From rags to riches:
On constant proportions investment strategies

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Abstract  
In this paper we study the performance of self-financing constant proportions trading strategies, i.e. dynamic asset allocation strategies that keep a fixed constant proportion of wealth invested in each asset in all periods in time, in a stationary financial market. We prove that any mixed constant proportions trading strategy yields a strictly positive exponential rate of growth of investors' wealth if prices are non-degenerated. This result can be regarded as being counterintuitive because any such strategy yields no increase of wealth under constant prices. We further show that the result also holds under small transaction costs, which is important for the viability of this approach because constant proportions strategies require frequent rebalancing of the portfolio.  

1 Introduction  
The problem of optimal investment in financial markets is central to any theory of portfolio selection. While investors’ objectives can be manifold, it is often useful to focus on certain optimality criteria as benchmarks. The theory of optimum-growth portfolio, or log-optimum investment, studies investment strategies that maximize the logarithmic growth rate of investors’ wealth, see e.g. Algoet and Cover (1988) and the survey by Hakansson and Ziemba (1995). When a strictly positive rate of growth can be achieved, wealth asymptotically

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becomes unbounded and overtakes any other investment strategy\textsuperscript{1}. The log-optimum investment strategy—commonly referred to as Kelly-rule (Kelly 1956) in the case of independent and identically distributed returns on investment—has proved quite successful in actual financial markets, Thorp (1971).

One might be tempted to consider investment strategies that yield exponential growth of wealth as being exceptional and in general difficult to find. However, as we show in this paper, \textit{any} constant proportions trading strategy yields unbounded and exponentially fast growth of wealth in a stationary financial market, provided prices are non-degenerated and the investor trades in at least two stocks. Constant proportions strategies require the investment of a fixed constant proportion of wealth in each asset in all periods in time. These trading strategies are self-financing and only call for a non-zero initial investment; hence investors following this rule go “from rags to riches.” Stationarity of the financial market rules out any systematic gain from investments e.g. through increasing prices. We regard this result as being counterintuitive because any constant proportions strategy yields no increase of wealth under constant prices. However, any persistent stationary variation of prices (not being identical over assets) yields unbounded growth of wealth under every mixed constant proportions strategy.

Constant proportions strategies have been studied—inspired by the optimality properties of the Kelly-rule—in many different frameworks, see e.g. Aurell, Baviera, Hammarlid, Serva, and Vulpiani (2000), Browne (1998), and Browne and Whitt (1996).

An issue of transaction costs is often ignored in the literature on optimal investment. In our approach, transaction costs disclose the major drawback of constant proportions strategies—the frequent rebalancing of the portfolio. We take this criticism into account by showing that our result also holds when transactions are costly—provided costs are sufficiently small.

The mathematical technique used in this paper adopts the notion of a balanced investment strategy, a concept first employed in the context of the von Neumann growth model by Radner (1971). This approach is relatively new to the financial market literature, see Evstigneev and Taksar (2000).

Due to the partial equilibrium character of our analysis, the impression of a money machine might arise and, moreover, it might be conjectured that constant proportions strategies are uninteresting when closing the model and dealing with a general equilibrium. This perspective has recently been explored from an evolutionary point of view in a different strand of literature, Blume and Easley (1992), Hens and Schenk-Hoppé (2001), Evstigneev, Hens, and Schenk-Hoppé (2001), and Amir, Evstigneev, Hens, and Schenk-Hoppé (2001).

The next section explains the model without transaction costs. Section 3 presents the main result on unbounded growth of wealth. The model with transaction costs is introduced and analyzed in Section 4.

\textsuperscript{1}Samuelson (1979) pointed out that from a utility-maximization perspective the log-optimum investment approach is only useful for investors with particular preferences.
2 Model

Let an investor observe prices and take actions in discrete periods in time, labelled \( t = 0, 1, 2, \ldots \). Uncertainty is modelled by states of nature that are determined by a stochastic process \( s_t, t = 0, \pm 1, \pm 2, \ldots \).

Consider a financial market with \( K \geq 2 \) assets whose prices \( p_t > 0, t = 0, 1, 2, \ldots \), form a sequence of strictly positive random vectors with values in the space \( R^K \). We assume that \( p_t \) depends on the history of the process \( s_t \) up to time \( t \), i.e.

\[
p_t = p_t(s^t), \quad s^t = (s_{t-1}, s_t).
\]

(All functions of \( s^t \) considered in the following are supposed to be measurable).

At each period in time \( t \) an investor chooses a portfolio \( h_t(s^t) = (h_1^t(s^t), \ldots, h_K^t(s^t)) \geq 0 \), where \( h_i^t \) is the number of units of asset \( i \) in the portfolio \( h_t \).

The assumption of non-negativity of \( h_t \) rules out short sales of the assets in our model. A sequence \( h_t(s^t) \), specifying a portfolio in each period in time \( t \) and in every random situation \( s^t \), is called a trading strategy.

We first restrict ourselves to the case of no transaction costs and derive our main result in this framework. We detail the necessary changes when transactions are costly in the section that presents the corresponding result with transaction costs.

Given \( w_0 > 0 \), we will say that \( h_t \) is a trading strategy with initial endowment \( w_0 \) if \( p_0(s^0) h_0 \leq w_0 \). A trading strategy is termed self-financing if

\[
p_t(s^t) h_t(s^t) \leq p_t(s^t) h_{t-1}(s^{t-1}), \quad t = 1, 2, \ldots \quad (a.s.). \tag{1}
\]

(Almost surely (a.s.) refers to the probability space underlying the stochastic process \( s^t \)) Equation (1) states that the budget constraint, imposing restrictions on the choice of the investor’s portfolio at every period in time, is determined by the value of the previous periods’ portfolio at the current prices.

Let us say that the market is stationary if the process \( s_t \) is ergodic (in particular, stationary) and the price vectors \( p_t \) do not explicitly depend on \( t \), i.e. \( p_t = p(s^t) \). When analyzing such markets, it is of interest to consider trading strategies of balanced growth (or, briefly, balanced strategies). These strategies are of the form

\[
h_t(s^t) = \gamma(s^t) \ldots \gamma(s^1) \tilde{h}(s^t), \quad t = 1, 2, \ldots, \tag{2}
\]

with \( \gamma(\cdot) > 0 \) being a scalar-valued function and \( \tilde{h}(\cdot) \) being a non-negative vector function such that \( \ln \gamma(s^t) \) and \( \ln |\tilde{h}(s^t)| \) are integrable. (For a vector \( h = (h^i) \), we write \( |h| = \sum_i |h^i| \).) In the case \( t = 0 \), we assume in (2) that \( h_0(s^0) = \tilde{h}(s^0) \).

The term “balanced” is justified because (2) implies that all ratios

\[
\frac{h_i^t(s^t)}{h_j^t(s^t)} = \frac{\tilde{h}_i(s^t)}{\tilde{h}_j(s^t)}, \quad i \neq j, \tag{3}
\]

describing the proportions between the amounts of different assets in the portfolio, form stationary stochastic processes. Furthermore, the random growth
rate of the amount of each asset $i = 1, ..., K$, in the portfolio

$$\frac{h_i^t(s^t)}{h_i^{t-1}(s^{t-1})} = \gamma(s^t) \frac{\tilde{h}^t(s^t)}{\tilde{h}^{t-1}(s^{t-1})}$$

(4)

is a stationary process. Clearly, for a balanced strategy the self-financing condition (1) is equivalent to

$$\gamma(s^t) p(s^t) \tilde{h}(s^t) \leq p(s^t) \tilde{h}(s^{t-1}), \quad (a.s.).$$

(5)

If (5) holds for some $t$ then it holds automatically for all $t$. Finally, provided $\ln p(s^t)$ is integrable, we can associate a balanced strategy (2) to any non-negative vector function $\tilde{h}(s^t)$ with $\ln |\tilde{h}(s^t)|$ being integrable by defining

$$\gamma(s^t) := \frac{p(s^t) h(s^{t-1})}{p(s^t) \tilde{h}(s^t)}.$$ 

(6)

For this strategy relations (1) and (5) hold as equalities.

Our analysis will be based on the following result. This proposition ensures that the growth rate of wealth of any investor employing a balanced trading strategy is completely determined by the expected value of $\gamma$. We further show that $E \ln \gamma(s^0) \equiv E \ln \gamma(s^t) > 0$ implies exponential growth of wealth, i.e. $p t h_t \to \infty$ a.s. exponentially fast.

**Proposition 1** For any balanced trading strategy (2), we have

$$\lim_{t \to \infty} \frac{1}{t} \ln(p_t h_t) = \lim_{t \to \infty} \frac{1}{t} \ln |h_t| = E \ln \gamma(s^0) \quad (a.s.).$$

(7)

**Proof.** We can write

$$\frac{1}{t} \ln |h_t| = \frac{1}{t} \sum_{m=1}^{t} \ln \gamma(s^m) + \frac{1}{t} \ln |\tilde{h}(s^t)|,$$

and so the second equality in (7) is an immediate consequence of the Birkhoff ergodic theorem because the term on the far right is asymptotically zero by integrability of $\ln |\tilde{h}(s^t)|$. The first equality in (7) follows from the relation $|\ln(p_t h_t) - \ln |h_t|| \leq |\ln \kappa|$ that is implied by (9).

The question which appears naturally when dealing with the above model is whether, in the present—stationary—context, there exist balanced strategies exhibiting possibilities for *unbounded growth*. Consider, for the moment, the deterministic case, where $S$ consists of a single point. Then the self-financing condition (1) reduces to

$$p h_t \leq p h_{t-1},$$

(8)

with some constant price vector $p > 0$. In this case, any balanced strategy is given by $h_t = \gamma \tilde{h}$, where $\gamma > 0$ is a constant scalar and $\tilde{h}$ is a constant strictly positive vector. We can immediately see from (8) that the maximum possible value for $\gamma$ is $1$, which rules out any possibility of a non-zero growth.
The above deterministic argument totally agrees with our intuition, and it would be natural to expect that it could be extended to the general, stochastic case. However, this intuition fails, and it turns out that, in a stochastic world, one can usually design a variety of balanced strategies exhibiting almost surely unbounded, and even exponential, growth. Moreover, the exponential growth is a typical phenomenon, which can be established for all so-called constant proportions trading strategies. Since the prices of the assets form stationary processes, no dividends are paid, and the strategies we deal with are purely self-financing, this result may look, at the first glance, counterintuitive.

3 Main Result

We show that any constant proportions trading strategy, i.e. any dynamic asset allocation strategy that keeps a fixed constant proportion of wealth invested in each asset in all periods in time, yields a strictly positive exponential growth of wealth in a stationary financial market.

We make the assumption that the price of every asset satisfies
\[ \kappa \leq p^k(s^t) \leq \kappa^{-1}, \]
where \( \kappa > 0 \) is a non-random constant\(^2\).

We further impose the following mild condition of non-degeneracy of the price process \( p(s^t) \).

**Assumption 2** With positive probability the variable \( \frac{p^k(s^t)}{p^k(s^{t-1})} \) is not constant with respect to \( k = 1, 2, \ldots, K \), i.e., there exist \( m \) and \( n \) (that might depend on \( s^t \)) for which
\[
\frac{p^m(s^t)}{p^m(s^{t-1})} \neq \frac{p^n(s^t)}{p^n(s^{t-1})}.
\]

A trading strategy \( h_t \) is called a constant proportions strategy if there exists a (non-random) vector \( \lambda = (\lambda_1, \ldots, \lambda_K) \in \text{int} \Delta^K \), i.e., \( \lambda_k > 0 \) for all \( k \) and \( \sum_k \lambda_k = 1 \), such that
\[
p_t^k h_t^k = \lambda_k p_t h_{t-1},
\]
for all \( t = 1, 2, \ldots \) and all \( k = 1, \ldots, K \).

In every period in time, the investor forms her portfolio by investing the constant share \( \lambda_k \) of her wealth \( p_t h_{t-1} \) into the \( k \)th asset. The wealth at the beginning of period \( t \) is determined by evaluating the portfolio \( h_{t-1} \) from the previous period at the current prices \( p_t^k \). Note that a constant proportions strategy requires the investor to rebalances her portfolio at the beginning of each period according to the proportions \( \lambda \).

\(^2\)In fact it suffices to assume that all prices \( p^k(s^t) \) are tempered random variables, i.e.
\[-\infty < \lim_{t \to \infty} (\ln p^k(s^t))/t < \infty \text{ a.s.} \]
This condition rules out that asset prices tend to zero or infinity exponentially fast.
Theorem 3  Given any \( \lambda = (\lambda_1, \ldots, \lambda_K) \in \text{int} \Delta^K \) and any initial wealth \( w_0 > 0 \). Then there exists an associated constant proportions strategy \( h_t \) that is self-financing and balanced, i.e. \( h_t \) is of the form (2) with some \( \gamma(\cdot) \) and \( \tilde{h}(\cdot) \) and fulfills (1) and (10). This strategy yields unbounded and exponentially fast growth of investors’ wealth, i.e. we have

\[
\lim_{t \to \infty} \frac{1}{t} \ln p_t h_t = \lim_{t \to \infty} \frac{1}{t} \ln |h_t| = E \ln \gamma(s^0) > 0 \quad (a.s.).
\] (11)

Proof. Without loss of generality assume that \( w_0 = 1 \). For any strictly positive initial wealth, the portfolio held in each period is simply a multiple (given by the ratio of actual initial wealth and \( w_0 \)) of the portfolio defined below. The asymptotic growth rate of investor’s wealth is independent of the initial endowment.

Define

\[
\tilde{h}(s^t) = \left( \frac{\lambda_1}{p^1(s^t)}, \ldots, \frac{\lambda_K}{p^K(s^t)} \right)
\] (12)

and let

\[
\gamma(s^t) = p(s^t) \tilde{h}(s^{t-1}) \left[ = \sum_{k=1}^K \frac{\lambda_k}{p^k(s^t)} \right],
\] (13)

cf. equation (6).

Consider the balanced strategy \( h_t \) with \( h_0 = \tilde{h}(s^0) \) described by formula (2) with the functions \( h \) and \( \gamma \) just defined. We have \( p(s^0) h_0(s^t) = p(s^0) \tilde{h}(s^0) = \sum_k \lambda_k = 1 = w_0 \), and, for each \( t = 0, 1, \ldots, \)

\[
p^k(s^{t+1}) h^k_{t+1} = p^k(s^{t+1}) \gamma(s^t) \ldots \gamma(s^{t+1}) \frac{\lambda_k}{p^k(s^{t+1})} = \gamma(s^t) \ldots \gamma(s^t) \gamma(s^{t+1}) \lambda_k = \gamma(s^t) \ldots \gamma(s^t) p(s^{t+1}) \tilde{h}(s^t) \lambda_k = \lambda_k p(s^{t+1}) h_t(s^t).
\]

Thus \( h_t \) is a constant proportions strategy with respect to the prescribed proportions \( \lambda = (\lambda_1, \ldots, \lambda_K) \).

In view of Proposition 1, it remains to establish strictly positivity of the growth rate, i.e. \( E \ln \gamma(s^t) > 0 \). To this end we observe that, by virtue of Jensen’s inequality (\( \lambda \) is a probability measure) and Assumption 2,

\[
\ln \sum_{k=1}^K \lambda_k \frac{p^k(s^t)}{p^k(s^{t-1})} > \sum_{k=1}^K \lambda_k \ln \frac{p^k(s^t)}{p^k(s^{t-1})}
\] (14)

with positive probability. The number \( E \ln \gamma \) is equal to the expected value of the expression on the left-hand side of (14). Consequently,

\[
E \ln \gamma(s^t) > \sum_{k=1}^K \lambda_k E \ln \frac{p^k(s^t)}{p^k(s^{t-1})} = 0.
\]
The equality on the far right holds because $E[\ln p^k(s^t)]$ is finite by condition (9). (This condition also ensures integrability of $\ln \gamma(s^t)$ and $\ln |\tilde{h}(s^t)|$.) \hfill □

Let us explain the intuition behind this result. Any constant proportions strategy ‘exploits’ the persistent fluctuation of prices in the following way. Keeping a fixed fraction of wealth invested in each asset implies that after a change in prices an investor sells those assets that are expensive relative to the other assets and purchases the relatively cheap assets. The stationarity of prices implies that this portfolio rule yields a strictly positive expected rate of growth, despite the fact that each asset price has growth rate zero. Hence, investors go “from rags to riches.”

4 Transaction Costs

This section extends the result of the previous section to markets with transaction costs. Transaction costs present the main obstacle in getting “from rags to riches” when using constant proportions strategies. This is due to the fact that their main disadvantage—the need to frequently rebalance the portfolio—becomes apparent when transactions are costly. However, since investors’ wealth grows exponentially fast in the absence of transaction costs, Theorem 3, there should be room for small losses in every period (through transaction costs or taxes) without eliminating the possibility of unbounded growth of wealth.

When transaction costs are present in the financial market, the self-financing condition (1) becomes

$$p_t(s^t) h_t(s^t) + \sum_{k=1}^{K} \delta_k p^k_t(s^t) |h^k_t(s^t) - h^k_{t-1}(s^{t-1})| \leq p_t(s^t) h_{t-1}(s^{t-1}), \tag{15}$$

for all $t = 1, 2, \ldots$ a.s. In (15) transaction costs for each asset is a fixed fraction $\delta_k$ of the order volume. For simplicity of presentation only we assume identical percentages for buying and selling.

We generalize the previous definition (10) by calling a trading strategy $h_t$ a constant proportions strategy if there exists a (non-random) vector $\lambda = (\lambda_1, \ldots, \lambda_K) \in \text{int} \Delta^K$ such that

$$p^k_t \ h^k_t + \lambda_k \left[ \sum_{n=1}^{K} \delta_n p^n_t(s^t) |h^n_t(s^t) - h^n_{t-1}(s^{t-1})| \right] = \lambda_k p_t h_{t-1}, \tag{16}$$

for all $t = 1, 2, \ldots$ and all $k = 1, 2, \ldots, K$. According to this definition, the investment (evaluated at the market price $p_t$) in asset $k$ in every period in time is equal to the fraction $\lambda_k$ of the beginning-of-period wealth $p_t h_{t-1}$ less the total transaction costs. If there are no transaction costs, i.e. $\delta_n = 0$ for all $n = 1, \ldots, K$, (16) coincides with (10). Any strategy satisfying (16) is self-financing. (Simply take the sum with respect to $k = 1, \ldots, K$ on both sides of (16) to obtain (15) with equality.)

We have the following result.
Theorem 4 Given any \( \lambda = (\lambda_1, \ldots, \lambda_K) \in \text{int} \Delta^K \) and any initial wealth \( w_0 > 0 \).
For any sufficiently small positive \( \delta = (\delta_1, \ldots, \delta_K) \) there exists an associated constant proportions strategy \( h_t \) that is self-financing and balanced, i.e. \( h_t \) is of the form (2) with some \( \gamma(\cdot) = \gamma_0(\cdot) \) and \( \hat{h}(\cdot) \) and fulfills (15) and (16). This strategy yields unbounded and exponentially fast growth of investors' wealth, i.e. we have
\[
\lim_{t \to \infty} \frac{1}{t} \ln p_t h_t = \lim_{t \to \infty} \frac{1}{t} \ln |h_t| = E \ln \gamma_0(s^0) > 0 \quad (a.s.) 
\] (17)
Proof. Analogous to the proof of Theorem 3 we assume without loss of generality that \( w_0 = \sum_k \lambda_k (p^n(s^0) + \delta_k)/p^n(s^0) \). As pointed out above, the asymptotic growth rate of investor’s wealth is independent of the initial endowment.

We define \( \hat{h}(s^t) = (\lambda_k/p^n(s^t))_{k=1,\ldots,K} \) as in the proof of Theorem 3, cf. equation (12).

In the model with transaction costs the random variable \( \gamma(s^t) \) is given by the solution to the equation
\[
f(\beta) := \beta + \sum_{n=1}^{K} \lambda_n \delta_n \left| \frac{\beta - p^n(s^t)}{p^n(s^{t-1})} \right| = \sum_{n=1}^{K} \lambda_n \frac{p^n(s^t)}{p^n(s^{t-1})} 
\]
in \( \beta \) for each fixed \( s^t \). If \( \delta_n = 0 \) for all \( n = 1, \ldots, K \), then the solution to (18) is equal to the quantity defined in the proof of Theorem 3. We need to show that \( \gamma(s^t) \) is well defined by (18) for all sufficiently small \( \delta \).

Suppose \( \delta_n > 0 \) for at least one \( n = 1, \ldots, K \), and let \( \delta_n < 1 \) for all \( n \). Then \( f(\beta) \) is strictly smaller (larger) than the right-hand side of (18) for \( \beta = 0 \) (\( \beta = \tilde{\beta} := \sum_n \lambda_n p^n(s^t)/p^n(s^{t-1}) \)). Since \( f \) is continuous in \( \beta \) it suffices to show that it is also strictly increasing in \( \beta \). Note that \( f \) is continuously differentiable in \( \beta \) at all points in \((0, \tilde{\beta})\) except at \( \beta = p^n(s^t)/p^n(s^{t-1}) \), \( n = 1, \ldots, K \). Since the derivative of \( f \) is bounded from below by \( 1 - \sum_n \lambda_n \delta_n > 0 \) at all points at which the derivative exists, we find that \( \gamma(s^t) \) is positive and uniquely defined by (18).

Next consider the balanced strategy \( h_t \) with \( h_0 = \tilde{h}(s^0) \) described by formula (2) with the functions \( \tilde{h} \) and \( \gamma \) defined above. Note that \( \gamma \) depends on the vector of transaction costs \( \delta \). For the sake of clarity we denote it by \( \gamma_0 \).

We now apply the dominated convergence theorem by Lebesgue to prove our assertion. It is clear from (18) that \( \gamma_0(s^t) \geq \min_n \lambda_n \frac{p^n(s^t)}{p^n(s^{t-1})} \). By condition (9) this yields \( E \ln |\gamma_0(s^t)| < \infty \). Further, from Theorem 3 we know that for \( \delta = 0 \), \( E \ln \gamma_0(s^t) > 0 \). Since, by definition of \( \gamma_0(s^t) \), we have that \( \gamma_0(s^t) \to 0 \) as \( \delta \to 0 \) for almost all \( s^t \), the dominated convergence theorem ensures that \( \lim_{\delta \to 0} E \ln \gamma_0(s^t) = E \ln \gamma_0(s^t) > 0 \). Therefore, \( E \ln \gamma_0(s^t) > 0 \) for all sufficiently small \( \delta > 0 \). \( \square \)

References


