

To prove the impact on consumption \(c_t\) it is sufficient to remark that from Proposition 5 it follows immediately \(g_f(X, t) > g_0(X, t)\) since \(\mathbb{E}[H_{t,T} | \mathcal{F}_t] > \mathbb{E}[H_{t,T} | \mathcal{F}_t]\). To prove the impact on the portfolio fractions we have to show consider the difference

\[
\frac{\partial g_0(X, t)}{\partial X} - \frac{\partial g_f(X, t)}{\partial X},
\]

under the given assumptions on \(n_1, n_2\). This sign is just the opposite of the sign of the derivative of

\[
M(X_t) := \int_t^T \mathbb{E}[\mathbf{1}_{\phi(X_s) < 0} \phi(X_s)^2 | \mathcal{F}_t] \, ds.
\]

Then

\[
\frac{\partial M(X_t)}{\partial X_t} = \int_t^T \mathbb{E}\left[ \frac{\partial X_s}{\partial X_t} \frac{\partial \phi(X_s)}{\partial X_s} \left( \mathbf{1}_{\phi(X_s) < 0} \phi(X_s)^2 \right) | \mathcal{F}_t \right] \, ds
\]

\[
= \int_t^T \mathbb{E}\left[ \frac{\partial X_s}{\partial X_t} \phi(X_s)^2 \frac{\partial \phi(X_s)}{\partial X_s} \left( \delta_{\phi(X_s) = 0} + \frac{2\mathbf{1}_{\phi(X_s) < 0}}{\phi(X_s)} \right) | \mathcal{F}_t \right] \, ds,
\]

\[
= 2 \int_t^T \mathbb{E}\left[ \frac{\partial X_s}{\partial X_t} \mathbf{1}_{\phi(X_s) < 0} \phi(X_s) \frac{\partial \phi(X_s)}{\partial X_s} | \mathcal{F}_t \right] \, ds,
\]

where by \(\delta_{\cdot}\) we denote the Dirac-delta function. For the \(X\)’s dynamics considered in the paper we have \(\partial X_s/\partial X_t \geq 0\) for all \(s \geq t\). Moreover, the sign of the expression within the expectation operator depends on the sign of \(\frac{\partial X_s}{\partial X_t} \mathbf{1}_{\phi(X) < 0} \phi(X)\). Since \(\phi(X_s)\) is multiplied by the index function, it immediately follows

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\textbf{Proof of Lemma 1}

The proof of Proposition 5 implies that the function \(g_{00}(\chi, t)\) must be of the form

\[
g_{00}(\chi, t) = \frac{1}{A_t} e^{-\delta(T-t)} \mathbb{E}[H_{t,T} | F_t] + \int_t^T \frac{e^{-\delta(s-t)}}{A_t} (\mathbb{E}[H_{s,t} | F_s] + \mathbb{E}[\log c_s | F_s]) ds,
\]

with

\[
\mathbb{E}[H_{t,T} | F_t] = \mathbb{E}\left[ \int_t^T \left( r_{\log} + w_{\log}(\alpha_{\log} - r_{\log}) - c_{\log}(s) - \frac{1}{2} w_{\log}^2 \sigma_{\log}^2 \right) ds | F_t \right].
\]

As \(g_{00}(\chi, t)\) is determined in the homogenous log-economy, the above equation can be simplified considerably. First, we note that in the log-economy, market clearing implies \(w_{\log} = 1\). Further, \(c_{\log} = 1/A_t\) and, as \(\gamma_I = \gamma_H = 0\),

\[
\alpha_{\log} = \delta + \mu_e(\chi, t), \quad \sigma_{\log} = \sigma_e(\chi, t).
\]

Inserting the above expressions in \(g_{00}(\chi, t)\) proves the claim. \(\square\)

\textbf{Proof of Proposition 6}

To obtain the equilibrium interest rate \(r^f(\chi, t)\) we start with the market clearing condition

\[
\omega_t^I \left( \frac{\alpha(\chi, t) - r(\chi, t)}{||\sigma(\chi, t)||^2} (1 + \gamma_I) + \gamma_I \eta_t \right) + (1 - \omega_t^I) \left( \frac{\alpha(\chi, t) - r(\chi, t)}{||\sigma(\chi, t)||^2} (1 + \gamma_H) + \gamma_H \eta_t \right) = 1.
\]

Solving for \(r(\chi, t)\) and expanding in \(\gamma\) gives the equilibrium interest rate. To determine \(d\omega_t^I\), recall that \(\omega_t^I = W_t^I / P_t\). Applying Itô’s Lemma to cross-sectional wealth, inserting the wealth differentials and dropping the \(O(\gamma^2)\) terms, we obtain the expression for \(d\omega_t^I\) up to second order in \(\gamma\). To derive the
price process, we consider
\[ \frac{dP_t}{P_t} = \frac{d(P_t/e_t)}{P_t/e_t} + \frac{de_t}{e_t} + \frac{d(P_t/e_t)}{P_t/e_t}, \tag{A.56} \]
with the ratio \( P_t/e_t \) given by
\[ P_t/e_t = \frac{1}{\omega_t'(c_t'/c_t + c_t''/c_t')} = A_t(1 + A_t\xi_t\Delta) + O(\gamma^2). \]

Then,
\[
\frac{dP_t}{P_t} = \frac{de_t}{e_t} + (\partial_t \log A_t + \xi_t\Delta \partial_t A_t) dt + (\gamma_I - \gamma_H) A_t \xi_t (d\omega_t'+ \sigma_e(X_t)d(Z^e, \omega')_t) \\
+ A_t \Delta (d\xi_t + \sigma_e(X_t)d(Z^e, \xi_t) + (\gamma_I - \gamma_H) A_t d(\omega', \xi)_t + O(\gamma^2). \tag{A.57} \]

Since the bracket processes can be written as
\[
d(Z^e, \omega')_t = \rho_{cX} \sigma_X(X_t) \frac{\partial \omega_t'}{\partial X_t} dt + e_t \sigma_e(X_t) \frac{\partial \omega_t'}{\partial e_t} dt, \\
d(Z^e, \xi_t) = \rho_{cX} \sigma_X(X_t) \frac{\partial \xi_t}{\partial X_t} dt, \\
d(\omega', \xi)_t = e_t \rho_{cX} \sigma_e(X_t) \frac{\partial \xi_t}{\partial X_t} \frac{\partial \omega_t'}{\partial e_t} dt + \sigma_e(X_t)^2 \frac{\partial \xi_t}{\partial X_t} \frac{\partial \omega_t'}{\partial X_t} dt,
\]
we insert the above expressions into (A.57) and (A.56). Since in an unconstrained homogeneous log-investor economy \( d\omega_t' = O(\gamma)dt + O(\gamma)dZ^X_t \), the instantaneous drift and the instantaneous volatility of the cumulative price process, \( (dP_t + e_t dt)/P_t \), follow as claimed. \( \square \)

**Proof of Proposition 7**

We assume without loss of generality \( \gamma_I < \gamma_H \) and that the constraint for the less risk averse investor II is binding. Market clearing implies
\[
\omega_t' \left( \frac{\alpha^c(\chi, t) - r^c(\chi, t)}{\lVert \sigma^c(\chi, t) \rVert^2} (1 + \gamma_I) + \gamma_I \eta_t^c \right) + (1 - \omega_t')w_t^c(\chi, t) = 1.
\]
Solving for the interest rate and expanding in \( \tau \) and \( \gamma \), we obtain the interest rate \( r(\chi^c, t) \) as given in the proposition with
\[
h(\gamma_I; \tau) = \frac{1}{\sqrt{\tau}} \theta_{\tau,0} + \sqrt{\tau} \theta_{\tau,1} + \tau \theta_{\tau,2} + \gamma_I \theta_{\gamma},
\]

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where

\[
\begin{align*}
\theta_{\tau,0} &= \|\sigma^c(\chi, t)\| \left( \frac{1 - \omega'_{t}}{2} \right) \left( v + \sqrt{B} \omega'_{t} \right), \\
\theta_{\tau,1} &= - \frac{(1 - \omega'_{t}) \|\sigma^c(\chi, t)\| \left( \frac{2B \left( \alpha^c(\chi', t) - \frac{1}{A_t} - \|\sigma^c(\chi, t)\|^2 \right) + v^2 \|\sigma^c(\chi, t)\|^2 (\omega'_{t} - 2) \right)}{2(B \omega'_{t}^2)^{3/2}}, \\
\theta_{\tau,2} &= \frac{v \|\sigma^c(\chi, t)\| \left( \omega'_{t} - 2 \right) B \theta_{\tau,1}, \\
\theta_\gamma &= - \frac{\|\sigma^c(\chi, t)\|^2 (1 + \eta_t^e)}{\omega'_{t} - 2} \\
&\quad \frac{\omega'_{t} (1 - \omega'_{t}) v \|\sigma^c(\chi, t)\|^2 \left( 2 \log(1 - \beta)(\omega'_{t} - 2) + (B(1 + \eta_t^e) - v^2(\omega'_{t} - 2)) \omega'_{t} \right)}{(\omega'_{t} - 2)(B \omega'_{t}^2)^{3/2}} \\
&\quad \frac{- 1}{\sqrt{\tau}} \frac{\|\sigma^c(\chi, t)\|^2 (1 - \omega'_{t}) \left( B \omega'_{t}^2 - \sqrt{B} \omega'_{t} \left( 2N + (\omega'_{t} - 2)(v^2 \omega'_{t} - 2 \log(1 - \beta)) \right) \right)}{B(\omega'_{t} - 2)^2 \omega'_{t}}.
\end{align*}
\]

To obtain the price dynamics we can use (A.57). By inspection of (A.57) we see that we need to calculate \(d\omega^t\) only up to \(O(\gamma, \tau)\), since the expression involving \(\omega\)-terms is pre-multiplied by the first order difference \(\gamma_t - \gamma_t^e\). This gives

\[
d\omega^t = \mu_{\omega}^c(\chi, t) dt + \sigma_{\omega}^c(\chi, t) dZ^c_t + O(\gamma, \tau), \quad (A.58)
\]

where

\[
\begin{align*}
\mu_{\omega}^c(\chi, t) &= \frac{\omega'_{t} (1 - \omega'_{t})^2}{\sigma_c^2(X_t)^2 C A_t} \left( A_t \left( EC^2 (C + v \sigma_c(X_t)) - D^2 v \sigma_c(X_t) \right) - 2C^2 \sigma_c(X_t)^2 (C + v \sigma_c(X_t)) \right) \\
&\quad + \frac{\omega'_{t} (1 - \omega'_{t})^2}{\sigma_c^2(X_t)} \left( \sigma_{\omega, 0}^2 - 2v \sigma_{\omega, 0} \sigma_c(X_t) + 2 \sigma_c(X_t) \left( v(C + \sigma_c(X_t)) - \sigma_c(X_t) \log(1 - \beta) \right) \right) \\
&\quad + \frac{2 \left( C + v \sigma_c(X_t) \right) \omega'_{t} (1 - \omega'_{t})^2 D}{\sigma_c(X_t)^2 \sqrt{\tau}}, \\
\sigma_{\omega}^c(\chi, t) &= - \frac{\omega'_{t} (1 - \omega'_{t})}{\sigma_c(X_t)} \left( \frac{D}{C} + \frac{C + v \sigma_c(X_t)}{\sqrt{\tau}} \right), \quad (A.59)
\end{align*}
\]
with

\[
C = \sqrt{\theta_{\tau,0}^2 - 2v\theta_{\tau,0}\sigma_e(X_t) + \sigma_e(X_t)^2(v^2 - 2\log(1 - \beta))},
\]

\[
D = \sigma_e(X_t)^2 \left( \theta_{\tau,0} + (v\sigma_e(X_t) - \theta_{\tau,0}) \left( 1 - \frac{v(1 - \omega_t^1)}{\sqrt{\omega_t^1 B}} \right) \right),
\]

\[
E = \sigma_e(X_t)^4 \left( 1 - \frac{v(1 - \omega_t^1)}{\sqrt{\omega_t^1 B}} \right)^2 + 2\sigma_e(X_t)^2 \left( \delta + \mu_e(X_t) + \sigma_e(X_t)^2 \left( 1 - \frac{v(1 - \omega_t^1)}{\sqrt{\omega_t^1 B}} \right) \right)
\]

\[+ 2c_{\tau,1} (\theta_{\tau,0} - v\sigma_e(X_t)).\]

We note that the \( dZ_t \) term for the dynamics of \( \omega_t^1 \) vanishes in \( O(\gamma^2, \tau^2) \). To obtain the price dynamics, we go back to equation (A.57). Moving from the unconstrained to the constrained economy adds additional terms, namely

\[
(\gamma_1 - \gamma_{II})A_t \xi_t \left( d\omega_t^1 + \sigma_e(X_t)d(Z^e, \omega_t^1)_t \right) + (\gamma_1 - \gamma_{II})A_t d(\xi, \omega_t^1)_t + O(\gamma^2, \tau^2)
\]

\[
= (\gamma_1 - \gamma_{II})A_t \xi_t \left( \left( \mu_0^{ce}(\chi, t) + \left( \rho_e \sigma_e(X_t) \frac{\partial X_t}{\xi_t} + \sigma_e(X_t) \right) \right) dt + \sigma_0^{ce}(\chi, t)dZ^e \right)
\]

\[+ O(\gamma^2, \tau^2).\]

In the unconstrained economy all terms of order \( O(\gamma^2, \tau^2) \) can be omitted, but this is not true in the constrained economy. Incorporating this term into the price dynamics, and noting from (A.57) that the price dynamics must be of the form

\[
\frac{dP_t + c_t dt}{P_t} = \alpha^e(\chi, t) dt + O(\gamma)dZ_t + \sigma_e(X_t)dZ_t,
\]

we obtain the drift and volatility as claimed in the proposition. Inserting these expressions together with \( \alpha^e(\chi, t) \) into the dynamics of \( \omega_t^1 \), we obtain the expressions for \( d\omega_t^1 \). Finally, the explicit functions for \( \mu_1^{ce}(\chi, t), \mu_2^{ce}(\chi, t), \sigma_1^{ce}(\chi, t), \) and \( \sigma_2^{ce}(\chi, t) \) follow. Since these are given by lengthy expressions they are not displayed here but can be obtained from the authors on request. 

\[\square\]
Notes

1See Artzner, Delbaen, Eber, and Heath (1999) for a definition of coherent risk measures.

2While Proposition 1 in Basak and Shapiro (2001) applies for a stochastic opportunity set, their analysis in the rest of the paper is confined to a constant opportunity set model.

3To simplify notation, we write $w_t$ for $w(X_t)$.

4Other forms of the VaR limit lead in general to wealth dependent VaR boundaries under the above approximation procedure.

5Given Assumption 2, the $k$th moment solves an ordinary partial differential equation of order $k$.

6If higher-order approximations of the initial VaR constraint are considered, we obtain a sequence of control problems $(P_n)_{n=1,2,...}$ which eventually converges to the original problem $(P)$. We do not consider this issue further in this paper.

7One can show that the solution of problem $(P1)$ exists, i.e. the gradient of the value function is positive and the Hessian is strictly negative definite.

8Note that the function $g^0_0(X,t)$ is available explicitly for $2(n_1 - n_2) \in \mathbb{N}^0$. To calculate $g^0_0(X,t)$ for explicit model settings we resort to simulation methods.

9The convexity of the constrained set $C(X)$ is ensured by the fact that $\partial^2 Q(c,w)/\partial w^2 < 0$.

10For brevity we write $w(X,t)$ instead of $w(X_t,t)$. No confusion should occur.

11By $\text{I}_A$ we denote the Heaviside function which equals one if $A$ is true and zero otherwise.

12In this discussion, we focus again for simplicity on the case without consumption in Proposition 4.

13This value holds for a model where volatility follows a lognormal process. For other processes, similar values were found.

14If $\gamma < 0$, the direction of the above effects just changes sign.

15The model can be easily extended to an arbitrary number of banks.

16Since the relevant expressions for these functions are very long and involved, they are omitted for brevity. A Mathematica file containing all such expressions is available from the authors on request.

17Alternatively, we could assume a square-root or a Gaussian process for $X$. However, the results do not change qualitatively.

18Only when the state $X$ increases further, volatility will eventually be lower than in the unconstrained economy.
References


