Accounting for Non-normality in Liquidity Risk

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August 4, 2009

It has been frequently discussed, that returns are not normally distributed. Liquidity costs, measuring market liquidity, are similarly non-normally distributed displaying fat tails and skewness. Liquidity risk models either ignore this fact or use the historical distribution to empirically estimate worst losses. We suggest a new and easily implementable, parametric approach based on the Cornish-Fisher approximation to account for non-normality in liquidity risk. We show how to implement this methodology in a large sample of stocks and provide evidence that it produces much more accurate results than an alternative empirical risk estimation.

Keywords: Asset liquidity, liquidity cost, Value-at-Risk, market liquidity risk, non-normality
JEL classification: G11, G12, G18, G32
Acknowledgments: Comments or questions are highly welcome. First version published on December 16, 2008.

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1 Introduction

Today, the most popular tool to measure, control and manage financial risk within corporations and financial institutions is the Value-at-Risk (VaR) concept.\(^1\) VaR measures the worst expected loss over a given horizon and a certain confidence level. In most institutions the standardized VaR-methodology is used to determine capital requirements.\(^2\)

One often criticized downside of the traditional VaR-model is its inability to capture liquidity risk, because its computation generally relies solely on market prices.\(^3\) Due to the neglect of liquidity risk the real risk of an institution is generally underestimated.\(^4\) In this context, liquidity risk, more specifically market liquidity risk, can be understood as the difficulty or cost of trading assets in crises. Market liquidity risk has to be distinguished from funding risk, which is the potential shortfall of meeting liabilities and having sufficient cash available.

Market liquidity risk has already acquired a great deal of attention. During the last few years several academic papers have been written on the consideration of liquidity risk in the VaR framework.\(^5\) The proposed solutions can be classified into two groups: The first one focuses on indirect risk measures by determining price quantity functions from transaction data. In this stream, the approaches of Cosandey (2001), Jarrow and Protter (2005), Berkowitz (2000), Jarrow and Subramanian (1997) and Almgren and Chriss (2000) are widely cited. In contrast, the second group makes use of direct liquidity cost measures such as the bid-ask-spread or the order-size-dependent weighted spread. For instance Bangia et al. (1999), Francois-Heude and Van Wynendaele (2001), Giot and Grammig (2005) and Stange and Kaserer (2008c) propose models that can be classified into this latter category.

A major issue in all liquidity risk models are the assumed distributional properties of asset returns and liquidity measures. For reasons of simplicity most often either a normal or empirical distribution is used such as in Bangia et al. (1999) or Stange and Kaserer (2008c). Since the distributions of continuous asset returns and liquidity costs are often skewed and leptocurtic or platycurtic, inappropriate normality assumptions necessarily lead to incorrect risk estimates. The use of empirical distributions might also be suboptimal, because large data sets are required and a the historical distributions might poorly proxy for the future.

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\(^{1}\)Cp. Dowd (2001), pp. 4-5.


\(^{4}\)Cp. Bangia et al. (1999); Stange and Kaserer (2008c).

\(^{5}\)Cp. Stange and Kaserer, 2008b; Ernst et al., 2009 for an overview.
2 Liquidity risk model

In other risk management contexts, non-normal distributions have already been addressed. Zangari (1996) and Mina and Ulmer (1999) suggested and analyzed the Cornish-Fisher approximation as method to account for the non-normality in the case of derivatives. Favre and Galeano (2002) propose to apply this method to hedge fund risk, Lee (2007) use it in the context of real estate asset allocation.

In this paper, we suggest to transfer the methodology of the Cornish-Fisher approximation to market liquidity risk measurement. As a classic case of non-normally distributed data, it should provide substantial improvements over existing parametric frameworks. At the same time it is implementable and more precise than other parametrization. We demonstrate the implementation of our methodological suggestion in a large sample of stocks and test its preciseness.

The remainder of this paper is organized as follows: Section 2 introduces a simple liquidity risk model and outlines our approach based on the Cornish-Fisher methodology. In section 3 we calculate and test the liquidity risk methods empirically, section 4 summarizes and concludes.

2 Liquidity risk model

Many studies show that the assumption of normally distributed returns is rejected for most financial time series, including those for individual stocks, stock indices, exchange rates, and precious metals. Specifically, continuous returns for these financial assets have empirical distributions which are leptocurtic relative to the normal distribution and in many cases skewed. Bollerslev (1987), for example, finds leptocurtosis in monthly S&P 500 returns, while French et al. (1987) report skewness in daily S&P 500 returns. Engle and Gonzales-Rivera (1991) find excess skewness and kurtosis in small stocks and in exchange rates.6

The argument of non-normality equally holds for liquidity costs. Stange and Kaserer (2008a) analyze the distributional properties of liquidity costs and show that they are heavily skewed and fat tailed. Bangia et al. (1999), Giot and Grammig (2005) and Stange and Kaserer (2008c) account for non-normality in the context of risk management.

In the following we outline a simple model of liquidity-adjusted risk based on Bangia et al. (1999) as a basis for discussion. Applicability to other risk models is

discussed later. We then propose an adapted model based on the Cornish-Fisher expansion which is a technique to correct the percentiles of a standard normal distribution for non-normality.\footnote{Mina and Ulmer (1999) investigate four possible methods to compute the Value-at-Risk for non-normally distributed assets: Johnson transformation, Fourier method, partial Monte-Carlo and Cornish-Fisher expansion. They find that Cornish-Fisher is simple to implement in practice, fast and traceable while the other three approaches requires a much larger implementation effort, but have higher precision for extreme distributions.}

\section*{2.1 Liquidity risk definition}

We use the straightforward liquidity risk model of Bangia, Diebold, Schuermann and Stroughair (1998, 1999) to show how the risk calculation proceeds. Bangia et al. include time-varying bid-ask-spreads into a parametric Value-at-Risk. They assume that liquidity costs of a transaction can be measured with the bid-ask-spread. The achievable transaction price - accounting for liquidity cost - is governed by the following process

\[ P_{\text{mid},t+1} = P_{\text{mid},t} \times e^{r_{t+1}} \times (1 - \frac{1}{2}S_{t+1}) \]

where \( P_{\text{mid}} \) is the mid-point of the bid-ask-spread\footnote{Calculated as \( P_{\text{mid}} = (P_{\text{bid}} - P_{\text{ask}})/2 \) with \( P_{\text{bid}} \) and \( P_{\text{ask}} \) being the bid and ask price respectively.}, \( r_{t+1} \) is the continuous mid-price return between \( t \) and \( t+1 \) and \( S_{t+1} \) is the (uncertain) bid-ask-spread in \( t+1 \). They assume that continuous mid-price returns are normally distributed. Relative liquidity risk can then be calculated as the mean-variance estimated price-return percentile and the empirically estimated spread percentile.\footnote{Notation is our own.}

\[ L - \text{VaR} = 1 - \exp(z_\alpha \times \sigma_r) + \frac{1}{2} (\mu_S + \hat{z}_\alpha(S) \times \sigma_S) \]  

\begin{equation}
L - \text{VaR} = 1 - \exp(z_\alpha \times \sigma_r) + \frac{1}{2} (\mu_S + \hat{z}_\alpha(S) \times \sigma_S)
\end{equation}

where \( \sigma_r \) is the volatility of the continuous mid-price return assuming zero daily mean returns, \( \mu_S \) and \( \sigma_S \) are the mean and volatility of the spread - all over the chosen horizon. \( z_\alpha \) denotes the percentile of the normal distribution for the given confidence level and \( \hat{z}_\alpha(S) \) is the empirically estimated percentile of the spread distribution.\footnote{The empirical percentile is calculated as \( \hat{z}_\alpha = (\hat{S}_\alpha - \mu_S)/\sigma_S \), where \( \hat{S}_\alpha \) is the percentile spread of the past 20-day historical distribution.}

By estimating the percentile empirically, Bangia et al. avoid distortions from the non-normality in spreads, which they show to be highly present in several currencies.

The model by Bangia et al. represents an intuitively plausible and simple way to incorporate liquidity risk into a conventional Value-at-Risk framework. Data requirements are manageable as mid-price data and spread information are usually easily accessible. Another merit of this model is the additivity of price risk and
liquidity risk which facilitates implementation in practice. There is no need to modify existing programs for determining VaR. The only necessary system change is to compute the cost of liquidity and add it to the existing VaR-figure.

Despite its appeal, the model has been subject to criticism in the literature. As extensively discussed and empirically analyzed in Stange and Kaserer (2008c), the assumption of perfect correlation between mid-price return and liquidity costs leads to distortions. In addition, the model does not account for the price impact of order size, i.e. the fact that liquidity costs strongly increase with the size of the order traded. Further, price risk is assumed as normally distributed and the use of empirical percentiles might not sufficiently capture the non-normality of the future spread distribution.

The following approach addresses this issue of non-normality, which is also present in other modeling solutions. Giot and Grammig (2005) assume a t-distribution in order to adjust for fat-tails in net returns, i.e. returns net of order-size-adjusted weighted spread. A t-distribution might, however, be similarly misleading than a normal distribution. Similar to the Bangia model, Stange and Kaserer (2008c) take empirical percentiles of the net-return distribution.

2.2 Cornish-Fisher expansion

A normal distribution is fully described by its first two moments, mean and variance. Higher centralized moments like skewness and excess-kurtosis are zero. However, if the distribution is non-Gaussian higher moments will also determine loss probabilities. For this reason it is not accurate to use standardized percentiles of a normal distribution for the calculation of L-VaR of non-normally distributed returns.

Cornish and Fisher (1937) have been the first to modify the standardized percentiles of a normal distribution in a way that higher moments are accounted for. They obtain explicit polynomial expansions for standardized percentiles of a general distribution in terms of its standardized moments and the corresponding percentiles of the standard normal distribution.11 Their proceeding is widely known as Cornish-Fisher expansion. The corresponding formula approximates percentiles of a random variable based on its first few cumulants.12 Using the first four mo-

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12 The cumulants of a distribution are closely related to its moments and can be informally thought of as standardized moments. For a detailed definition of cumulants see Cornish and Fisher (1937).
ments (mean, variance, skewness and kurtosis), the Cornish-Fisher expansion for approximate \( \alpha \)-percentile \( \tilde{z}_\alpha \) of a standardized random variable is calculated as\(^{13}\)

\[
\tilde{z}_\alpha \approx z_\alpha + \frac{1}{6} (z_\alpha^2 - 1) \gamma + \frac{1}{24} (z_\alpha^3 - 3z_\alpha) \kappa - \frac{1}{36} (2z_\alpha^3 - 5z_\alpha) \gamma^2
\]  \hspace{1cm} (2)

where \( z_\alpha \) is the \( \alpha \)-percentile of a \( N(0,1) \) distribution, \( \gamma \) denotes the skewness and \( \kappa \) the excess-kurtosis of random variable.\(^{14}\) Note that in case of a normal distribution, skewness \( \gamma \) and excess-kurtosis \( \kappa \) are equal to zero, which leads to \( \tilde{z}_\alpha = z_\alpha \). The approximate \( \alpha \)-percentile \( \tilde{z}_\alpha \) can now be used in a classic Value-at-risk approach.

### 2.3 Modified liquidity-adjusted Value-at-Risk

Substituting \( z_\alpha \) and \( \tilde{z}_\alpha(S) \) from equation (1) with the modified percentile \( \tilde{z}_\alpha \) from (2) we obtain the following modified VaR estimate

\[
L - VaR = 1 - \exp(\mu_r + \tilde{z}_\alpha(r) \times \sigma_r) \times \left( 1 - \frac{1}{2} (\mu_S + \tilde{z}_\alpha(S) \times \sigma_S) \right)
\]  \hspace{1cm} (3)

where \( \tilde{z}_\alpha(r) \) is the percentile of the return distribution accounting for its skewness and kurtosis and \( \tilde{z}_\alpha(S) \) and the corresponding spread distribution percentile.\(^{15}\)

This approach constitutes a simple parametric approach accounting for mean, variance, skewness and kurtosis of the underlying non-normal distributions. Although skewness and kurtosis are also difficult to estimate it induces less heavy data requirements than any ad-hoc or empirical approach and might possibly more accurately determine the distribution of future returns. However, the expansion is, after all, only a proxy for the real distribution. Therefore, if the real distribution is not sufficiently described by the first four moments or those moments cannot be estimated with sufficient accuracy, this method yields false risk estimates.\(^{16}\)

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\(^{14}\) The skewness of \( y \) is computed from historical data over \( n \) days as \( \gamma = \frac{1}{n} \sum_{t=1}^{n} (y_t - \bar{y})^3 / \hat{\sigma}^3 \) with \( \bar{y} \) being the expected value and \( \hat{\sigma} \) the volatility of \( y \). The excess kurtosis for \( y \) is \( \kappa = \frac{1}{n} \sum_{t=1}^{n} (y_t - \bar{y})^4 / \hat{\sigma}^4 - 3 \).

\(^{15}\) We integrated the critique by Loebnitz, 2006, that worst spreads need to be deducted from worst, not from current mid-prices. To keep the sample as large as possible, we reduced the rolling window up to 20 days at the beginning of the sample, in order to also include the first two years into the results period. This discriminates models using skewness and kurtosis, but they nevertheless show superior performance as will be shown in section 3. Using continuous spread returns does not further improve results (cp. table 8 in appendix).

\(^{16}\) Cp. also Zangari (1996), p. 10f.
3 Empirical performance

Table 1: Descriptive statistics of relative bid-ask-spreads

<table>
<thead>
<tr>
<th>Bid-ask-spread</th>
<th>11/2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>DAX Mean</td>
<td>0.52%</td>
<td>0.34%</td>
<td>0.13%</td>
<td>0.09%</td>
<td>0.08%</td>
<td>0.03%</td>
<td>0.17%</td>
</tr>
<tr>
<td>Avg. Vola</td>
<td>0.21%</td>
<td>0.21%</td>
<td>0.21%</td>
<td>0.21%</td>
<td>0.21%</td>
<td>0.21%</td>
<td>0.21%</td>
</tr>
<tr>
<td>Obs.</td>
<td>3,711</td>
<td>3,712</td>
<td>3,745</td>
<td>3,473</td>
<td>3,710</td>
<td>3,736</td>
<td>40,727</td>
</tr>
<tr>
<td>MDAX Mean</td>
<td>1.39%</td>
<td>0.83%</td>
<td>0.43%</td>
<td>0.33%</td>
<td>0.29%</td>
<td>0.25%</td>
<td>0.53%</td>
</tr>
<tr>
<td>Avg. Vola</td>
<td>0.73%</td>
<td>0.48%</td>
<td>0.46%</td>
<td>0.45%</td>
<td>0.44%</td>
<td>0.42%</td>
<td>0.31%</td>
</tr>
<tr>
<td>Obs.</td>
<td>8,246</td>
<td>12,300</td>
<td>11,640</td>
<td>11,940</td>
<td>11,704</td>
<td>11,590</td>
<td>67,422</td>
</tr>
<tr>
<td>SDAX Mean</td>
<td>2.60%</td>
<td>1.83%</td>
<td>1.12%</td>
<td>0.81%</td>
<td>0.72%</td>
<td>0.65%</td>
<td>1.15%</td>
</tr>
<tr>
<td>Avg. Vola</td>
<td>2.15%</td>
<td>1.15%</td>
<td>1.05%</td>
<td>0.95%</td>
<td>0.89%</td>
<td>0.79%</td>
<td>1.04%</td>
</tr>
<tr>
<td>Obs.</td>
<td>4,827</td>
<td>10,300</td>
<td>11,897</td>
<td>12,114</td>
<td>12,506</td>
<td>12,345</td>
<td>63,560</td>
</tr>
<tr>
<td>TECDA X Mean</td>
<td>n/a</td>
<td>3.69%</td>
<td>0.68%</td>
<td>0.47%</td>
<td>0.40%</td>
<td>0.32%</td>
<td>0.55%</td>
</tr>
<tr>
<td>Avg. Vola</td>
<td>n/a</td>
<td>0.69%</td>
<td>0.49%</td>
<td>0.46%</td>
<td>0.43%</td>
<td>0.43%</td>
<td>0.70%</td>
</tr>
<tr>
<td>Obs.</td>
<td>0</td>
<td>5,773</td>
<td>7,335</td>
<td>7,622</td>
<td>7,337</td>
<td>7,685</td>
<td>34,459</td>
</tr>
<tr>
<td>All Mean</td>
<td>1.74%</td>
<td>1.03%</td>
<td>0.62%</td>
<td>0.46%</td>
<td>0.40%</td>
<td>0.33%</td>
<td>0.65%</td>
</tr>
<tr>
<td>Avg. Vola</td>
<td>1.00%</td>
<td>0.67%</td>
<td>0.64%</td>
<td>0.62%</td>
<td>0.58%</td>
<td>0.56%</td>
<td>0.60%</td>
</tr>
<tr>
<td>Obs.</td>
<td>15,984</td>
<td>35,845</td>
<td>38,369</td>
<td>38,944</td>
<td>38,993</td>
<td>38,745</td>
<td>206,170</td>
</tr>
</tbody>
</table>

While we applied the Cornish-Fisher approximation to the basic spread model of Bangia et al. (1999), analogous use in other liquidity models such as the weighted spread model in Giot and Grammig (2005) or Stange and Kaserer (2008c) is also easily feasible.

3 Empirical performance

3.1 Description of data and implementation

For the empirical part of our analysis, we obtained price and bid-ask-spread data from Datastream for the 160 stocks in the main German stock indices in the period of 7/2002 to 12/2007. Table 1 presents some descriptive statistics. The DAX was the most liquid index with the smallest spread, followed by the MDAX and the TecDAX. SDAX was the least liquid index. The spread volatility is of the same order of magnitude than the spread level. The spread was not only lowest in the DAX, it also varied considerably less over time than in the other indices. In all indices, liquidity improved over the sample period. In 2007, SDAX and TecDAX were at least as liquid as the DAX was in 2002/2003.

For the risk estimation we chose a 1-day horizon and a 99 % confidence level, conforming to the standard Basel framework, to calculate daily risk forecasts. Mean continuous mid-price return in the Bangia model is set to zero. Spread means as well as returns means in the modified L-VaR model are estimated using a 20-day rolling procedure. We account for volatility clustering using a common exponential
weighted moving average (EWMA) method. Volatilities are also calculated rolling over 20 days as

$$\sigma_t^2 = (1 - \delta) \sum_{i=1}^{20} \delta^{i-1} r_{t-i}^2 + \delta^{20} r_{t-20}^2$$

with a weight $\delta$ of 0.94.\(^{17}\) Skewness and excess-kurtosis are calculated simple and straightforward as 500-day rolling estimates. We choose a very long estimation horizon, because moment estimates with the standard method gain significantly in accuracy with the length of the horizon.\(^{18}\) Our approach therefore aims to evaluate the potential of the Cornish Fisher methodology. Other estimation procedures are available and might generate more precise results with smaller estimation samples.\(^{19}\) This might be explored in future research.

Table 2 provides an overview on our skewness and excess-kurtosis estimates. Continuous mid-price returns are only very slightly skewed. However, excess-kurtosis can become quite substantial with values of around 6, which is far from zero of the normal distributions. Spreads are heavily right-skewed and also exhibit fat tails. As exemplary illustration figure 1 shows the sample period histogram of the spread for Comdirect, an SDAX stock. It is clear, that the normal distribution hardly fits

\(^{17}\)In correspondence with JP Morgan (1996) and practical implementation of the last term as squared return instead of squared volatility in Hull (2006), p. 575. We neglected the GARCH model class, because it is less common in practice and has higher computational requirements.

\(^{18}\)Shorter rolling windows produced weak results, especially for the kurtosis estimates, which were heavily influenced by outliers.

\(^{19}\)Cp. for example Sengupta and Zheng (1997); Joonas and Gill (1998); Kim and White (2004).
3 Empirical performance

Figure 1: Sample period spread histogram and fitted normal distribution for the Comdirect stock

Table 3: Empirical percentile estimates for the Bangia model

<table>
<thead>
<tr>
<th>Emp. spread percentile estimate z(S)</th>
<th>DAX</th>
<th>MDAX</th>
<th>SDAX</th>
<th>TECDA</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>4.01</td>
<td>3.55</td>
<td>3.35</td>
<td>3.43</td>
<td>3.59</td>
</tr>
<tr>
<td>Median</td>
<td>4.02</td>
<td>3.66</td>
<td>3.40</td>
<td>3.30</td>
<td>3.72</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>22.8%</td>
<td>37.8%</td>
<td>63.8%</td>
<td>53.7%</td>
<td>52.3%</td>
</tr>
</tbody>
</table>

Table 3: Empirical percentile estimates for the Bangia model

the empirical distribution and that the distribution is right-skewed with skewness of 2.24 and fat-tailed with excess kurtosis of 6.97.

In the Bangia framework, we determined the empirical percentiles and calculated $\hat{z}_\alpha(s)$. Descriptive statistics are shown in table 3 and are similar to the range of results between 2 and 4.5 for currencies in the original paper. Average empirical percentiles range from 3.35 to 4.01 with a mean of 3.69, which is far from the 2.33 for the normal distribution. Worst losses are much more probable than would be expected from the normal distribution.

For the modified L-VaR estimation, we calculated percentiles based on the Cornish-Fisher approximation (2). Statistics are shown in table 4. Estimates also deviate from the 2.33 expected from the normal distribution. The spread estimates are, however, substantially different from the empirical estimates of the original Bangia approach and in general slightly lower.

3.2 Magnitude of liquidity risk

Table 5 shows the mean and median risk levels by index and year for each risk estimate. This allows to compare the magnitude of estimates when calculating normally
3 Empirical performance

Table 4: Cornish-Fisher percentile estimates

<table>
<thead>
<tr>
<th>Return</th>
<th>DAX</th>
<th>MDAX</th>
<th>SDAX</th>
<th>TEC-DAX</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>-2.60</td>
<td>-2.35</td>
<td>-2.84</td>
<td>-2.87</td>
<td>-2.79</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>125.9%</td>
<td>209.6%</td>
<td>161.3%</td>
<td>157.3%</td>
<td>173.3%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Spread</th>
<th>DAX</th>
<th>MDAX</th>
<th>SDAX</th>
<th>TEC-DAX</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>3.67</td>
<td>3.17</td>
<td>3.08</td>
<td>3.10</td>
<td>3.23</td>
</tr>
<tr>
<td>Median</td>
<td>3.48</td>
<td>3.09</td>
<td>2.97</td>
<td>3.08</td>
<td>3.13</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>26.3%</td>
<td>65.2%</td>
<td>28.3%</td>
<td>42.2%</td>
<td>25.7%</td>
</tr>
</tbody>
</table>

Table 5: Risk estimates by index

<table>
<thead>
<tr>
<th>Risk estimates</th>
<th>DAX</th>
<th>MDAX</th>
<th>SDAX</th>
<th>TEC-DAX</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price Risk</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>6.14%</td>
<td>4.53%</td>
<td>3.03%</td>
<td>5.59%</td>
<td>4.81%</td>
</tr>
<tr>
<td>Median</td>
<td>3.31%</td>
<td>3.97%</td>
<td>4.56%</td>
<td>4.89%</td>
<td>4.99%</td>
</tr>
<tr>
<td>L-VaR (Bangia et al.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>4.33%</td>
<td>5.34%</td>
<td>5.68%</td>
<td>6.36%</td>
<td>5.72%</td>
</tr>
<tr>
<td>Median</td>
<td>3.47%</td>
<td>4.60%</td>
<td>5.75%</td>
<td>5.65%</td>
<td>4.93%</td>
</tr>
<tr>
<td>Modified L-VaR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>4.48%</td>
<td>7.40%</td>
<td>8.18%</td>
<td>9.38%</td>
<td>7.47%</td>
</tr>
<tr>
<td>Median</td>
<td>3.95%</td>
<td>5.78%</td>
<td>6.97%</td>
<td>6.94%</td>
<td>5.83%</td>
</tr>
</tbody>
</table>

distributed mid-price risk, liquidity-adjusted total risk of the original Bangia model (equation (1)) and our new modified L-VaR (equation (3)). Liquidity adjusted total risk estimated with the original Bangia model is naturally higher at an average 5.57% than normally estimated price risk at 4.81%, which neglects liquidity. The modified L-VaR provides the highest risk estimates with 7.47% average daily VaR. As could be expected, SDAX and TecDAX are the indices with the highest overall risk level. The DAX has the lowest risk level, especially pronounced if liquidity risk is also taken into account.

Overall, neglecting liquidity risk leads to a severe underestimation of the total risk of an asset. The deviation between the Bangia method and the Cornish-Fisher method is largest for the less liquid indices SDAX and TecDAX. We will now analyze which liquidity adjustment is more precise.

3.3 Preciseness of liquidity-adjusted risk models

3.3.1 Backtesting framework

L-VaR models are only useful insofar as they predict risk reasonable well. Therefore we will evaluate their validity through a comparison between actual and predicted loss levels. If a model is perfectly calibrated, the percentage of days where losses exceed the VaR-prediction exactly matches the confidence level. If there are more exceedances than predicted, the model underestimates risk and too little regulatory capital is allocated to the position. However, too little exceedance, hence overestima-
3 Empirical performance

tion, leads to inefficient use of capital.\textsuperscript{20} Since parameters are backward-estimated, the backtesting is, of course, out-of-sample.

Since we calculate L-VaR for a confidence level of 99\%, we expect the frequency of exceedances to equal 1\%. We use the standard test by Kupiec (1995) to determine if the realized frequency deviates from the predicted level of 1\% on a statistically significant basis.\textsuperscript{21} Kupiec (1995) shows, that the probability of observing losses in excess of VaR on $N$ days over the forecast period $T$ is governed by the binomial process\textsuperscript{22}

\[ P(N) = \left(1 - \frac{N}{T}\right)^{T-N} \left(\frac{N}{T}\right)^N \]

We expect the frequency of exceedances to equal $\alpha = 1\%$. The question if the realized frequency of losses in excess of VaR $N/T$ is significantly different from the predicted $\alpha$ can be answered with the likelihood ratio (LR) test statistic

\[ LR_{\alpha c} = -2 \ln \left[ (1 - \alpha)^{T-N} \alpha^N \right] + 2 \ln \left[ (1 - N/T)^{T-N} (N/T)^N \right] \]

(4)

which is chi-squared distributed with one degree of freedom under the null hypothesis that $\alpha = N/T$. Taking a confidence interval of 95\% for the test statistic, the null hypothesis would be rejected for $LR_{\alpha c} < 3.84$.\textsuperscript{23}

Realized losses are calculated as realizable net return when liquidating the position

\[ r_{\text{net}} = \ln \left( \frac{P_{t}^{\text{mid}}}{P_{t-1}^{\text{mid}}} \right) + \ln \left( 1 - \frac{1}{2} S_{t} \right) \]

For our sample of stocks, we calculated the percentage of stocks where the realized loss frequency did not deviate from the predicted frequency on a statistically significant basis. For the fraction of stocks, where the Kupiec-statistic could not accept the VaR-approach, we further investigate the reason. Either the model has been rejected due to too many L-VaR-exceedances and overestimates risk or too few, i.e. it underestimates risk. We also determine the respective fraction of stocks with under- and overestimation.

\textsuperscript{23}Please note that the choice of the confidence region for the test statistic is not related to the confidence level selected for the L-VaR-calculation, but merely refers to the decision rule to accept or reject the model.
In addition we will analyze the magnitude $M$ of VaR-exceedances calculated as difference between realized and predicted loss

$$M = (r_{net,t} - LVaR_t | r_{net,t} < -LVaR_t)$$

$M$ can be seen as the unpredicted loss and characterizes the level of underestimation of the risk measure. It shows if the realized loss is only marginally or substantially larger than estimated, therefore being an additional indicator for the accuracy of a L-VaR-approach.

### 3.3.2 Backtesting results

Figure 2 shows the breakdown of stocks, where risk has been correctly, under- or overestimated. The risk measure is defined as incorrect, if the Kupiec statistic (equation (4)) produces a statistically significant deviation between risk forecast and return realization. The results demonstrate the vast improvement of the Cornish-Fisher parametrization over the original Bangia model. Risk has been correctly predicted for 83 % of the stocks with the modified L-VaR compared with 44 % with the empirical percentiles model of Bangia et al. The Bangia model seems to generally underestimate risk in all indices, although there is also slight overestimation in the MDAX and SDAX as well. The modified L-VaR model produces around 8 %

Table 8 in the Appendix shows that improvement stems from accounting for non-normality in price but mostly in liquidity costs.
3 Empirical performance

Table 6: Magnitude of exceedances

<table>
<thead>
<tr>
<th>Magnitude of exceedances</th>
<th>Index</th>
<th>DAX</th>
<th>MDAX</th>
<th>SDAX</th>
<th>TecDAX</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>Bangia et al.</td>
<td>733</td>
<td>1198</td>
<td>1031</td>
<td>630</td>
<td>3,612</td>
</tr>
<tr>
<td></td>
<td>Modified L-VaR</td>
<td>477</td>
<td>869</td>
<td>771</td>
<td>354</td>
<td>2,471</td>
</tr>
<tr>
<td>Mean</td>
<td>Bangia et al.</td>
<td>1.05%</td>
<td>1.82%</td>
<td>2.39%</td>
<td>2.39%</td>
<td>1.92%</td>
</tr>
<tr>
<td></td>
<td>Modified L-VaR</td>
<td>0.98%</td>
<td>3.68%</td>
<td>2.23%</td>
<td>2.23%</td>
<td>2.23%</td>
</tr>
<tr>
<td>Median</td>
<td>Bangia et al.</td>
<td>0.61%</td>
<td>0.86%</td>
<td>1.22%</td>
<td>1.20%</td>
<td>0.91%</td>
</tr>
<tr>
<td></td>
<td>Modified L-VaR</td>
<td>0.55%</td>
<td>1.17%</td>
<td>1.42%</td>
<td>1.07%</td>
<td>1.07%</td>
</tr>
<tr>
<td>Max.</td>
<td>Bangia et al.</td>
<td>45.12%</td>
<td>52.39%</td>
<td>47.85%</td>
<td>87.07%</td>
<td>87.07%</td>
</tr>
<tr>
<td></td>
<td>Modified L-VaR</td>
<td>33.22%</td>
<td>103.76%</td>
<td>33.11%</td>
<td>71.27%</td>
<td>103.76%</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>Bangia et al.</td>
<td>1.83%</td>
<td>9.03%</td>
<td>2.96%</td>
<td>4.56%</td>
<td>6.01%</td>
</tr>
<tr>
<td></td>
<td>Modified L-VaR</td>
<td>1.03%</td>
<td>1.16%</td>
<td>2.07%</td>
<td>8.33%</td>
<td>5.71%</td>
</tr>
</tbody>
</table>

The performance in the TecDAX is worst compared to all other indices. Especially from a regulatory perspective, the substantial underestimation by the Bangia model poses a significant problem. Risk does not seem to be adequately measured by the normal return distribution and empirical spread percentiles.

Table 6 shows the magnitude of exceedances $M$ as calculated by equation (5). As shown already by the acceptance rate, the number of exceedances is much higher in the Bangia than in the modified L-VaR model. The level of exceedance is comparable between both models. There seem to a large number of small exceedances in the Bangia model, which slightly downward biases the mean exceedance for DAX and MDAX stocks. This is also underlined by the higher standard deviation in the Bangia model. The maximum exceedance can reach 90-100% of the estimation, slightly higher for the modified L-VaR model than for the original Bangia. However, maximum exceedance is only higher in the MDAX, in the other indices, the Bangia model surpasses the modified L-VaR. All in all, exceedances seem to be similar in both models.

While results already seem robust when looking at different indices, we will also look at time sub-samples as further robustness test. We split the full period and calculated the percentage of stocks with correct risk estimation separately for each sub-period. While the absolute acceptance level of the Kupiec statistic is less reliable, because the sample is smaller, the relative level between the two models is of interest. Results are shown in table 7. In both sub-periods, the modified L-VaR model performs consistently better than the original Bangia model across all indices. This provides an indication, that the higher preciseness of our suggested model is not specific to the period used in our comparison.
In this paper we proposed a new method to adjust a Value-at-Risk risk measure for liquidity risk. To account for non-normality in price and liquidity cost data we employed a Cornish-Fisher approximation, which takes skewness and kurtosis of a distribution into account. We tested our modified L-VaR, as well as a standard specification by Bangia et al. (1999) in a sample of daily frequency for 160 stocks over 5.5 years. Our modified L-VaR proves to be highly superior. The Kupiec test statistic indicates that risk is correctly estimated for substantially more stocks. Accounting for non-normality via the Cornish-Fisher approximation provides significantly more accurate results than with the empirical method of Bangia et al. (1999).

While these results are restricted to situations where positions can be traded at the bid-ask-spread the method can be analogously applied to other liquidity measures such as weighted spread of Stange and Kaserer (2008c). As other liquidity cost measures like weighted spread are similarly non-normal, we hypothesize that our method is also superior in other liquidity risk approaches. This, however, remains to be tested. Future research should also test more precise moment estimation techniques, which are likely to further improve the preciseness of our liquidity risk estimation.

### Table 7: Percent of stocks with correct risk estimation by sub-period

<table>
<thead>
<tr>
<th></th>
<th>DAX</th>
<th>MDAX</th>
<th>SDAX</th>
<th>TECDA</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sub-period H/2002 - 1/2005</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L-VaR (Bangia et al.)</td>
<td>50.00%</td>
<td>62.50%</td>
<td>73.50%</td>
<td>67.50%</td>
<td>66.12%</td>
</tr>
<tr>
<td>Modified L-VaR</td>
<td>64.38%</td>
<td>85.33%</td>
<td>81.15%</td>
<td>85.90%</td>
<td>84.32%</td>
</tr>
<tr>
<td><strong>Sub-period H/2005 - 11/2007</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L-VaR (Bangia et al.)</td>
<td>33.33%</td>
<td>33.33%</td>
<td>51.25%</td>
<td>52.50%</td>
<td>45.07%</td>
</tr>
<tr>
<td>Modified L-VaR</td>
<td>83.75%</td>
<td>80.00%</td>
<td>83.50%</td>
<td>90.00%</td>
<td>85.31%</td>
</tr>
</tbody>
</table>

Overall, the higher acceptance rate and the comparable level of exceedance magnitude make the modified L-VaR model highly superior to the original Bangia et al. specification. While these results are restricted to situations where positions can be liquidated at bid-ask-spread costs, we hypothesize that results similarly improve when using other, possibly more comprehensive risk measures. Results might also be further improved if more sophisticated estimation techniques for skewness and kurtosis are incorporated. We leave this point for further research. Overall, backtesting results demonstrate the vast superiority of our suggested liquidity risk estimation technique based on a Cornish-Fisher approximation.

### 4 Conclusion

In this paper we proposed a new method to adjust a Value-at-Risk risk measure for liquidity risk. To account for non-normality in price and liquidity cost data we employed a Cornish-Fisher approximation, which takes skewness and kurtosis of a distribution into account. We tested our modified L-VaR, as well as a standard specification by Bangia et al. (1999) in a sample of daily frequency for 160 stocks over 5.5 years. Our modified L-VaR proves to be highly superior. The Kupiec test statistic indicates that risk is correctly estimated for substantially more stocks. Accounting for non-normality via the Cornish-Fisher approximation provides significantly more accurate results than with the empirical method of Bangia et al. (1999).

While these results are restricted to situations where positions can be traded at the bid-ask-spread the method can be analogously applied to other liquidity measures such as weighted spread of Stange and Kaserer (2008c). As other liquidity cost measures like weighted spread are similarly non-normal, we hypothesize that our method is also superior in other liquidity risk approaches. This, however, remains to be tested. Future research should also test more precise moment estimation techniques, which are likely to further improve the preciseness of our liquidity risk estimation.
4 Conclusion
Appendix

Table 8: Detailed backtest results
Table shows detailed backtest results for all models tested. Accepted is the percentage of stocks where Kupiec (1995) could not reject correct risk estimation, underestimated/overestimated is percentage of remaining stocks with estimated risk lower/higher than real risk. Modified L-VaR is proposed risk model according to equation (3). Modified L-VaR (cont. spread) is the proposed risk model calculated with continuous liquidity returns. Modified L-VaR (normal price) is the proposed risk model where price risk is assumed normal as in the original Bangia et al. model. L-VaR (Bangia et al.) is the original risk model according to equation (1).

<table>
<thead>
<tr>
<th>Detailed backtest results</th>
<th>DAX</th>
<th>MDAX</th>
<th>SDAX</th>
<th>TECAX</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Underestimated</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modified L-VaR</td>
<td>8%</td>
<td></td>
<td>10%</td>
<td>13%</td>
<td>8%</td>
</tr>
<tr>
<td>Modified L-VaR (cont. spread)</td>
<td>0%</td>
<td></td>
<td>11%</td>
<td>13%</td>
<td>11%</td>
</tr>
<tr>
<td>Modified L-VaR (normal price)</td>
<td>57%</td>
<td></td>
<td>24%</td>
<td>46%</td>
<td>40%</td>
</tr>
<tr>
<td>L-VaR (Bangia et al.)</td>
<td>82%</td>
<td>57%</td>
<td>40%</td>
<td>69%</td>
<td>56%</td>
</tr>
<tr>
<td><strong>Accepted</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modified L-VaR</td>
<td>91%</td>
<td>81%</td>
<td>85%</td>
<td>78%</td>
<td>83%</td>
</tr>
<tr>
<td>Modified L-VaR (cont. spread)</td>
<td>91%</td>
<td>80%</td>
<td>84%</td>
<td>78%</td>
<td>82%</td>
</tr>
<tr>
<td>Modified L-VaR (normal price)</td>
<td>43%</td>
<td>69%</td>
<td>63%</td>
<td>54%</td>
<td>58%</td>
</tr>
<tr>
<td>L-VaR (Bangia et al.)</td>
<td>17%</td>
<td>42%</td>
<td>53%</td>
<td>49%</td>
<td>44%</td>
</tr>
<tr>
<td><strong>Overestimated</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modified L-VaR</td>
<td>3%</td>
<td>11%</td>
<td>9%</td>
<td>6%</td>
<td>8%</td>
</tr>
<tr>
<td>Modified L-VaR (cont. spread)</td>
<td>3%</td>
<td>11%</td>
<td>9%</td>
<td>6%</td>
<td>7%</td>
</tr>
<tr>
<td>Modified L-VaR (normal price)</td>
<td>0%</td>
<td>2%</td>
<td>3%</td>
<td>0%</td>
<td>2%</td>
</tr>
<tr>
<td>L-VaR (Bangia et al.)</td>
<td>0%</td>
<td>1%</td>
<td>1%</td>
<td>0%</td>
<td>0%</td>
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References


References


References


ZANGARI, P. (1996): A VaR methodology for portfolios that include options. RiskMetrics(TM) Monitor First quarter, JPMorgan