Capm under ambiguity

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Abstract

In this paper we applied the model of individual choice under ambiguity proposed by Zhang (2002) in the context of the market model of asset returns of Kwon (1985). The ambiguity is introduced via unknown volatilities of assets’ residual leading to two factor CAPM. We test this model on US stock data and find that the ambiguity factor has a significant impact on the cross section of asset expected returns. This effect seems to be economically meaningful.

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1 Introduction

The successful theory of choice under uncertainty should satisfy at least two criteria. First, it should be based on axioms that are "reasonable" given our understanding of the individual rationality and, second, it should be analytically tractable and suitable for applications. Expected utility theory is the one that satisfies both criteria. Although, the "reasonability" of its axioms has been questioned by many researchers based on specific examples, it still remains the workhorse of the contemporary economics.

Von Neumann and Morgenstein (1944) stated four axioms that allow to derive the expected utility representation of individual choice under uncertainty. The theory deals with so called objective probabilities and initially the individual preferences are defined over the lotteries. The expected utility theory has been first criticized by Allais (1953), who showed that Independence axiom is violated when lotteries with extreme gains and losses are considered. The analytical advantage of the expected utility theory appear also its main weakness: the objective function is linear in probabilities. Several non-expected utility theories has been proposed that deal with non-linear objective functions. These generalization were done mainly at the expense of the analytical tractability.

Expected and non-expected utility theories assume the existence of objective probabilities known to individuals. Savage (1954) proposed an axiomatic model starting from individual preferences over payoffs (not lotteries) that specify an outcome in each state of the world. He shows that individual preferences can be represented via expected utility maximization under subjective probabilities. Among the axioms assumed by Savage (1954) is the so called "Sure-thing principle" stating that if two payoffs have the same outcome in some state of the world then changing this outcome will not affect the ordering of preferences. Ellsberg (1961) showed that this principle is not as obvious as it might seem. Consider two urn that have 25 black and 50 white and yellow balls in unknown proportion. There are three states of the world corresponding to the color of the ball drawn from the urn. The following table describes four payoffs $A$, $B$, $C$ and $D$:

<table>
<thead>
<tr>
<th></th>
<th>Black</th>
<th>White</th>
<th>Yellow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>$C$</td>
<td>100</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$D$</td>
<td>0</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>
An individual will prefer payoff \( A \) dominates payoff \( B \) and payoff \( D \) to payoff \( C \). Payoff \( A \) is better than \( B \) since the individual do not know the proportion of white and yellow balls in the urn. Payoff \( D \) is preferred to \( C \) for exactly the same reason. Sure-thing principle is violated since \( C \) and \( D \) differ from \( A \) and \( B \) by the outcome in one state of the world. Note that the "Sure-thing principle" should always hold whatever is the underlying probability space provided that there is no ambiguity. Paradoxically, the fact that it holds for any probabilistic distribution does not guarantee that it holds when some events have unknown probability. We will find a similar paradoxical situation in the context market model of asset returns (see Section 3).

Ellsberg paradox shows that an individual may not attach (subjective) probabilities to all the possible states of the world and introduces the concept of ambiguity. It led to an emergence of many theories that model the choice under ambiguity, most notably models with non-additive probabilities (capacities) or Choquet expected utility theories. The generalization is again done mainly at the expense of analytical tractability. Although these theories explain a broad range of phenomena, many of them leave no hope for applications.

One of few successful financial application has been made by Epstein and Chen (2002). Unfortunately, the choice model is rather ad hoc: it is assumed that an individual maximizes minimum expected utility over a set of priors (probability measures).

In this paper we are mainly interested in financial applications of the models of individual choice under ambiguity. We will introduce the ambiguity via unknown asset return volatility. There is an area in the quantitative finance pioneered by Avellaneda et al (1995), which actually deals with models of uncertain volatility. It is assumed that the volatility process is not known but the volatility always lies within known bounds. This literature is mostly concerned with finding riskless bounds on derivatives prices but not with the choice under ambiguity. In this paper we assume similar (in spirit) uncertain volatility setup and will study implications of the ambiguity for equilibrium asset prices. We adopt the approach of Zhang (2002) to modeling individual choice, which in our view, is sufficiently restrictive to produce interesting results. We will intentionally select the simplest model setup so that we can easily quantity the effect of the ambiguity and produce testable implications.

The paper is organized as follows. In the next section we provide a brief overview of Zhang (2002) approach and its major implications. In the section
that follows we derive the implications of this approach in the framework market model of assets’ returns with unknown volatilities. In particular, we obtain a two factor CAPM with ambiguity of asset return being the second factor. In the forth section we provide an empirical test of the CAPM under ambiguity on US stock data. The last section provide concluding remarks.

2 Zhang subjective ambiguity approach

In this section we describe the approach to modelling the choice under ambiguity proposed by Zhang (2002).

As in the pure probabilistic setup there is a $\sigma$-algebra $\Sigma$ of events. The subset $\Sigma^u$ of this algebra is a set of unambiguous events, that is, events which are assigned (subjective) probabilities by an individual. The set $\Sigma^u$ is assumed to be $\lambda$-system, which differs from $\sigma$-algebra in that it is not closed with respect to the intersection. Naturally, if two events are unambiguous, their intersection need not to be unambiguous.

Let $A$ be an ambiguous event. The idea of the approach proposed by Zhang (2002) is that an individual estimate a likelihood of this event using the probabilities of unambiguous events. The inner measure of ambiguous event $A$, representing its likelihood, is defined in the following way:

$$p^*(A) = \sup \{p(B) : B \in \Sigma^u, B \subseteq A\}$$ (1)

The definition implies that the individual knows that event $A$ will happen "at least" with probability $p^*(A)$ and this is also the maximum probability that he can be sure of given his beliefs. The individual does not assume more about likelihood of this event, which reflects his total of "infinite" aversion to the ambiguity.

Note, that using the inner measure to evaluate the likelihood of an event can be viewed as a pessimistic judgement. These pessimistic beliefs are not consistent in a sense that $p^*$ is not a probability measure: $p^*(A) \neq 1 - p^*(A^c)$. For an individual to be truly pessimist it is necessary that he is pessimistic about events that favor him and optimistic about complementary events. As we will see below, this is reflected in the definition of Choquet integral.

The inner measure is an example of capacity - a generalization of a probability measure. Zhang (2002) states nine axioms for individual preference over payoffs. Given these axioms, the preference relationship on the set of payoffs can be represented via "expected utility" under the above defined
capacity. More precisely, a payoff $f$ is equivalent or preferred to a payoff $g$ if and only if:

$$\int u(f)dp_* \geq \int u(g)dp_*$$

where the integrals are defined in the sense of Choquet:

$$\int \phi dp_* = \int_0^\infty p_*(\phi \geq t)dt + \int_{-\infty}^0 [p_*(\phi \geq t) - 1]dt$$

(2)

From (2) it is clear that the individual is "pessimistic" about the likelihood of events of the type "utility is higher than some value". This pessimism reflects his aversion to the ambiguity. Note, that this type of pessimism is different from that of an individual under Epstein approach who chooses according to minimal utility over possible probability measures on the set of all events (see more discussion in the next section).

3 Theoretical model

In this section we will derive CAPM under the market model with unknown volatilities of asset returns. We will first consider a simple example to illustrate the implications of this new approach and than derive CAPM under ambiguity in a static setting. We will demonstrate that in this case the problem of choice under ambiguity can be transformed to the classical utility maximization problem by changing the probability measure.

3.1 A simple example

Let us consider a simple example that will illustrate the notion of the inner measure. We assume that an individual faces a project that generate random return:

$$\tilde{r} = \alpha + \sigma \tilde{\varepsilon}$$

(3)

where $\tilde{\varepsilon} \sim N(0,1)$, $\alpha$ and $\sigma$ are some constants.

By writing (3) we implicitly assume that there exist an objective probability distribution of the future wealth. The $\sigma-$algebra $\Sigma$ of all possible events is formally defined as a Borel algebra on $\mathbb{R}^2$. An example of event in $\Sigma$ is $\{\tilde{r} \leq r_0, \tilde{\varepsilon} \leq \varepsilon_0\}$. The deterministic relationship (3) suggests that the
two dimensional σ-algebra can be trivially reduced to a one dimensional one. For our purpose, it is important to keep the first formal definition.

The individual has some believes about likelihood of some events (unambiguous events) but not about all. We assume that these subjective believes coincide with the objective probabilities. By assumption, the individual knows α and has correct believes about the distribution of ε. However, he does not know the precise value of the volatility parameter σ. The only thing he knows about the volatility is that it lies between high and low bounds σ^h and σ^l respectively.

The set of unambiguous events Σ^u, corresponding to our assumptions, is the one dimensional Borel algebra generated by the random variable ε and an example of an unambiguous event is {ε ≤ ε_0}. Let us now compute inner measure of ambiguous event \( B = \{ r \geq r_0 \} \). According to the definition of the inner measure (see previous section) we should look for all the unambiguous events \( B^u \subseteq B \). For any event \( A \in \Sigma \) define \( \bar{x}_A \equiv r_0 1_A \). Let us note that that \( B^u = \{ \bar{x}_{B^u} \geq r_0 \} \) and that the statement \( B^u \subseteq B \) is equivalent to:

\[
\bar{x}_{B^u} \geq r_0 \Rightarrow \bar{x}_{B^u} \leq \bar{r}.
\]

Since any unambiguous event \( B^u \) is measurable with respect to ε, then \( \bar{x}_{B^u} \) is a measurable function of ε. This permits us to reformulate the problem of finding the inner measure of \( B \) in the following way:

\[
p^*(B) = \sup_{\varphi: \varphi(\bar{r}) \geq r_0 \Rightarrow \varphi(\bar{e}) \leq \bar{r}} p \{ \varphi(\bar{e}) \geq r_0 \}.
\]

The solution to the problem (4) \( \varphi^* \) is independent of \( r_0 \):

\[
\varphi^*(\bar{e}) = \alpha + \sigma(\bar{e}) \bar{e} \leq \bar{r},
\]

where

\[
\sigma(\bar{e}) = \sigma^l 1_{\{\bar{e} > 0\}} + \sigma^h 1_{\{\bar{e} \leq 0\}}.
\]

and it follows that

\[
p^*(B) = p \{ \varphi^*(\bar{e}) \geq r_0 \}.
\]

Suppose, for example, that the individual has a choice between investing in this project that costs zero and not investing (zero future wealth in all
states of the world). Following Zhang (2002), the individual chooses to invest in the project if:

\[ \int u(\widetilde{r})dp_\ast \geq u(0). \]

Now assume that \( u \) satisfies the conditions of a risk-averse utility function (a positive convex function) then using (2) and (6), we can write:

\[ \int u(\widetilde{r})dp_\ast = \int_0^\infty p_\ast(u(\widetilde{r}) \geq t)dt \]
\[ = \int_0^\infty p_\ast(\widetilde{r} \geq u^{-1}(t))dt \]
\[ = \int_0^\infty p(\varphi_\ast(\varepsilon)) \geq u^{-1}(t))dt \]
\[ = \int_0^\infty p(u(\varphi_\ast(\varepsilon)) \geq t)dt \]
\[ = \mathbb{E}^P u(\varphi_\ast(\varepsilon)) \]

As we see, the individual evaluates the expected utility (under historical probability \( P \)) of the modified wealth (5). Equivalently, we may say that he evaluates expected utility of wealth under the new probability \( Q \) under which residual returns are distributed as \( \sigma_i(N(0,1)) \cdot N(0,1) \).

In the next section we will see that these results can be easily generalized to the case of a portfolio that consists of several assets with future prices satisfying (3) and having unknown but bounded volatilities. Under this specific framework the problem of portfolio choice under ambiguity will be reduced to the classical utility maximization by transformation of the probability measure. The transformed probability is independent of the composition of the portfolio (provided that there is no short selling of risky assets) and is constructed in the manner as in the single asset case. It should be noted that the equivalence between portfolio choice under ambiguity and the classical portfolio choice is specific feature of the model setup under consideration and does not hold generically.

3.2 A multiasset model (static case)

In this section we consider the case of multiple assets. We will introduce ambiguity into the market model of Kwon (1985), which allows to derive CAPM in classical case without restrictive assumptions on the utility function.
or asset return distribution. We show that in the presence of ambiguity, the classical version of CAPM should be modified to include the second factor that measures the degree of the ambiguity in returns.

Let us postulate a simple linear one-factor static model of asset returns:

$$\tilde{r}_i = \alpha_i + \beta_i \tilde{f} + \sigma_i \tilde{\varepsilon}_i \quad 0 \leq i \leq n$$

(7)

here $\tilde{r}_i$ is the return on asset $i$, $\tilde{r}_0 \equiv r_0$ - the return on the risk-free asset. Residuals $\tilde{\varepsilon}_i$ are independent and have standard normal distributions. Following Kwon (1985) we assume fair game condition:

(A1) $E(\tilde{\varepsilon}_i | \tilde{f}) = 0$

The return of the market portfolio is equal to the weighted sum of returns on all the traded risky assets with weights being equal to their market shares:

$$\tilde{r}_M = \sum_{i=1}^{n} \omega_i \tilde{r}_i$$

Since the common factor is defined up to an affine transformation, we can assume that $\alpha$s sum to zero and $\beta$s sum to one when weights $\omega_i$ are applied. Consequently, we will have:

$$\tilde{r}_M = \tilde{f} + \sum_{i=1}^{n} \omega_i \sigma_i \tilde{\varepsilon}_i = \tilde{f}$$

if $n$ is sufficiently large.

Equation (7) describes the objective (or historical) probability measure, which is not perfectly known to individuals (see the example in the previous section). They are assumed to know $\alpha$s and $\beta$s of all the traded assets, the probability distribution of the factor but not volatilities $\sigma_i$. The only piece of information they have about volatilities is that they lie inside some bounds, specific to each asset and which are common knowledge:

$$\sigma_i^l \leq \sigma_i \leq \sigma_i^h$$

(8)

Let us say a few words about the rationale for assuming this type of an uncertainty. The true volatility parameter may follow some process, which

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2Note, that asset returns are not necessarily normally distributed since we do not specify the distribution of the market return.

3We implicitly assume that subjective probabilities of unambiguous events coincide with objective ones.
has complicated structure and which is impossible to deduce from the historical asset returns. For example, volatility might be path dependent in a sophisticated way. The fact that volatility is not constant and not known can be consistent with the perfect knowledge of $\alpha$ and $\beta$s. Indeed, an ordinary linear regression of asset returns on the market return will produce unbiased estimates of these parameters nevertheless. If there is a sufficiently long history of data available then we may assume that they are known with good precision.

The $\sigma-$algebra $\Sigma$ of all events is generated by residual returns $\tilde{e}_i$, the common factor $\tilde{f}$ and asset returns $\tilde{r}_i$. If the volatilities were known without ambiguity then this $\sigma-$algebra would trivially be reduced to the one generated by the residuals and the common factor given the deterministic relationship (7).

From assumptions made above, the set of unambiguous events $\Sigma^u$ is naturally defined as the $\sigma$-algebra generated by residuals $\tilde{e}_i$ and the factor $\tilde{f}$. In this particular case the $\lambda-$system of unambiguous events is $\sigma-$algebra. The set of payoffs is represented by the random value of the portfolio of assets held by an individual in the end of the period. An example of an ambiguous event is the event of the type $\{\tilde{r}_i \geq a\}$. Let us denote by $\Pi$ the set of all possible portfolios of assets with non-negative weight of each asset. An element $\pi$ of this set can be characterized by a string of non-negative portfolio weights $\{\theta_i\}_{i=0}^n$ summing to one. Let us define:

$$ F = \{\{\pi \geq a\}; \pi \in \Pi, a \in \mathbb{R}\} $$

the set of all ambiguous events of the type: the portfolio value is greater than some number.

Before we proceed with some mathematics let us note that the common diversification argument will not work in this setup with ambiguity. The traditional intuition suggests that if a portfolio is well diversified then the ambiguity over residual return disappears. As it will be made clear shortly, this intuition is not correct provided that we assume an individual to be "infinitely averse" to ambiguity like in Zhang (2002).

The following Lemma provides a formal basis for the rest of the paper.

**Lemma 1** The inner measure $p_*$ on the set of events $F$ coincides with the historical probability measure under modified system of asset returns:

$$ \tilde{r}_i = \alpha_i + \beta_i \tilde{f} + \sigma_i(\tilde{e}_i)\tilde{e}_i \quad 0 \leq i \leq n \quad (9) $$
where

\[ \sigma_i(\tilde{\varepsilon}) = \sigma_i^l 1_{\{\tilde{\varepsilon}_i > 0\}} + \sigma_i^h 1_{\{\tilde{\varepsilon}_i \leq 0\}}. \]

and \( \tilde{\varepsilon}_i \) as well as \( \tilde{f} \) have objective probability distributions.

**Proof.** Let \( \pi \) be an element of \( \Pi \) characterized by asset weights \( \{\theta_i\}_{i=0}^n \) than since:

\[
\tilde{\pi} = \sum_{i=0}^n \theta_i \tilde{r}_i = \sum_{i=0}^n \theta_i \left( \alpha_i + \beta_i \tilde{f} + \sigma_i(\tilde{\varepsilon}_i) \tilde{\varepsilon}_i \right)
\]

we have:

\[
p_* \{ \tilde{\pi} \geq a \} \geq p \left\{ \left( \alpha_i + \beta_i \tilde{f} + \sigma_i(\tilde{\varepsilon}_i) \tilde{\varepsilon}_i \right) \geq a \right\}
\]

where \( p \) is the objective probability measure and the event

\[
\left\{ \left( \alpha_i + \beta_i \tilde{f} + \sigma_i(\tilde{\varepsilon}_i) \tilde{\varepsilon}_i \right) \geq a \right\}
\]

on the right hand side of (10) is unambiguous. Following the same logic as in the previous section, it is easy to see that this is also the maximum ambiguous even that is contained in \( \{ \tilde{\pi} \geq a \} \). ■

The lemma can be alternatively stated in the following way. The inner measure on the set of events \( F \) coincides with modified objective probability \( Q \). Under \( Q \) the residual returns are independent and distributed as \( \sigma_i(N(0,1)) \cdot N(0,1) \). It is important that the inner measure can be substituted by the same modified measure \( Q \) for all events in \( F \). This unique feature (specific to the model selected) allows to establish the equivalence between the choice under ambiguity and the classical probabilistic choice via transforming the probability as in the example of the previous section.

According to Zhang (2002), the choice of an individual can be represented via maximization of "expected utility" with expectation understood in the general sense of Choquet and inner measure \( p_* \) as the capacity. Utility function \( u(x) \) is defined over the set of outcomes, which is the set of real numbers - values of the portfolio of assets in the end of the period. Let us make three assumptions that will enable us to reduce the problem of choice under ambiguity to the standard problem of choice under probabilistic uncertainty:
(A2) Utility function \( u(x) \) is non-negative and monotone.

(A3) It is convex function: \( u''(x) \leq 0 \).

(A4) There is no short selling of risky assets in equilibrium.

The last assumption does not seem particularly restrictive in the adopted setup. In classical case this immediately follows from two-fund separation property. The second assumption is crucial and ensures that an individual is risk-averse in the transformed problem.

Now let us denote by \( \theta_i \) the weight of \( i \) asset in the portfolio of an individual. Then from (2) and (A2) the "expected utility" of this portfolio is equal to

\[
\int_0^\infty p_* \left\{ \sum_{i=0}^n \theta_i \tilde{r}_i \geq t \right\} dt = \int_0^\infty p_* \left\{ \sum_{i=0}^n \theta_i \tilde{r}_i \geq u^{-1}(t) \right\} dt. \tag{11}
\]

To find the inner measure inside the integral on the right hand side of (11), we use Lemma 1:

\[
p_* \left\{ \sum_{i=0}^n \theta_i \tilde{r}_i \geq u^{-1}(t) \right\} = p \left\{ \sum_{i=0}^n \theta_i (\alpha_i + \beta_i \tilde{f} + \sigma_i (\tilde{e}_i) \tilde{e}_i) \geq u^{-1}(t) \right\}. \tag{12}
\]

Now combing (11) and (12) and integrating by parts, we have:

\[
\int_0^\infty p_* \left\{ \sum_{i=0}^n \theta_i \tilde{r}_i \geq t \right\} dt = \int_0^\infty p \left\{ \sum_{i=0}^n \theta_i (\alpha_i + \beta_i \tilde{f} + \sigma_i (\tilde{e}_i) \tilde{e}_i) \geq u^{-1}(t) \right\} dt \tag{13}
\]

The last integral on the right hand side of (13) is equal to the expectation of the utility function under the modified probability measure defined by the system of asset returns (9).

Hence we proved the following Proposition.

**Proposition 2** Given axioms of choice stated in Zhang (2002) and assumptions A1- A4, the problem of choice of an individual under model (7) with
ambiguity is equivalent to that of under the standard probabilistic model with transformed probability measure $Q$ defined by the system (9).

**Remark** The ambiguity is not diversified away.

Indeed, let us compute the distribution of the market portfolio return under the measure $Q$. We have:

$$
\tilde{r}_M = \sum_{i=0}^{n} \omega_i \tilde{r}_i = \sum_{i=0}^{n} \beta_i \omega_i \tilde{f} + \sum_{i=0}^{n} \omega_i \sigma_i (\tilde{e}_i) \tilde{e}_i
$$

(14)

$$
= - \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{n} \omega_i \Delta_i + \tilde{f},
$$

where $\Delta_i = \sigma^h_i - \sigma^l_i$ and the last equality holds in the limit when $n$ is large due to the Law of large numbers and the fact that:

$$
E[\sigma(\tilde{e})\tilde{e}] = \frac{\sigma^h}{\sqrt{2\pi}} \int_{-\infty}^{0} se^{-s^2/2} ds - \frac{\sigma^l}{\sqrt{2\pi}} \int_{0}^{\infty} se^{-s^2/2} ds
$$

$$
= - \frac{\Delta_i}{\sqrt{2\pi}} \int_{0}^{\infty} se^{-s^2/2} ds = - \frac{\Delta_i}{\sqrt{2\pi}}
$$

As we see, the market portfolio return is not distributed exactly as the factor under the transformed measure as it was under the historical measure. This result seems indeed very puzzling. Although whatever are the actual volatilities of assets' returns the common factor always coincides with the market return (in the limit), the investors demand a negative premium for the market return being equal to the (weighted) average of the premia on individual asset returns. Mathematically we can state that the measure $Q$ is not equivalent to $P$ asymptotically as $n$ goes to infinity since, asymptotically, zero $P$–measure event $\{\tilde{r}_M < \tilde{f}\}$ is a non-zero $Q$–measure event.

Here we can draw a parallel with the famous Ellsberg paradox discussed in the introduction. Recall that "The Sure Thing principle" naturally holds whatever the distribution of white and yellow balls is, which, however, does not imply that it holds if the distribution is not known. In our case, although the investors perfectly know that whatever is the historical probability, the market portfolio return is equal to the common factor value, under
the decision measure \( Q \) they treat it differently. This observation shows that ambiguity is different from unknown probability.

Using (14), under \( Q \) (9) can be equivalently written as:

\[
\tilde{r}_i = \left[ \alpha_i + \frac{1}{\sqrt{2\pi}} \beta_i \sum_{j=0}^{n} \omega_j \Delta_j - \frac{1}{\sqrt{2\pi}} \Delta_k \right] + \beta_i \tilde{r}_M + \xi_i \quad 0 \leq i \leq n, \quad (15)
\]

where

\[
E^Q(\xi_i | \tilde{f}) = 0.
\]

### 3.3 CAPM in the static setting

Kwon (1985) showed that under model (7) with "fair game" condition A1 and risk averse investors CAPM must hold in equilibrium.

In the previous section we showed how to transform the problem of choice under ambiguity to the standard problem of maximizing expected utility under transformed probability measure \( Q \). Using the result of Kwon (1985) and the representation (15), we conclude that CAPM holds under \( Q \):

\[
E^Q(\tilde{r}_i) - r_0 = \beta_i (E^Q(\tilde{r}_M) - r_0). \quad (16)
\]

Recall, that by assumption the market portfolio return has the same distribution under \( Q \) and \( P \). Using the result stated in Proposition 2, we obtain an expression for the expected return on asset \( i \) under \( Q \):

\[
E^Q(\tilde{r}_i) = \alpha_i + \beta_i E(\tilde{f}) + E[\sigma_i(\tilde{f}) \epsilon_i] \\
= \alpha_i + \beta_i E(\tilde{f}) - \frac{1}{\sqrt{2\pi}} \Delta_i \\
= E^P(\tilde{r}_i) - \frac{1}{\sqrt{2\pi}} \Delta_i. \quad (17)
\]

From (14) we obtain:

\[
E^Q(\tilde{r}_M) = -\frac{1}{\sqrt{2\pi}} \sum_{i=0}^{n} \omega_i \Delta_i + E^P(\tilde{r}_M) \quad (18)
\]

Now putting together (17), (18) and (16) we obtain:

\[
E^P(\tilde{r}_i) - r_0 = \beta_i \left( E^P(\tilde{r}_M) - r_0 - \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \omega_j \Delta_j \right) + \frac{\Delta_i}{\sqrt{2\pi}}
\]

Hence we proved the following Proposition:
Proposition 3 Assume that
1) asset returns are described by model (7),
2) volatilities of residual returns are known to satisfy (8),
3) investors’ preferences comply 9 axioms stated in Zhang (2002),
4) assumptions A1-A4,
then generalized version of CAPM holds:
\[
E(\tilde{r}_i) - r_0 = \beta_i \left( E(\tilde{r}_M) - r_0 - \sum_{j=0}^{n} \omega_j \gamma_j \right) + \gamma_i \quad 0 \leq i \leq n,
\]
where \( \gamma_i = \frac{\Delta_i}{\sqrt{2\pi}} \)

Proposition 2 suggests that the systematic risk is not the only factor that determines risk premium in the equilibrium. Equality (19) states that the risk premium is a linear function of the asset beta, which measures its systematic risk, and the asset gamma, which represents a measure of the degree of ambiguity of this asset’s return. The immediate implication of this result is that standard CAPM may not hold in reality due to the uncertainty about the risk of assets (here represented by volatilities of residual returns). For example, the empirical risk factors found to be significant in Fama-Macbeth type regressions might reflect missing "ambiguity factor".

It is interesting to see what happens if we take the approach of Epstein in modeling the choice under ambiguity (see, for example, Epstein and Chen (2002)). According to Epstein, investors evaluate attractiveness of a portfolio of assets by the minimum expected utility attainable over all possible probability measures. In the context of the market model that we assumed, the set of possible probability measures consists of all possible specifications (7), satisfying condition (8). It is obvious that minimum expected utility is always attained under probability measure defined by:
\[
\tilde{r}_i = \alpha_i + \beta_i \tilde{r}_M + \sigma_i^k \tilde{e}_i \quad 0 \leq i \leq n
\]
that is the one where volatilities of residual returns take the highest possible values.

Since residual returns have zero expectations under the prior corresponding to the minimum utility, we conclude that Epstein approach yields the classical version of CAPM. This result could be expected. Under Epstein approach investors resolve the ambiguity by choosing the worst probabilistic
scenario and than act as classical utility maximizers. To put it in other words Epstein restricts the pessimism of investors by requesting that the resulting beliefs form a probability measure. In our case the subjective believes about likelihood of different events might be inconsistent \((p^*(A) \neq 1 - p^*(A^c))\) and, consequently, the decision making cannot generically be reduced to the classical case. Only under the very specific setup considered in this paper we are able to establish this equivalence.

Here we can see an interesting similarity between \(Q\) and risk-neutral distributions that also typically have fatter left tails. To draw a parallel, the probability measure \(Q\) can be viewed as an "ambiguous-free" measure. There are, however, two important differences. First, the ambiguous-free measure is uniquely determined. Second, it is not (asymptotically) equivalent to the historical measure. These features are consequences of the assumption that investors are "infinitely" averse to ambiguity.

4 Empirical testing

In this section we will discuss the implications of the two factor CAPM and provide some empirical tests in the spirit of Fama and Macbeth (1973).

4.1 Implications of the ambiguity

In this section we discuss the empirical implications of the presence of the ambiguity in asset returns. The main implication of the model considered in the previous sections is that CAPM should be modified to include an ambiguity factor along with the systematic risk to account for the risk premium. The ambiguity factor is shown to depend on the width of the volatility band. In reality, volatility bounds can not be observed and some proxy measures should be used instead.

The traditional way of testing CAPM is running Fama-Macbeth regressions (see Fama and Macbeth (1973)), which are cross sectional regressions of realized asset returns on risk factors including asset betas:

\[
r_{it} = \lambda_{0t} + \lambda_{1t}\beta_i + \sum_{j=2}^{J} \lambda_{jt}\eta_{ijt} + \epsilon_{it},
\]

where \(\eta_{ijt}\) denote a time series of realizations of other \(J - 1\) risk factors for
The classical version of CAPM implies the following hypotheses:

\[ E(\hat{\lambda}_0) = r_0, \]
\[ E(\hat{\lambda}_1) = E(r_{Mt}) - r_0 > 0, \]
and
\[ E(\hat{\lambda}_j) = 0 \quad \text{for} \quad j \geq 2. \]

Here \( \hat{\lambda} \) denotes the regression estimates.

Let us now use equations (19) and (18) to see how the presence of the ambiguity changes these hypotheses. As it has been noted before, asset betas can be estimated by running ordinary regressions of realized asset returns on the market return the way it is done traditionally. Denoting \( \frac{1}{\sqrt{2\pi}} \Delta_i \) by \( \eta_{2i} \) we have the following new hypotheses implied by the CAPM under ambiguity:

\[ E(\hat{\lambda}_0) = r_0, \]
\[ E(\hat{\lambda}_1) = E(r_{Mt}) - r_0 - \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \omega_{jt} \Delta_j \]
\[ = E^Q(r_{Mt}) - r_0 > 0, \]
\[ E(\hat{\lambda}_2) = 1 \]
and
\[ E(\hat{\lambda}_j) = 0 \quad \text{for} \quad j \geq 3. \]

The first hypothesis is unchanged. The relationship between assets’ betas and expected returns is still positive but the coefficient is now equal to the market risk premium under the transformed (not historical) measure. The major difference is that another factor starts to matter in the cross sectional variation of expected returns. This factor is proportionate to the width of the return volatility band of an asset return.

Based on the CAPM under ambiguity we can make two predictions concerning the results of Fama-Macbeth regressions. First, omitting the second factor \( \eta_{2i} \) will lead to an upward bias in estimates of regression intercepts, hence, the average \( \bar{\lambda}_0 = \frac{1}{T} \sum_t \hat{\lambda}_{0t} \) will be (statistically) significantly larger then the average risk free rate \( r_0 \):

\[ \bar{\lambda}_0 = r_0 + \frac{1}{n} \sum_{i=1}^{n} \eta_{2i} = r_0 + \frac{1}{\sqrt{2\pi}} \frac{1}{n} \sum_{i=1}^{n} \Delta_i. \]
Second, the average of estimates \( \bar{\lambda}_1 \) will biased downwards relative to average market risk premium since the market premia under the transformed measure is less than that under the historical measure:

\[
\bar{\lambda}_1 = r_M - r_0 - \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \omega_{jt} \Delta_j
\]

An intercept estimate \( \bar{\lambda}_0 \) can be interpreted as zero-\( \beta \) portfolio return for month \( t \) so general version of CAPM (Black (1972)) cannot be rejected from the observation of these biases. One can test the zero-beta CAPM using its prediction that:

\[
\bar{\lambda}_0 + \bar{\lambda}_1 = r_M.
\]

The version of CAPM with ambiguity predicts that:

\[
\begin{align*}
\bar{\lambda}_0 + \bar{\lambda}_1 &= \bar{r}_0 + \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n} \Delta_i + r_M - r_0 - \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \omega_{jt} \Delta_j \\
&= \bar{r}_M + \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{n} \sum_{i=1}^{n} \Delta_i - \sum_{j=0}^{n} \omega_j \Delta_j \right].
\end{align*}
\]

We expect that the arithmetic average of volatility band width across stocks is greater than their weighted average since higher weights are placed on larger stocks (with presumably less ambiguity). Hence the sum of two averages should be greater than the average market return.

Table 1 presents some results given in table 4 of Fama and Macbeth (1973) using our notation. As we observe the biases in \( \bar{\lambda}_0 \) and \( \bar{\lambda}_1 \) are clearly present (an exception is the last row). Although \( \bar{\lambda}_0 + \bar{\lambda}_1 \) is greater than the average market return for the whole period as well as most of subperiods, the difference is insignificant taking into account that the standard deviation estimates of coefficient averages is of order of several percentage points. The last observation suggests that either ambiguity is unrelated to the capitalization of a stock or it does not vary much across stocks. More information can be collected by direct testing of CAPM with the ambiguity factor.

### 4.2 Proxy for the volatility bandwidth

Direct testing of the CAPM under ambiguity involves running Fama-Macbeth regressions with \( \eta_{2i} \) included. There are two problems associated with such
a test: one is conceptual and the other is pure technical. From the model we expect the average estimate \( \hat{\lambda}_{2t} \) to be significantly different from zero and, ideally, not significantly different from one. This might be considered as a null hypothesis for testing CAPM under ambiguity. This test, in fact, may be misleading since non-significant average value of \( \hat{\lambda}_{2t} \) implies that either the CAPM under ambiguity is rejected or \( \eta_{2t} \) does not significantly vary across assets. Apparently, one has also to look at the estimates of the intercept.

The technical problem originates from the fact that volatility is unobserved and the volatility bounds should be inferred indirectly from asset returns. From the econometric point of view this is a rather demanding and almost impossible task. In practice, we can use some measures of the degree of the spread of the underlying distribution of the residual volatility as proxies for the volatility bandwidth. Alternatively, we may assume a class of parametric distribution for the volatility and find estimates of these bounds implicit in asset return data. We prefer the first approach and propose using the estimate of the standard deviation of the volatility as a proxy. This choice makes perfect sense if, for example, the volatilities of residual returns of different assets have the same distribution up to a scale factor.

The standard deviation of the residual return volatility can be estimated in the following way. The first step is to construct a series of residual returns on the selected asset. This can be accomplished by running the ordinary regression of realized asset returns on the realized market return and finding the residuals \( \tilde{\xi}_t \) (from \( t \) to \( t+1 \)). These residuals are estimates of model residual returns \( \tilde{\xi}_t = \tilde{\sigma}_t \tilde{\epsilon}_t \), which, are conditionally normal with volatility \( \tilde{\sigma}_t \) having unknown distribution with sure bounds \( \sigma^d \) and \( \sigma^h \). We can easily establish the relationship between unconditional moments of \( |\tilde{\xi}_t| \) and unconditional moments of \( \tilde{\sigma}_t \).

Indeed, \( |\tilde{\xi}_t| = \tilde{\sigma}_t |\tilde{\epsilon}_t| \) with \( |\tilde{\epsilon}_t| \) being equal to an absolute value of normally distributed random variable conditionally (hence unconditionally also) independent of \( \tilde{\sigma}_t \). Assuming that the distributions of \( \tilde{\sigma}_t \) and \( \tilde{\epsilon}_t \) are independent of time we have the following relationships:

\[
E \left[ |\tilde{\xi}| \right] = E \left[ |\tilde{\sigma}| \right] E \left[ |\tilde{\epsilon}| \right] = \sqrt{2\pi} E \left[ \tilde{\sigma} \right],
\]

\[
E \left[ \tilde{\xi}^2 \right] = E \left[ \tilde{\sigma}^2 \right] E \left[ \tilde{\epsilon}^2 \right] = E \left[ \tilde{\sigma}^2 \right]. \tag{22}
\]

If we are able to accurately measure moments of \( |\tilde{\xi}| \) then obtaining mo-
ments of $\tilde{\sigma}$ is straightforward from (22):

$$E[\tilde{\sigma}] = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} |\tilde{\xi}_t|,$$

$$E[\tilde{\sigma}^2] = \frac{1}{T} \sum_{t=1}^{T} \tilde{\xi}_t^2.$$

Consider the following estimate of the variance of $\tilde{\sigma}$:

$$V(\tilde{\sigma}) = E[\tilde{\sigma}^2] - \left(E[\tilde{\sigma}]\right)^2,$$  \hspace{1cm} (23)

In fact, this is a biased estimate of the variance. Taking, however, into account large number of observations ($T = 372$) we may ignore the bias as being intelligibly small. The standard deviation of the volatility can now be computed by taking the square root of the estimate of its variance.

### 4.3 Testing CAPM under ambiguity for US stocks

In this section we will empirically test the two factor CAPM developed above. As a database we will use a series of 48 industry portfolios constructed by Fama and French and available in the Internet site http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. We select monthly observations of the returns of these portfolios for the period of 30 years 1973 - 2003. The benchmark market return minus risk free return is also available for the same period. By considering portfolios instead of single assets we admit a certain degree of inconsistency. We may rationalize this by assuming that the residual returns of assets within one industry have the same volatility bounds.

Following the steps proposed in the previous section we estimated residual returns of these industry portfolios and calculated the estimates of their volatility standard deviations\(^4\). Table 2 presents the summary statistics for this time series. The average value of the volatility standard deviation appear to be equal to 1.5% and its average amplitude of the variation is 0.55%. The maximum and minimum values of the proxy are 2.6% and 0.15%. The minimum value appears to be much less then the mean minus two standard

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\(^4\)We ran regressions $r_{it} = \alpha_i + \beta_i(r_M - r_f) + \xi_{it}$ implicitly assuming that the excess market return is the common factor.
deviations. The next minimum value is already 0.68% so we can consider 0.15% as an outlier. In the empirical tests below we will consider full and the reduced sample of portfolios (excluding one industry) when testing CAPM so that we can see the impact of this outlier on the results. The average volatility of monthly returns is equal to 4.4% suggesting that the volatility of monthly returns typically vary from 2.9% to 5.9% or from 10% to 20% on annual basis.

The appropriate methodology for testing CAPM involves running cross sectional Fama-Macbeth regression of realized (here monthly) returns on assets’ beta and the proxy for the volatility bandwidth:

\[ r_{it} = \lambda_0 + \lambda_1 \beta_i + \lambda_2 \eta_i + \epsilon_{it}, \]  

(24)

where \( \eta_i = \sqrt{\hat{\sigma}_i} \) is the estimate of the standard deviation of asset \( i \) volatility.

In the table 3 we provide the results of the test of CAPM using Fama-Macbeth regressions (24). We observe that the cross sectional effect of the ambiguity factor is statistically significant for both samples. In the reduced sample the significance is noticeably higher. The effect of market beta appears to be insignificant in both cases.

The average contribution of the ambiguity factor to the cross section variability of expected returns can be measured by 1.5% \cdot 0.17 = 0.25% that is one standard deviation product the coefficient estimate. This might seem to be not very significant economically given that the average standard deviation of realized returns across assets is estimated to be 4.18%. However, one should note that the latter figure must significantly overestimate the variability of expected returns. For the purpose of comparison, the cross sectional standard deviation of average realized returns is only 0.22%.

5 Concluding remarks

In this paper we applied the model of individual choice under ambiguity proposed by Zhang (2002) in the context of the market model of asset returns of Kwon (1985). The ambiguity is introduced via unknown volatilities of assets’ residual leading to two factor CAPM. We test this model on US stock data and find that the ambiguity factor has a significant impact on the

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5Monthly returns are expressed in percentage.
cross section of asset expected returns. This effect seems to be economically meaningful.

The model setup considered in this paper is very convenient in that it permits reducing the problem of the choice under ambiguity to the classical utility maximization via a non-equivalent change of probability measure. One of the area of future research is to extend this setting to multiple periods, and study the effect of the ambiguity on dynamic portfolio allocation and consumption.

References


