Value-at-Risk for Nonlinear Financial Instruments – Linear Approximation or Full Monte-Carlo?

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1 Computing Value-at-Risk

The most widely used tool to measure, gear and control market risk is value-at-risk (VaR). VaR quantifies the worst loss over a specified target horizon with a given statistical confidence level. In other words, it represents a quantile of an estimated profit-loss distribution. Various organizations and interest groups have recommended VaR as a portfolio risk-measurement tool. Moreover, since the publication of the market-risk-measurement system RiskMetrics™ of J.P. Morgan in 1994 VaR has gained increasing acceptance and can now be considered as the industry's standard tool to measure market risks.

While the basic concept of VaR is simple, many complications can arise in practical use. An important complication is caused by nonlinearity in the portfolio payoff structure. This problem arises for all portfolios that include assets with highly nonlinear payoffs, such as option positions. For such nonlinear portfolios, VaR can not be computed directly from a risk factor distribution. Instead, the risk factor distribution first needs to be converted into a profit-loss distribution for the portfolio. VaR is then computed from this profit-loss distribution.

Several methods for computing VaR of nonlinear portfolios have been proposed. Parametric models such as delta-normal are based on statistical parameters such as the mean and the standard deviation of the risk factor distribution. Using these parameters and the delta of the position, VaR is calculated directly from the risk factor distribution. In other words, the delta of the position serves as an approximation for the conversion from the risk factor distribution to the profit-loss distribution. Non-parametric models are simulation or historical models. Among simulation approaches, we distinguish between full valuation and partial valuation models. A full simulation approach creates a number of scenarios for the risk factors and then, for each scenario, performs a complete revaluation of the portfolio, thus giving the profit-loss distribution of the portfolio. A partial valuation approach uses simulations to create the distribution of risk factors but does not fully revalue the portfolio. Instead, it makes use of delta or delta-gamma approximations to obtain the portfolio value. Figure 1 gives an overview of the various approaches to compute VaR.

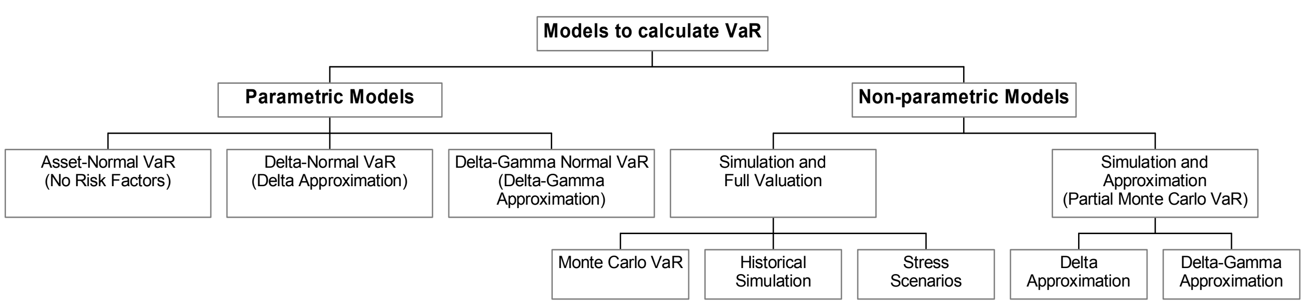

Figure 1: Approaches to VaR Computation

1.1 Asset-Normal VaR, Delta-Normal VaR, and Extensions

The basic model for calculating value-at-risk is the Asset-Normal VaR[1]. It makes the assumption that the respective values of the positions in the portfolio are normally distributed. The VaR can then be computed as

$$(1) \quad VAR_{\alpha}(t,T) = z_{\alpha} \cdot \sqrt{\mathbf{w}' \cdot \Sigma \cdot \mathbf{w}} \cdot \sqrt{T-t} \cdot PF(t),$$

where Σ is the variance-covariance matrix of the individual positions or assets, z_{α} represents the α-quantile of the standard normal distribution, $\mathbf{w}$ is the vector with the portfolio weights, (T - t) is the time horizon and PF(t) is the actual value of the portfolio. To reduce the dimension of the problem, however, a model known as delta-normal[2] is often used. This model is based on a risk factor representation of the individual positions. In other words, every position in the portfolio is modeled exclusively on the basis of market risk factors[3]. The assumptions of the delta-normal model are:

1. The changes in the value of the portfolio are linearly dependent on the respective changes in the value of the market risk factors. In other words, a Taylor-approximation of first order of the change of the value of the portfolio is used. As a consequence, the VaR of linear derivatives such as forward contracts or interest rate swaps can be calculated exactly. For nonlinear instruments, such as options, the Delta-Normal VaR is only a local approximation of the exact VaR.
2. The relative changes in the value of the market risk factors follow a joint normal distribution.
3. The composition of the portfolio is constant over time.

In the case of a portfolio, the Delta-Normal VaR can be computed as

$$(2) \quad VAR_{\alpha}(t,T) = z_{\alpha} \cdot \sqrt{\mathbf{D}' \cdot \Sigma \cdot \mathbf{D}} \cdot \sqrt{T-t}$$

with

$$(3) \quad \Sigma = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,M} \\ \vdots & \ddots & \vdots \\ \sigma_{M,1} & \cdots & \sigma_{M,M} \end{pmatrix}$$

as the variance-covariance matrix of the risk factors and

(4) $$\mathbf{D} = \begin{pmatrix} d_1 \\ \vdots \\ d_M \end{pmatrix} = \begin{pmatrix} \delta_1 \cdot S_1(t) \\ \vdots \\ \delta_M \cdot S_M(t) \end{pmatrix}$$

as the $(M \times 1)$-vector of modified portfolio sensitivities relative to the changes of the market risk factors, with δ_i as the i-th portfolio sensitivity and $S_i(t)$ as the value of the i-th market risk factor at time t.

As a straightforward extension to the Delta-Normal VaR, we introduce the Delta-Exact VaR. The Delta-Exact VaR assumes a lognormal distribution of the risk factor[4]. For some risk factors, such as the price of a stock, a lognormal distribution of the value (i.e., normal distribution of the continuously compounded returns) is a more appropriate assumption than a normal distribution.

It is important to realize that neither model takes leptokurtic or skewed distributions into consideration. Moreover, the statistical distribution of the market risk factors is assumed to be known and the covariance structure is assumed to be deterministic. Furthermore, in expression (2) we make use of the square root of time rule, which relies on the assumptions that the volatility of the changes in the risk factors is constant and that there is no serial correlation. These two assumptions are often violated in practice because of time series of risk factors exhibiting mean reversion (e.g. interest rates) or mean reversion in volatility.

Because of the simplifying assumptions, the Delta-Normal VaR can be calculated easily in a spreadsheet. The computational cost is low and the model is easy to understand and to interpret. For portfolios with a highly linear payoff structure, i.e. no substantial option components, the delta-normal model is a reasonably accurate way of calculating VaR. However, the assumption of normally distributed risk factors is no longer appropriate if the distributions are leptokurtic. To take this problem into account, ALBANESE / LEVIN / CHAO (1997) suggest the use of a stochastic rather than a deterministic variance-covariance matrix with known probability distribution but unknown parameters. The parameters are estimated using a Bayesian approach. The uncertainty is being taken into account by considering a stochastic variance-covariance matrix. The authors provide a closed form solution for their so-called Bayesian VaR (BVaR):

(5) $$\mathrm{VAR}_\alpha(t,T) = t_{n,\alpha} \cdot \sqrt{\mathbf{d}' \cdot \mathbf{C}^\wedge \cdot \mathbf{d}} \cdot \sqrt{T - t}$$

where $\mathbf{C}^\wedge$ is the adjusted variance-covariance matrix, **d** the vector of the M sensitivies of the portfolio and $t_{n,\alpha}$ the α-quantile of a univariate central t-distribution with n degrees of freedom. The parameter n is used to determine the fatness of the tails. An alternative approach is suggested, for example, in HULL / WHITE (1998) and in VENKATARAMAN (1997). They use mixtures of normal distributions[5] to obtain empirically realistic non-normal return distributions. However, no closed form solution is available for the mixture approaches.

Instead of modeling the leptokurtic distribution directly, it can be modeled as a conditional normal distribution with time-varying volatility. To improve the estimation of the covariance structure, a number of alternatives to the ordinary variance estimator have been proposed.

Generally, historical and implied models can be considered to estimate the variance-covariance matrix. While the latter seem theoretically superior because, being based on aggregate market expectation as opposed to only historical time series, they incorporate more information, their practical use is hampered by the fact that they need to be derived from option price data. For many practical situations, option markets are not liquid enough or even inexistent and thus the estimation of a complete implied variance-covariance matrix can be difficult. Furthermore, implied volatility cannot be derived from option price data in a model-independent fashion.

Popular historical methods for estimating volatility aside from the ordinary estimator are the EWMA-model (exponentially weighted moving average model) or the various ARCH/GARCH models. A brief overview of frequently used models can be found in HULL (1999).

For portfolios with substantial option components and relatively few sources of risk, the extension of the delta-normal to the delta-gamma normal model is sometimes recommended[6]. This model is based on a Taylor-expansion of second order of the future value of the portfolio around the current value. It provides increased precision at relatively (compared to Monte Carlo simulation) low computational cost. However, there exists no closed form solution if considering more than one risk factor. ROUVINEZ (1997) distinguishes between quantile-based approaches and optimization approaches. Quantile-based approaches make use of the quantile of the distribution of the changes in the portfolio. There exist several approaches to calculate the quantile of the change of the portfolio with the moments. ZANGARI (1996a) uses the so-called moment matching approach. He looks for a distribution which approximates the effective distribution as well as possible. This method is sometimes called the Delta-Gamma-Johnson VaR[7]. However, moments higher than kurtosis cannot be taken into account. ZANGARI (1996c) suggests a method to calculate the quantile of the change of the value of the portfolio directly with the use of a Cornish-Fisher expansion. This is therefore called a direct quantile approach. The resulting VaR is called Delta-Gamma-Cornish-Fisher VaR[8]. This approach has the advantage that higher moments can be considered, too.

The optimization approach is an alternative way for calculating VaR suggested in WILSON (1998)[9]. The resulting Delta-Gamma-Maximum-Loss is an alternative to the Delta-Gamma Normal VaR to measure the market risk of a portfolio with nonlinear instruments. For a further discussion of this approach see STUDER (1995) and STUDER (1997).

The purpose of this work is to show differences between simple approximation models and more involved simulation models. We therefore do not use the extensions of the delta-normal model.

1.2 Monte Carlo VaR and Partial Monte Carlo VaR

A Monte Carlo VaR is computed by simulating the changes in the values of the market risk factors and revaluing the entire portfolio of positions for each simulation trial. The simulation of the risk factors makes assumptions about their underlying processes.

We make the standard assumption of geometric Brownian motion[10] for the stock price process. The dynamics of $S(t)$ are given by the stochastic differential equation

(6) $$\frac{dS}{S} = \mu(t) \cdot dt + \sigma(t) \cdot dz,$$

where $dz = \varepsilon_i \cdot \sqrt{dt}$ is the infinitesimal increment of the Brownian motion (or Wiener process) and ε_i a standard normal random variable. The solution of expression (6) for the asset price at time T is given by

(7) $$S_j(T) = S_j(0) \cdot \exp \cdot \left[\left(\mu_j - \frac{\sigma_j^2}{2} \right) \cdot T + \sigma_j \cdot \varepsilon_i \cdot \sqrt{T} \right],$$

where μ_j can be interpreted as the annualized expected value of the relative change, σ_j the annualized standard deviation of the relative change and T the time horizon in years. The use of geometric Brownian motion to model the dynamics of the stock price implies that the price of the underlying is lognormally distributed and the continuously compounded return is normally distributed.

Because of the short horizon of VaR calculations, it is standard practice to assume the expected change in the price of a market risk factor over the time period to be zero. This is usually considered a reasonable simplification because the expected change in the price of a market risk factor tends to be small when compared with the volatility of the respective changes[11].

Given the specification of the stochastic processes, future scenarios for the different market risk factors can be generated. To generate random normal variables which are correlated according to our estimated variance-covariance matrix, we make use of the so-called Cholesky factorization[12]. First, we estimate the historical correlation matrix. Second, we decompose the correlation matrix to get the Cholesky matrix, a lower triangular matrix. Third, we generate a vector with uncorrelated random normal variables. Finally, we multiply the lower triangular matrix with the vector from the third step to get a vector with normal random variables which are correlated according to our estimated correlation matrix. For each simulation trial we then revalue the single positions and the whole portfolio.

The respective VaR can then be read as the α-quantile of the change of the portfolio. The higher the number of simulation trials, the more precise the estimation we get, and in the case of infinite simulation trials the estimated probability distribution converges to the theoretical distribution.

One of the advantages of the Monte Carlo simulation method is that arbitrary processes can be assumed for the underlying asset. In addition, the generation of different correlated scenarios is easily possible. When combining Monte Carlo simulation with full valuation, the portfolio is fully revalued for each simulated price path. Because of its flexibility the Monte Carlo method is the most powerful method for calculating VaR. Various risks such as price risk or credit risk can be dealt with simultaneously. Moreover, in combination with full valuation, Monte Carlo simulation gives the most accurate results for portfolios with substantial option components. However, the advantages must be weighed against the drawback of relatively high technical requirements. The calculation tends to be slow because the portfolio has to be revalued many times.

The **Partial Monte Carlo VaR** approach differs slightly from the conventional Monte Carlo approach. On the basis of a Monte Carlo simulation of the market risk factors the portfolio is revalued with a delta or delta-gamma approximation. The advantage of the Partial Monte Carlo VaR compared to the Monte Carlo VaR consists in faster computation. However, the Partial Monte Carlo VaR does not improve approximations over parametric models if it applies the same approximation methods and assumes the same processes for the market risk factors. In fact, aside from the simulation error, the Partial Monte Carlo VaR and Delta-Exact VaR are equivalent in terms of precision if the same approximation method is used.

It should be noted that the applied one-dimensional Monte Carlo simulation with one risk factor (e.g. the stock price) is a simplification. When dealing with option positions, other risks such as the volatility risk or the interest rate risk are important, too. Furthermore, the assumption of normally distributed risk factors is, as discussed above, sometimes problematic. In order to have a more realistic benchmark distribution, one could perform a Monte Carlo simulation with realistic (empirical) third and fourth moments. However, the question we want to address is how to quantify the difference between simple approximation models and the Monte Carlo model and the mentioned extensions are beyond the scope of this work. For this reason, we also do not consider alternative approaches such as VaR based on historical simulation. Instead, we focus in the following sections on the application of Delta-Normal VaR, Delta-Exact VaR, and Monte Carlo VaR.

2 Calculation of VaR for Portfolios with Linear and Nonlinear Financial Instruments

2.1 Portfolio with Stocks

In this section we calclate the VaR for a portfolio with shares of company X and Y, respectively. The basic formula for a two-asset portfolio can be applied, i.e.,

$$(8)\qquad VAR_{\alpha}(t,T) = z_{\alpha} \cdot \sigma_{day,PF} \cdot \sqrt{T-t} \cdot PF(t)$$

where $PF(t)$ is the actual value of the portfolio and the daily portfolio volatility is given by

$$(9)\quad \sigma_{day,PF} = \cdot\sqrt{w_{share1}^{2} \cdot \sigma_{day,share1}^{2} w_{share2}^{2} \cdot \sigma_{day,share2}^{2} + 2 \cdot \rho \cdot w_{share1} \cdot \sigma_{day,share1} \cdot w_{share2} \cdot \sigma_{day,share2}}$$

with w_{share1} and w_{share2} as the respective portfolio weights.

Table 1 shows the Delta-Normal VaR, the Delta-Exact VaR and the Monte Carlo VaR for a pure share portfolio for different confidence levels and time horizons. The weights are chosen as to be equal. The relative VaR's are given in percent of the invested amount[13]. For the Monte Carlo VaR, 1,000,000 simulations were performed. In addition, the difference between the relative Delta-Exact VaR and the relative Monte Carlo VaR divided by the relative Monte

Carlo VaR is calculated in order to have a measure for the relative deviation of the simple approximation method from the Monte Carlo benchmark:

$$(10) \qquad \Delta = \frac{(\text{rel. Delta} - \text{Exact VaR}) - (\text{rel. Monte Carlo VaR})}{\text{rel. Monte Carlo VaR}} .$$

This number can be interpreted as the VaR approximation error (VaR error).

Table 1: VaR of a Stock Portfolio. Delta-Normal VaR (DN), Delta-Exact VaR (DE), Monte Carlo VaR (MC) and relative deviations between Delta-Exact VaR and Monte Carlo VaR (Δ). VaR figures are relative (in percent of portfolio value).

	1 Day				**10 Days**				**25 Days**			
	DN	DE	MC	Δ	DN	DE	MC	Δ	DN	DE	MC	Δ
90%	2.68	2.64	2.66	-0.00	8.48	8.13	8.33	-0.02	13.42	12.55	13.04	-0.03
95%	3.44	3.38	3.40	-0.00	10.89	10.32	10.50	-0.01	17.22	15.82	16.28	-0.02
99%	4.87	4.75	4.77	-0.00	15.40	14.27	14.47	-0.01	24.36	21.62	22.03	-0.01

As expected, the VaR is increasing with the confidence level and with the time horizon. Because we have no nonlinearity in this share portfolio, we expect the VaR to be the same in the two models used, the Delta-Normal VaR and the Monte Carlo VaR. However, the simple Delta-Normal VaR relies on the assumption that the prices of the market risk factors are normally distributed whereas in the Monte Carlo simulation we assume that the risk factor (stock price) is lognormally distributed and the continuously compounded returns are normally distributed. To account for this we recalculate the VaR for the shares portfolio in the following way:

$$(11) \qquad R = z_\alpha \cdot \sigma_{day,PF} \cdot \sqrt{T-t} ,$$

and use this result to calculate

$$(12) \qquad VAR_\alpha(t,T) = PF(t) \cdot e^R - PF(t) .$$

The resulting VaR's are also shown in Table 1, denoted as Delta-Exact VaR. In most cases Delta-Exact values are closer to the Monte Carlo simulation results. However, a small empirical estimation error remains even for 1,000,000 simulation trials. Obviously, Delta-Exact VaR represents an improvement over Delta-Normal VaR (relative to the Monte Carlo benchmark). This effect is an immediate consequence from the assumption that the price of the market risk factors follows a lognormal rather than a normal distribution. The remaining error is caused by the simulation error. For linear portfolios such as pure stock portfolios, Delta-Exact VaR and Monte Carlo VaR give the same results. In the following sections we choose several portfolios containing nonlinear instruments, but with varying degree of nonlinearity. Higher nonlinearity is expected to give greater VaR approximation error, i.e., a greater difference between the simple approximation model and the simulation approach.

2.2 Option Portfolio

The basic formula to calculate VaR becomes

$$(13) \qquad VAR_{\alpha}(t,T) = z_{\alpha} \cdot \sigma_{day} \cdot \sqrt{T-t} \cdot S(t) \cdot \eta \cdot \kappa \cdot \delta_{Option} \ ,$$

where σ_{day} is the daily volatility of the underlying, η is the number of contracts, κ equals the contract size, $S(t)$ is the actual price of the underlying and δ_{option} is the delta of the option. According to theory, the delta-normal model should give more distorted results than in the case of the pure stock portfolio because it does not account for nonlinearity. To compare the applied models we again correct formula (13) and calculate the Delta-Exact VaR, i.e.,

$$(14) \qquad VAR_{\alpha}(t,T) = \left[S(t) \cdot \eta \cdot \kappa \cdot e^{R} - S(t) \cdot \eta \cdot \kappa\right] \cdot \delta_{Option} \ .$$

If one hundred percent are invested in call options on company Y[14], the VaRs shown in Table 2 arise. As apparent, values (in percent of the invested amount) are very high relative to the values we calculated for the linear portfolio. The daily VaR is about four times as high as the daily VaR for the pure stock portfolio. Furthermore, the deviation of the Delta-Normal VaR, as well as of the Delta-Exact VaR, from the Monte Carlo VaR increases with the time horizon and with the confidence level. High confidence levels and longer time horizons imply a larger potential change in the value of the underlying. Because the approximations are local linear approximations, larger changes in the underlying result in poorer approximations. It is also interesting to note that the Delta-Exact VaR is higher (with one exception) than the Monte Carlo VaR. This effect can be attributed to the convexity of the instruments in the portfolio.

Table 2: VaR of Call Option Portfolio. Delta-Normal VaR (DN), Delta-Exact VaR (DE), Monte Carlo VaR (MC) and relative deviations between Delta-Exact VaR and Monte Carlo VaR (Δ). VaR figures are relative (in percent of portfolio value).

	1 Day				10 Days				25 Days			
	DN	DE	MC	Δ	DN	DE	MC	Δ	DN	DE	MC	Δ
90%	9.53	9.40	9.25	0.01	30.13	28.85	28.92	-0.00	47.65	44.48	44.44	0.00
95%	12.23	12.02	11.74	0.02	38.69	36.58	35.41	0.03	61.17	56.01	52.84	0.05
99%	17.29	16.87	16.26	0.03	54.70	50.55	46.27	0.09	86.49	76.39	65.95	0.15

The simple Delta-Normal VaR relies on the assumption that relative changes in the market risk factors are normally distributed and is based on a linear approximation. Even for the Delta-Exact VaR, the calculated VaRs differ substantially from the Monte Carlo VaR because of the nonlinearity. Considering the relative deviations (Δ) between the Delta-Exact VaR and the Monte Carlo VaR, we clearly see that the deviations between the Delta-Exact VaR and the Monte Carlo VaR are higher than in the case of linear instruments, e.g. the pure stock portfolio.

Table 3: VaR of Portfolio with Short-Maturity Call Options. Delta-Normal VaR (DN), Delta-Exact VaR (DE), Monte Carlo VaR (MC) and relative deviations between Delta-Exact VaR and Monte Carlo VaR (Δ). VaR figures are relative (in percent of portfolio value).

	1 Day				**10 Days**				**25 Days**			
	DN	DE	MC	Δ	DN	DE	MC	Δ	DN	DE	MC	Δ
90%	17.92	17.69	16.78	0.05	56.68	54.30	52.14	0.04	89.63	83.72	76.65	0.09
95%	23.01	22.63	21.16	0.06	72.77	68.86	61.22	0.12	115.06	105.42	84.75	0.24
99%	32.53	31.76	29.00	0.09	102.89	95.14	74.59	0.27	162.69	143.79	93.83	0.53

Table 3 shows call options on the same underlying Y. However, in this case, options with shorter time to maturity (a quarter of a year) and higher convexity are chosen. The VaR values increase and become nearly twice as high as the values before. As in Table 2, the Delta-Exact VaR overestimates the VaR if compared with the Monte Carlo VaR. Moreover, the deviations increase even more significantly when short-maturity options are considered. The error caused by the linear approximation becomes so high as to make the use of linear approximations hard to justify.

2.3 Straddle and Strangle

Straddles and strangles are also highly nonlinear instruments. The (long-)straddle portfolio is constructed by investing half of the total value in long call options and long put options, respectively. Both options have a time to maturity of one year, a strike price of 230 (i.e., at-the-money options) and the same underlying Y.

Table 4: VaR of Straddle Portfolio. Delta-Normal VaR (DN), Delta-Exact VaR (DE), Monte Carlo VaR (MC) and relative deviations between Delta-Exact VaR and Monte Carlo VaR (Δ). VaR figures are relative (in percent of portfolio value).

	1 Day				**10 Days**				**25 Days**			
	DN	DE	MC	Δ	DN	DE	MC	Δ	DN	DE	MC	Δ
90%	1.99	1.78	1.85	-0.03	6.30	5.47	5.23	0.04	9.96	8.43	7.62	0.10
95%	2.55	2.27	2.26	0.00	8.09	6.93	5.46	0.26	12.79	10.61	7.72	0.37
99%	3.61	3.19	2.92	0.09	11.44	9.58	5.55	0.72	18.08	14.48	7.76	0.86

In the case of the (long-)strangle portfolio we invest in a call with a higher strike price (235) than the strike of the put option (225).

Table 5: VaR of Strangle Portfolio. Delta-Normal VaR (DN), Delta-Exact VaR (DE), Monte Carlo VaR (MC) and relative deviations between Delta-Exact VaR and Monte Carlo VaR (Δ). VaR figures are relative (in percent of portfolio value).

	1 Day				**10 Days**				**25 Days**			
	DN	DE	MC	Δ	DN	DE	MC	Δ	DN	DE	MC	Δ
90%	2.13	1.88	1.82	0.03	6.74	5.78	5.80	-0.00	10.66	8.91	7.56	0.17
95%	2.73	2.40	2.26	0.06	8.65	7.33	5.71	0.28	13.69	11.22	7.67	0.46
99%	3.87	3.38	2.94	0.14	12.24	10.12	5.47	0.85	19.35	15.30	7.70	0.98

The formula applied for the straddle as well as the strangle portfolio is

$$(15) \qquad VAR_{\alpha}(t,T) = z_{\alpha} \cdot \sigma_{day} \cdot \sqrt{T-t} \cdot S(t) \cdot \eta \cdot \kappa \cdot (\delta_{Call} + \delta_{Put}),$$

where η is the number of straddle and strangle contracts, respectively. Table 4 and Table 5 show the respective VaR values for the two portfolios. The VaR of both the straddle and strangle portfolio is smaller than in the case of the call option portfolio. However, the relative deviation between the approximation model and the Monte Carlo VaR is much higher, meaning that the VaR approximation error is higher. Note that we again observe an increasing relative deviation between the two applied models if the time horizon or the confidence level increases. In addition, the Delta-Exact VaR again overestimates the true VaR (as measured by the Monte Carlo VaR).

2.4 Digital Options

As a last example, we investigate digital options as a representative of exotic derivatives. A digital option pays the amount Q if the option ends in the money and nothing otherwise. Note that the price function of such an option is both convex and concave. We calculate the Delta-Exact VaR for a portfolio with digital options on company X, applying the following formula for digital options:

$$(16) \qquad VAR_{\alpha}(t,T) = z_{\alpha} \cdot \sigma_{day} \cdot \sqrt{T-t} \cdot S(t) \cdot \eta \cdot \kappa \cdot \delta_{DigitalOption}$$

with the delta of the digital option being

$$(17) \qquad \delta_{DigitalOption} = Q \cdot e^{-rt} \cdot N'(d_2) / (S(t) \cdot \sigma \cdot \sqrt{T-t}),$$

where Q is the money amount received if the option ends in the money, $N'(d_2)$ is the density function at the value d_1, and d_2 itself is

$$(18) \quad d_2 = \frac{\ln(S(t)/X) + (r - \sigma^2/2) \cdot (T - t)}{\sigma \cdot \sqrt{T - t}}.$$

Table 6 shows the VaR for the digital option portfolio. We note that the deviations between the applied models again increase with longer time horizons. However, in contrast to the portfolios with nonlinear financial assets in the previous sections, we observe here that the Monte Carlo VaR is higher than the Delta-Exact VaR.

Table 6: VaR of Portfolio with Digital Options. Delta-Normal VaR (DN), Delta-Exact VaR (DE), Monte Carlo VaR (MC) and relative Deviations between Delta-Exact VaR and Monte Carlo VaR (Δ).VaR figures are relative (in percent of portfolio value).

	1 Day				**10 Days**				**25 Days**			
	DN	DE	MC	Δ	DN	DE	MC	Δ	DN	DE	MC	Δ
90%	5.98	5.89	6.00	-0.01	18.92	18.04	19.23	-0.06	29.91	27.76	30.79	-0.09
95%	7.68	7.53	7.68	-0.01	24.29	22.86	24.27	-0.05	38.41	34.90	38.38	-0.09
99%	10.86	10.56	10.82	-0.02	34.34	31.52	33.45	-0.05	54.30	47.48	51.35	-0.07

Table 7: VaR of Portfolio with Short-Maturity Digital Options. Delta-Normal VaR (DN), Delta-Exact VaR (DE), Monte Carlo VaR (MC) and relative Deviations between Delta-Exact VaR and Monte Carlo VaR (Δ).VaR figures are relative (in percent of portfolio value).

	1 Day				**10 Days**				**25 Days**			
	DN	DE	MC	Δ	DN	DE	MC	Δ	DN	DE	MC	Δ
90%	11.33	11.33	11.33	-0.00	35.83	34.17	36.71	-0.06	56.66	52.57	60.01	-0.12
95%	14.54	14.26	14.49	-0.01	46.00	43.28	45.80	-0.05	72.74	66.10	71.49	-0.07
99%	20.57	20.01	20.29	-0.01	65.05	59.70	60.76	-0.01	102.85	89.91	86.51	0.03

For short-maturity options (Table 7) as well as for long-maturity options, the delta-normal method overestimates the Monte Carlo VaR for the 95% and 99% confidence level. The opposite can occur for the 90% confidence level. The Delta-Exact VaR, in contrast, tends to underestimate the Monte Carlo VaR. These effects are caused by the price structure of the digital options, which can be convex or concave depending on the location of the exercise price relative to the price of the underlying.

3 A "global" Measure for Nonlinearity

In this section we propose a simple measure for nonlinearity. Because the use of approximation methods can be justified for some nonlinear portfolios but not for others, it would be useful to have a measure for the appropriateness of a method for a specific portfolio. Such a measure has many applications. For example, if an approximation method is used for the computation of VaR, such a measure can give an indication of the accuracy of the VaR approximation. It can thus raise a warning flag whenever a portfolio is evaluated for which the VaR approximation may not be accurate. This application is particularly useful for portfolios with nonlinearities, if minimizing computational cost is critical and, therefore, a full simulation approach cannot be used. It is even conceivable that such a measure could be used for real-time switching between different methods for VaR computation. If the measure indicates low levels of nonlinearity, approximations could be used for fast computation, whereas computationally more costly simulation methods might be used whenever pronounced nonlinearities are detected in a specific portfolio.

An immediate candidate for such a nonlinearity measure is a standard convexity measure in the form of the second partial derivatives with respect to the risk factor (for options, this measure is called gamma if the risk factor is the underlying asset). However, local nonlinearity measures such as gamma tend to give inaccurate results because local nonlinearity can vary greatly across the distribution. VaR seems to be a particularly inappropriate application of local measures because it is concerned with extreme changes in the values of risk factors [15]. Figure 2 illustrates the poor prediction of VaR errors by a local convexity measure (gamma). The figure shows gamma values and corresponding VaR approximation errors for all example portfolios considered in the previous section. It can be seen that VaR approximation errors vary between zero percent and as high as one hundred percent for the same gamma value (approximately 0.08 in the figure). Gamma is measured at the current value of the risk factor and is scaled by a factor of 100.

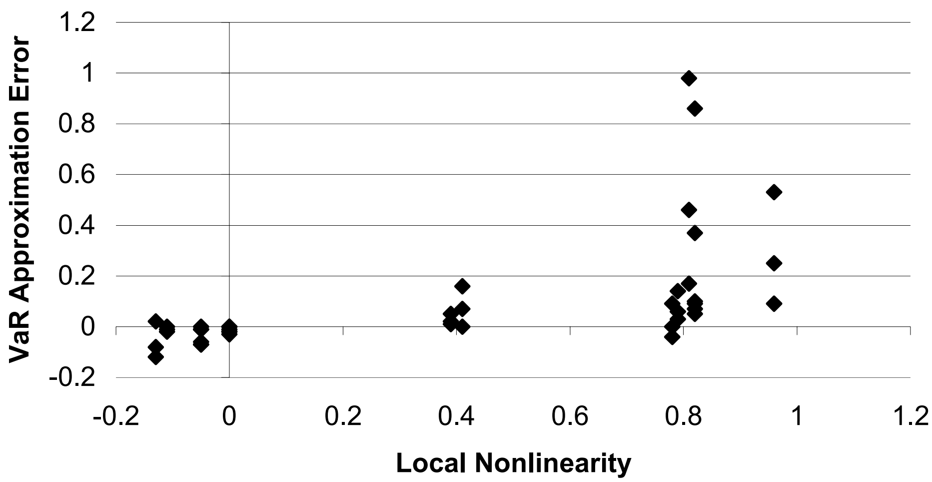

Figure 2: Local Nonlinearity and VaR Errors

For improved VaR error prediction, we propose a measure that uses delta differences between several quantiles of the risk factor distribution as a measure for nonlinearity. First, we fix a two-sided quantile, denoted by α, of the risk factor distribution. Second, we calculate the deltas of the portfolio at the chosen quantile (i.e., the α-quantile and the $(1-\alpha)$-quantile) of the distribution of a market risk factor as well as at the current value of the market risk factor. Third, we calculate delta differences, i.e.,

(19) $\chi_1 = \delta_{Option}(S) - \delta_{Option}(S - \Delta S)$ (convexity 1),

(20) $\chi_2 = \delta_{\text{Option}}(S + \Delta S) - \delta_{\text{Option}}(S)$ (convexity 2),

where δ_{Option} (S - ΔS) is the portfolio delta at the value of the market risk factor corresponding to the α-quantile of its distribution, δ_{Option} (S + ΔS) is the respective value at the (1-α)-quantile and δ_{Option} (S) is the option delta at the current value of the market risk factor. Convexity 1 and convexity 2 measure downside and upside nonlinearity, respectively, in the risk factor considered. Positive and negative values indicate convexity and concavity, respectively. If the two measures are added up, i.e.,

(21) $\chi_{\text{global}} = \chi_1 + \chi_2$ (sum),

then the delta differences of the portfolio at quantiles α and (1-α) are obtained. Obviously, such a measure is not local unless the quantiles are set very close to the current value of the risk factor. We therefore call it "global" convexity or "global" nonlinearity.

The question arises immediately how the appropriate quantile α can be determined. It seems intuitive that the quantile level depends on the confidence level of the VaR calculation. Although it is important to keep in mind that VaR cannot be computed directly off the risk factor distribution (for nonlinear financial instruments, losses may not be greatest for extreme changes but for very small or no changes at all in the risk factor), the example with the straddle indicates that, nonetheless, it may be reasonable to use lower α values for higher confidence levels in the VaR calculation. We use, somewhat arbitrarily, α values of 10%, 5%, 1% for VaR confidence levels of 90%, 95%, 99%, respectively.

Table 8 shows global convexity values for the portfolios for which VaR was calculated in the previous section. It can be seen that the values increase for a longer time horizon (25 days vs. 1 day). Furthermore, they also tend to increase with decreasing α. Negative values indicate a concave structure. This can most easily be seen for the digital options, where upside nonlinearity is pronouncedly concave. Overall, when comparing the global nonlinearity values of Table 8 with the VaR errors (Δ) of the delta-exact method (relative to Monte Carlo VaR) of the previous Tables, the measure appears to be a good indicator for the inaccuracy of the linear VaR approximation. This impression is confirmed by Figure 3, which relates global convexity values with the VaR error caused by linear approximation for all examples from Section 2. The improvement over local nonlinearity as measured by gamma (Figure 2) is apparent. Figure 3 also shows, however, that the measure is not perfect in capturing the adverse effects of nonlinearity on VaR accuracy. For the same degree of nonlinearity as indicated by the measure, variations of accuracy in the order of magnitude of 20% are possible for the portfolios examined.

When determining which methods to use for VaR, a critical value needs to be found above which nonlinearity is considered too high for linear VaR approximations. For example, if 20% VaR error is deemed acceptable, a critical nonlinearity value of approximately 0.50 might be chosen (see Figure 3). Productwise, such a critical value implies that straddles, strangles, and the short-maturity call option cannot be evaluated with a linear approximation for a VaR horizon of 25 days. For a VaR horizon of 1 day, however, none of the instruments considered exceed the critical value of 0.5. Therefore, it is concluded that linear approximation is acceptable for all instruments in that case.

Table 8: Global Convexity for different Portfolios containing Nonlinear Assets

		1 Day			25 Days		
		Convexity 1	Convexity 2	Sum	Convexity 1	Convexity 2	Sum
Option portfolio	**10%**	0.03	0.03	0.06	0.14	0.12	0.26
	5%	0.03	0.03	0.06	0.18	0.15	0.33
	1%	0.05	0.05	0.10	0.25	0.20	0.45
Option short	**10%**	0.05	0.05	0.10	0.31	0.26	0.57
	5%	0.06	0.07	0.13	0.38	0.31	0.69
	1%	0.10	0.10	0.20	0.46	0.38	0.84
Digital option	**10%**	0.00	-0.01	-0.01	-0.01	-0.04	-0.05
	5%	0.00	-0.01	-0.01	-0.01	-0.05	-0.06
	1%	-0.01	-0.01	-0.02	0.01	-0.08	-0.07
Digital short	**10%**	0.00	-0.02	-0.02	0.13	-0.19	-0.06
	5%	0.00	-0.02	-0.02	0.21	-0.27	-0.06
	1%	0.00	-0.03	-0.03	0.36	-0.42	-0.06
Straddle portfolio	**10%**	0.06	0.06	0.12	0.28	0.25	0.53
	5%	0.06	0.06	0.12	0.36	0.30	0.63
	1%	0.10	0.10	0.20	0.50	0.40	0.90
Strangle portfolio	**10%**	0.06	0.05	0.11	0.28	0.23	0.51
	5%	0.06	0.06	0.12	0.36	0.29	0.65
	1%	0.10	0.10	0.20	0.50	0.41	0.91

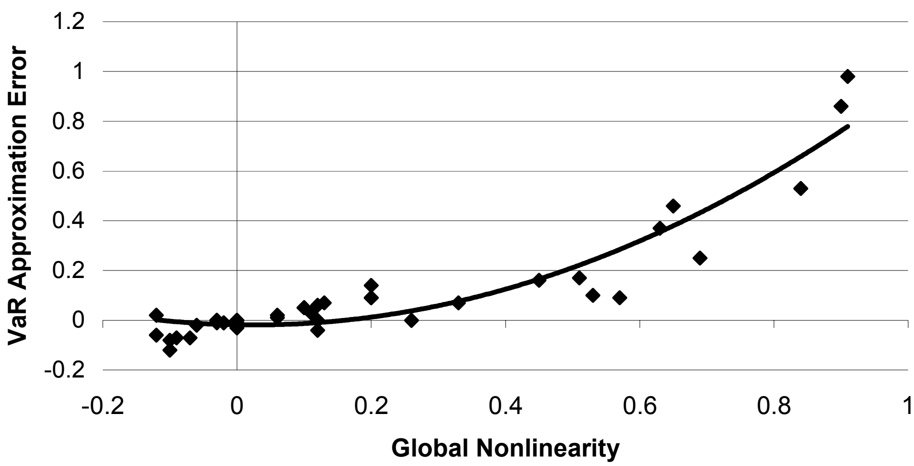

Figure 3: Global Nonlinearity and VaR Errors

Ideally, a measure for nonlinearity is simple, generally applicable, accurate, and computationally inexpensive. However, simplicity and general applicability as well as accuracy and computational inexpensiveness tend to be opposing goals. Therefore, depending on the application, a compromise solution has to be found. The measure proposed above is very simple and can be calculated at minimal cost. Clearly, although the measure performs reasonably well in the examples above, it cannot be concluded that such a simple measure will give accurate results for any nonlinear portfolio.

Of course, many variations of the measure proposed above are conceivable. For example, it is arguable that concave and convex structures should not offset each other in the calculation of total convexity. To avoid such an offsetting effect, the absolute values of convexity 1 and convexity 2 could be used. Instead of expression (22) we would have

$$\chi_{\text{global}} = |\chi_1| + |\chi_2| \tag{22}$$

Other variations might, for example, evaluate the nonlinearity measure at several quantiles to compute an "average" value for nonlinearity. Although we illustrate the application of the convexity measure using several examples, our analysis does not represent a thorough evaluation of the measure proposed. Further research will have to evaluate, based on a much larger number of representative portfolios, which measure provides the best prediction of the approximation error.

The measure proposed above represents a pragmatic approach to predicting errors of linear VaR approximations. Using such a measure involves several difficult calibration decisions (measure quantile, critical level of nonlinearity). Nonetheless, our examples give an indication that, with some calibration work using example portfolios, it appears to be possible to predict errors of linear VaR approximations for specific portfolios with an accuracy that is high enough to make such a measure a practical tool in the risk management of nonlinear portfolios.

4 Conclusion

We implement different methods of computing VaR for portfolios containing nonlinear financial instruments. We show that simple linear approximation models (e.g. the Delta-Exact VaR) are reasonably accurate in many cases. For some heavily optioned portfolios, however, only simulation methods such as Monte Carlo simulation with full revaluation are appropriate. In particular, we find that:

- For portfolios without substantial option components, the (linear) Delta-Normal and particularly the Delta-Exact VaR represent a good approximation for the Monte Carlo VaR. The results deteriorate somewhat for increasing VaR time horizons and confidence levels.
- For portfolios with more pronounced nonlinearities, such as pure option portfolios, Delta-Exact VaR and Monte Carlo VaR tend to differ substantially, especially for longer VaR time horizons and high confidence levels. Portfolios containing other highly nonlinear instruments such as straddles or strangles also produce large approximation errors.
- Furthermore, the difference between the relative Delta-Exact VaR and the Monte Carlo VaR increases for options with a short time to maturity.

Because the use of approximation methods can save considerable computational resources over full simulation methods, it would be useful to know for which portfolios approximation methods can be used without sacrificing the accuracy of the resulting VaR. Because errors from the use of approximation methods are caused by nonlinear payoff structures, we propose a simple measure ("global" convexity) based on delta differences between different quantiles of the risk factor distribution to quantify the extent of nonlinearity in a portfolio. Using several example portfolios, we show that high nonlinearity as indicated by our measure tends to coincide with large errors from the use of approximation methods. Thus, the measure proposed can be useful in determining the accuracy of linear approximation methods for a specific portfolio. In practice, such a measure may therefore prove useful as a portfolio-specific evaluation tool for VaR approximation methods.

Endnotes

[1] The systematization of the different VaR models is based on WILSON (1998). The Asset-Normal VaR can be found in WILSON (1998) or JORION (1996).
[2] According to READ (1998).
[3] Since we are only considering the market risk, the expressions market risk factors and risk factors are used as synonyms throughout the rest of the paper.
[4] The Delta-Exact VaR should give results which are closer to the Monte Carlo VaR than the Delta-Normal VaR does. This is due to the fact that in calculating the Monte Carlo VaR, we assume a lognormal price process rather than a normal price process.
[5] See also ZANGARI (1996d).
[6] For a broad discussion about the Delta-Gamma Normal VaR models see READ (1998) or WILSON (1998).
[7] See ZANGARI (1996a) and ZANGARI (1996b). The name Johnson stems from the famous mathematician Johnson who described the process to find a distribution which approximates the effective distribution as good as possible. BRITTEN JONES / SCHÄFER (1997) suggest to approximate the effective distribution with a central chi-squared distribution which has the same first four moments.
[8] See FALLON (1996); ZANGARI (1996c).
[9] The concept of WILSON (1998) was developed in 1994. It is a special case of the Maximum Loss Optimization and was developed further by STUDER (1995) and STUDER (1997).
[10] For a model with jump-diffusion processes, see BRUAND (1996).
[11] This assumption is also made in Risk Metrics™ models.
[12] For a discussion about Cholesky Factorization see JORION (1996).
[13] Note that all the given VaR values in this paper are relative values, in percent of the invested amount. As a consequence, the expressions relative VaR and VaR can be regarded as synonyms.
[14] The options are chosen to be at the money to have high convexity, which should result in higher distortions between the two applied models.
[15] This is also the reason why delta-gamma (and possibly additional, higher, derivatives) approximations, although superior to simple delta-approximations, are not a solution to the nonlinearity problem.

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